<u>Site graphs</u> Throughout we assume a set of agent names ~~Ag~~ Nme, a set of site identifiers SiteId and a signature

$$\Sigma : Nme \longrightarrow \mathcal{P}_{fin}(SiteId).$$

A <u>site graph</u> comprises

- a set of <u>agents</u> Ag with a name function $nme : Ag \longrightarrow Nme$;

- a set of <u>sites</u> $Sts \subseteq \{(A,i) \mid i \in \Sigma\, nme(A)\}$, with a function $ag : Sts \to Ag$ s.t. $ag(A,i) = A$;

- a set of <u>links</u> $Lnk \subseteq Sts \times Sts$ forming a partial function $Sts \rightharpoonup Sts$ — when $((A,i),(B,j)) \in Lnk$ we write $src((A,i),(B,j)) = (A,i)$ and $tar((A,i),(B,j)) = (B,j)$ for its source and target.

We say a site graph is <u>complete</u> when $Sts = \{(A,i) \mid A \in Ag \;\&\; i \in \Sigma\, nme(A)\}$.

Remark The presentation ignores natural axioms ~~[illegible]~~ on site graphs (eg. that the Lnk partial function is irreflexive), extra information such as link state and internal state of a site.

A <u>map</u> from site graph $T = (Ag, \ldots)$ to $T' = (Ag', \ldots)$ is a triple of partial functions $(f_{Ag}, f_{Sts}, f_{Lnk})$ where

$f_{Ag} : Ag \to Ag'$, $f_{Sts} : Sts \to Sts'$ where $f_{Sts}(A,i) = (A',i)$ for some $A' \in Ag'$ when defined,
and $f_{Lnk} : Lnk \to Lnk'$

which satisfy certain (partial) commutation relations.

$$\begin{tikzcd}
Nme \arrow[d, equal] \arrow[dr, phantom, "="] & Ag \arrow[l, "nme"'] \arrow[d, "f_{Ag}"] \arrow[dr, phantom, "\supseteq"] & Sts \arrow[l, "ag"'] \arrow[d, "f_{Sts}"] \arrow[dr, phantom, "\supseteq"] & Lnk \arrow[l, "sce"'] \arrow[r, "tar"] \arrow[d, "f_{Lnk}"] \arrow[dr, phantom, "\subseteq"] & Sts \arrow[d, "f_{Sts}"] \\
Nme' & Ag' \arrow[l, "nme'"] & Sts' \arrow[l, "ag'"] & Lnk' \arrow[l, "sce'"] \arrow[r, "tar'"'] & Sts'
\end{tikzcd}$$

The commutation relations say

$$nme' \circ f_{Ag} = nme$$

$$ag' \circ f_{Sts} \subseteq f_{Ag} \circ ag.$$

$$sce' \circ f_{Lnk} \subseteq f_{Sts} \circ sce \quad \text{and}$$

$$tar' \circ f_{Lnk} \subseteq f_{Sts} \circ tar.$$

Maps compose componentwise to form a category

We call such a map of site graphs an <u>embedding</u> when f_{Ag}, f_{Sts} and f_{Lnk} are injective (total) functions (for this injectivity of f_{Ag} is sufficient to ensure injectivity of f_{Sts} and f_{Lnk}) and

$$ag' \circ f_{Sts} = f_{Ag} \circ ag,$$

$$sce' \circ f_{Lnk} = f_{Sts} \circ sce,$$

$$tar' \circ f_{Lnk} = f_{Sts} \circ tar, \text{ and}$$

$((f_{Ag}(A), i), (f_{Ag}(B), j)) \in Lnk'$, where $A, B \in Ag$, implies $((A, i), (B, j)) \in Lnk$.

We call the map an <u>action map</u> when f_{Ag}, f_{Sts} and f_{Lnk} are partial injective functions (for this the partial injectivity of f_{Ag}, viz. $f_{Ag}(A) = f_{Ag}(B)$, where both are defined, implies $A = B$., is sufficient) and

$$ag' \circ f_{Sts} = f_{Ag} \circ ag$$

& $f_{Ag}(A) = A'$ & $(A', i) \in Sts' \Rightarrow (A, i) \in Sts$

To signify that $\psi : S \to T$ a map of site graphs is an embedding we'll write

$$\psi : S \rightarrowtail T$$

and that $\alpha : S \to T$ is an action map write

$$\alpha : S \twoheadrightarrow T.$$

Embeddings compose to give embeddings and include all isomorphisms. Likewise action maps compose and include isomorphisms.

Rules A rule denotes an action map

$$S \xrightarrow{\alpha} S'.$$

An application of such a rule is determined by an embedding

$$\varphi : S \rightarrowtail T$$

into a ~~fixed~~ complete site graph T. Its effect is given by a pushout

$$\begin{tikzcd}
S \arrow[r, "\alpha"] \arrow[d, tail, "\varphi"'] & S' \arrow[d, tail, "\varphi'"] \\
T \arrow[r, "\alpha'"'] \arrow[ur, phantom, "\ulcorner", very near end] & T'
\end{tikzcd}$$

in the category of site graphs and their maps. That the pushout maps φ' and α' take the form of an embedding and an action map, respectively, relies on the following proposition.

<u>Proposition</u> A pair of maps

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, tail, "\varphi"'] & S' \\ T & \end{tikzcd}$$

where φ is an embedding and α is an action map has a pushout in the category of site graphs and their maps. Moreover, the pushout takes the form

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, tail, "\varphi"'] & S' \arrow[d, tail, "\varphi'"] \\ T \arrow[r, "\alpha'"'] & T' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where φ' is an embedding & α' is an action map.

~~[illegible]~~

The proposition is driven by the following construction of a special pushout in the category of sets with partial functions; — it is this construction which determines the form of agents in T', and from these its other structure. Consider maps

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, tail, "\varphi"'] & S' \\ T & \end{tikzcd}$$

in the category of sets with partial functions, assuming φ is ~~[illegible]~~ total and injective, and α is a partial injective function i.e.

$$\alpha(s_1) = \alpha(s_2), \text{ both defined, } \Rightarrow s_1 = s_2 .$$

Then the pushout can be constructed as follows:

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, tail, "\varphi"'] & S' \arrow[d, tail, dashed, "\varphi'"] \\ T \arrow[r, dashed, "\alpha'"'] & T' \end{tikzcd}$$

where

$$T' = \Big(\big(T \setminus \{\varphi(s) \mid \alpha(s) \text{ is undefd.}\}\big) \mathbin{\dot\cup} S' \Big) \big/ \sim$$

the equivalence classes of a disjoint union under equivalence relation $\sim$ ~~[illegible]~~ s.t.

$$(1,t) \sim (2,s') \text{ if } \exists s \in S.\ t = \varphi(s) \ \&\ s' = \alpha(s).$$

[I'm using 1 and 2 to tag the two components of the disjoint union]. Then define

$$\alpha'(t) = \begin{cases} \{(1,t)\}_\sim & \text{if } t \in T \\ \text{undefined} & \text{otherwise,} \end{cases}$$

and $\varphi'(s') = \{(2,s')\}_\sim$.

Compression lemma

$$
\begin{tikzcd}
T_1' \arrow[rrrr, dashed, "\gamma'"] & & & & T_2' \\
& T_1 \arrow[ul, "\alpha_1"'] \arrow[rr, "\gamma"] & & T_2 \arrow[ur, "\alpha_2"'] & \\
& & S \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] \arrow[d, "\alpha"] & & \\
& & S' \arrow[uull, bend left, "\varphi_1'"] \arrow[uurr, bend right, "\varphi_2'"'] & &
\end{tikzcd}
$$

Given the (undotted) situation above, where γ is a site-graph map there is a unique map γ' (dotted map as shown) s.t.

(i) $\alpha_2 \circ \gamma = \gamma' \circ \alpha_1$ &

(ii) $\varphi_2' = \gamma' \circ \varphi_1'$.

Proof. Existence of a site-graph map γ' s.t. (i) and (ii) follows directly from universality of the pushout of α, φ_1. ~~That γ' is an action map will require a detailed check.~~ $\boxtimes$

(?) Think γ' is an action map if γ is an action map. (needs checking)

Naive compression. Let

$$T_0 \xrightarrow{\alpha_1'} T_1 \xrightarrow{\alpha_2'} \cdots \xrightarrow{\alpha_n'} T_n$$

be determined by a sequence of applications of rules

$$\begin{tikzcd} S_1 \arrow[r, "\alpha_1"] \arrow[d, tail, "\varphi_1"'] & S_1', & S_2 \arrow[r, "\alpha_2"] \arrow[d, tail, "\varphi_2"'] & S_2', & \cdots, & S_n \arrow[r, "\alpha_n"] \arrow[d, tail, "\varphi_{n-1}"'] & S_n'. \\ T_0 & & T_1 & & & T_{n-1} & \end{tikzcd}$$

A naive-compression step is determined by

$$\begin{tikzcd} & & \vdots \\ & & T_j \arrow[u, "\alpha_{j+1}'"'] \\ T_i \arrow[r, "\alpha_{i+1}'"] & \cdots \arrow[r, "\alpha_{j-1}'"] & T_{j-1} \arrow[u, "\alpha_j'"'] \\ \vdots \arrow[u, "\alpha_i"] & & \end{tikzcd}$$

where for some ψ,

$$\begin{tikzcd} T_i \arrow[r, "\alpha_{i+1}'"] & \cdots \arrow[r, "\alpha_{j-1}'"] & T_{j-1} \\ & S_j \arrow[ul, tail, bend left, "\psi"] \arrow[ur, tail, bend right, "\varphi_j"'] \arrow[u, phantom, "="] & \end{tikzcd}$$

Then by the compression lemma we can replace the original sequence by

$$
\begin{tikzcd}
 & \vdots \\
T_j' \arrow[r, "\gamma"] & T_j \arrow[u, "\alpha'_{j+1}"'] \\
T_i \arrow[u, "\alpha''_j"] & \\
\vdots \arrow[u, "\alpha_i"] &
\end{tikzcd}
$$

~~A~~ The sequence $T_0 \xrightarrow{\alpha_1'} \cdots \xrightarrow{\alpha_n'} T_n$ is naive compressed if it does not permit any weak compression step.

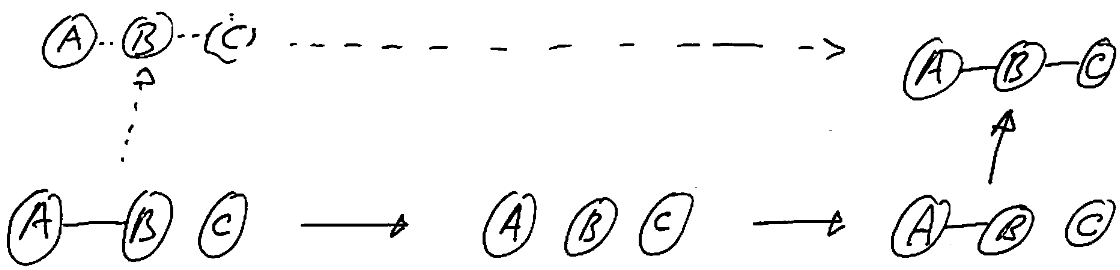
A problem with naive compression

Consider rules

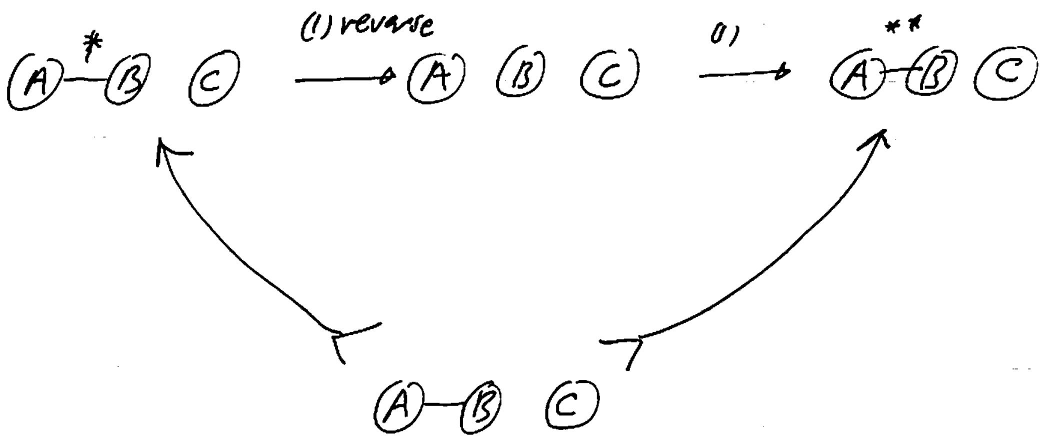
(1) $A \quad B \longleftrightarrow A - B$

(2) $A - B \quad C \longrightarrow A - B - C$

Starting from ~~the~~ site graph $A - B \quad C$ we obtain a sequence of rule applications:



We would like to compress as shown by the dotted part. But we do ~~not~~ have



commutes. ~~as~~ The link * goes to undefined under map '(1) reverse' ~~sam~~ & does not get sent to the link ** under the composite ~~no~~ action map (A)—(B) (C) ⟶ (A) (B) (C) ⟶ (A)—(B) (C).

However this map does express that agents (A), (B) on the left are identical to agents (A), (B) on the right, so the ~~same~~ (A)—(B) links must be the same.

By making the maps of site graphs

treat ~~[illegible]~~ identity of sites and links so separately ~~[illegible]~~ — and we need this for the pushouts in applying rules — we have failed to capture that the identity of sites and links is determined by the agents they belong to (once they are present).

<u>The solution</u> A less exacting condition ~~definition~~ of commutativity. Let $f, g : S \longrightarrow T$ be maps of site graphs. Write

$$f \sim_{Ag} g \quad \text{iff} \quad f_{Ag} = g_{Ag}$$

ie their constituent maps on agents are the same.

Let $S = (Ag_S, \ldots)$, $T = (Ag_T \ldots)$ be site graphs. Let $h : Ag_S \longrightarrow Ag_T$. Define site-graph map $\tilde{h} : S \longrightarrow T$ by

$$(\tilde{h})_{Ag} = h ,$$

$$h_{Sts}(A,i) = \begin{cases} (h(A),i) & \text{provided } (h(A),i) \in Sts_T \\ \text{undefined} & \text{otherwise,} \end{cases}$$

$$h_{Lnk}((A,i),(B,j)) = \begin{cases} ((h(A),i),\ (h(B),j)) \\ \quad \text{provided } ((h(A),i),(h(B),j)) \in Lnk_T \\ \text{undefined otherwise.} \end{cases}$$

~~Welllllle~~ <u>Proposition</u>

$$\begin{tikzcd} T \arrow[rr, "\gamma"] & & T' \\ & S \arrow[ul, tail, "\varphi"] \arrow[ur, tail, "\varphi'"'] \arrow[u, phantom, "\simeq_{Ag}"] & \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} T \arrow[rr, "\widetilde{(\gamma_{Ag})}"] & & T' \\ & S \arrow[ul, tail, "\varphi"] \arrow[ur, tail, "\varphi'"'] \arrow[u, phantom, "="] & \end{tikzcd}$$

Weak compression lemma

$$\begin{tikzcd}
T_1' \arrow[rrrr, dashed, "\gamma'"] & & & & T_2' \\
& T_1 \arrow[ul, "\alpha_1"] \arrow[rr, "\gamma"] \arrow[urr, phantom, "\sim_{Ag}"] & & T_2 \arrow[ur, "\alpha_2"'] & \\
& & S \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] \arrow[u, phantom, "\sim_{Ag}"] \arrow[d, "\alpha"] & & \\
& & S' \arrow[uuull, bend left, "\varphi_1'"] \arrow[uuurr, bend right, "\varphi_2'"'] & &
\end{tikzcd}$$

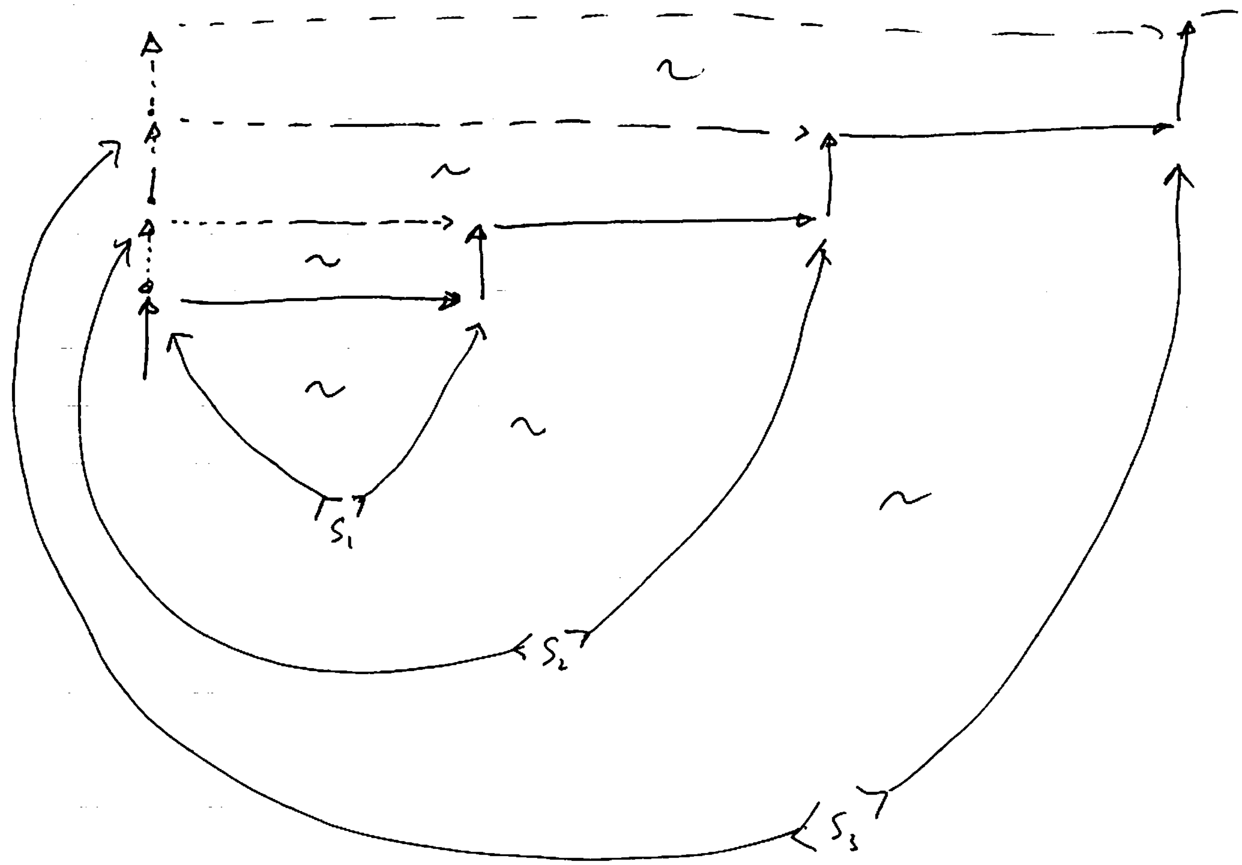
Given the (undotted) situation above there is a (not necessarily unique) site graph map γ' (dotted) s.t.

(i) $\alpha_2 \circ \gamma \sim_{Ag} \gamma' \circ \alpha_1$ &

(ii) $\varphi_2' \sim_{Ag} \gamma' \circ \varphi_1'$.

<u>Proof</u> Replace γ by $\widetilde{\gamma}_{Ag}$ and then use the compression lemma. ☒

A series of weak compressions:



Taking account of symmetry of agents

Let $f, g : S \longrightarrow T$ be maps of site graphs. Let $\sigma : Ag_T \cong Ag_T$ be a permutation (bijection) of agents of T. Write

$$f \overset{\sigma}{\Rightarrow} g, \quad \text{or} \quad \begin{tikzcd} S \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & T \end{tikzcd},$$

if $g_{Ag} = \sigma \circ f_{Ag}$.

Strong compression lemma

$$
\begin{tikzcd}
T_1' \arrow[rrrr, dashed, "\gamma'"] & & & & T_2' \\
& T_1 \arrow[ul, "\alpha_1"'] \arrow[rr, "\gamma"] & & T_2 \arrow[ur, "\alpha_2"] & \\
& & S \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] \arrow[u, phantom, "\Rightarrow \sigma"] \arrow[d, "\alpha"] & & \\
& & S' \arrow[uuull, bend left, "\varphi_1'"] \arrow[uuurr, bend right] & &
\end{tikzcd}
$$

Given the (undotted) situation where γ is a site-graph map, there is a site-graph map γ' s.t.

(i) $\alpha_2 \circ \tilde{\sigma} \circ \gamma \sim_{Ag} \gamma' \circ \alpha_1$ &

(ii) $\varphi_2' \sim_{Ag} \gamma' \circ \varphi_1'$.

Proof. From the weak-compression lemma replacing γ there by $\tilde{\sigma} \circ \gamma$. $\boxtimes$