Event Structures, Stable Families and Concurrent Games

Notes for "Distributed Games and Strategies"
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Glynn Winskel
gw104@cl.cam.ac.uk

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Preface

These notes introduce a theory of two-party games still under development. A lot can be said for a general theory to unify all manner of games found in the literature. But this has not been the main motivation. That has been the development of a generalized domain theory, to lift the methodology of domain theory and denotational semantics to address the highly interactive nature of computation we find today.

There are several arguments why the next generation of domain theory should be an intensional theory, one which pays careful attention to the ways in which output is computed from input. One is that if the theory is to be able to reason about operational concerns it had better address them, albeit abstractly. Another is that sometimes the demands of compositionality force denotations to be more intensional than one would at first expect; this occurs for example with nondeterministic dataflow—see the Introduction. These notes take seriously the idea that intensional aspects be described by strategies, and, to fit computational needs adequately, try to understand the concept of strategy very broadly.

This idea comes from game semantics where the domains and continuous functions of traditional domain theory and denotational semantics are replaced by games and strategies. Strategies supercede functions because they give a much better account of interaction extended in time. (Functions, if you like, have too clean a separation of interaction into input and output.) In traditional denotational semantics a program phrase or process term denotes a continuous function, whereas in game semantics a program phrase or process term denotes a strategy.

However, traditional game semantics is not always general enough, for instance in accounting for nondeterministic or concurrent computation. Rather than extending traditional game semantics with various bells and whistles, these notes attempt to carve out a general theory of games within a general model of nondeterministic, concurrent computation. The model chosen is the partial-order model of event structures, and for technical reasons, its enlargement to stable families. Event structures have the advantage of occupying a central position within models for concurrency, and the development here should suggest analogous developments for other 'partial-order' models such as Mazurkiewicz trace languages, Petri nets and asynchronous transition systems, and even 'interleaving' models based on transition systems or sequences.

In their present state, these notes are incomplete in several ways. First, they don't account for games with back-tracking, games where play can revisit previous positions. While a little odd from the point of view of everyday games, this feature is very important in game semantics, for instance in order to re-evaluate the argument to a function.[1] Second, the notes don't have enough examples. Third, the notes say too little on the *uses* of games and strategies in semantics,

[1]The theory has been extended to allow back-tracking and copying via event structures with symmetry, which support a rich variety of pseudo (co)monads to achieve this.

types, logic and verification. Fourth, they don't address the issue of parallel causes thoroughly. I hope to some extent to make up for these inadequacies in the lectures and some are addressed in the broader "ECSYM Notes" [1]. What I claim the notes do do, is begin to unify a variety of approaches and provide canonical general constructions and results, which leave the student better placed to structure and analyse critically the often arcane world of games and strategies in the literature.

Contents

Chapter 1

Introduction

Games and strategies are everywhere, in logic, philosophy, computer science, economics, in leisure and in life.

Slogan: Processes are nondeterministic concurrent strategies.

1.1 Motivation

We summarise some reasons for developing a theory of nondeterministic concurrent games and strategies.

1.1.1 What is a process?

In the earliest days of computer science it became accepted that a computation was essentially an (effective) partial function $f : \mathbb{N} \to \mathbb{N}$ between the natural numbers. This view underpins the Church-Turing thesis on the universality of computability.

As computer science matured it demanded increasingly sophisticated mathematical representations of processes. The pioneering work of Strachey and Scott in the denotational semantics of programs assumed a view of a process still as a function $f : D \to D'$, but now acting in a continuous fashion between datatypes represented as special topological spaces, 'domains' D and D'; reflecting the fact that computers can act on complicated, conceptually-infinite objects, but only by virtue of their finite approximations.

In the 1960's, around the time that Strachey started the programme of denotational semantics, Petri advocated his radical view of a process, expressed in terms of its events and their effect on local states—a model which addressed directly the potentially distributed nature of computation, but which, in common with many other current models, ignored the distinction between data and process implicit in regarding a process as a function. Here it seems that an adequate notion of process requires a marriage of Petri's view of a process and the vision of Scott and Strachey. An early hint in this direction came in answer to the following question.

What is the information order in domains? There are essentially two answers in the literature, the '*topological*,' the most well-known from Scott's work, and the '*temporal*,' arising from the work of Berry:

• *Topological*: the basic units of information are *propositions* describing finite properties; more information corresponds to more propositions being true. Functions are ordered pointwise.

• *Temporal*: the basic units of information are *events*; more information corresponds to more events having occurred over time. Functions are restricted to 'stable' functions and ordered by the intensional 'stable order,' in which common output has to be produced for the same minimal input. Berry's specialized domains 'dI-domains' are represented by event structures.

In truth, Berry developed 'stable domain theory' by a careful study of how to obtain a suitable category of domains with stable rather than all continuous functions. He arrived at the axioms for his 'dI-domains' because he wanted function spaces (so a cartesian-closed category). The realization that dI-domains were precisely those domains which could be represented by event structures, came a little later.

1.1.2 From models for concurrency

Causal models are alternatively described as: causal-dependence models; independence models; non-interleaving models; true-concurrency models; and partial-order models. They include Petri nets, event structures, Mazurkiewicz trace languages, transition systems with independence, multiset rewriting, and many more. The models share the central feature that they represent processes in terms of the events they can perform, and that they make explicit the causal dependency and conflicts between events.

Causal models have arisen, and have sometimes been rediscovered as *the* natural model, in many diverse and often unexpected areas of application:
Security protocols: for example, forms of event structure, strand spaces, support reasoning about secrecy and authentication through causal relations and the freshness of names;
Systems biology: ideas from Petri nets and event structures are used in taming the state-explosion in the stochastic simulation of biochemical processes and in the analysis of biochemical pathways;
Hardware: in the design and analysis of asynchronous circuits;
Types and proof: event structures appear as representations of propositions as types, and of proofs;
Nondeterministic dataflow: where numerous researchers have used or rediscovered causal models in providing a compositional semantics to nondeterministic dataflow;
Network diagnostics: in the patching together local of fault diagnoses of com-

munication networks;
Logic of programs: in concurrent separation logic where artificialities in Brookes' pioneering soundness proof are obviated through a Petri-net model;
Partial order model checking: following the seminal work of McMillan the unfolding of Petri nets (described below) is exploited in recent automated analysis of systems;
Distributed computation: event structures appear both classically,*e.g.* in early work of Lamport, and recently in the Bayesian analysis of trust and modelling multicore memory.

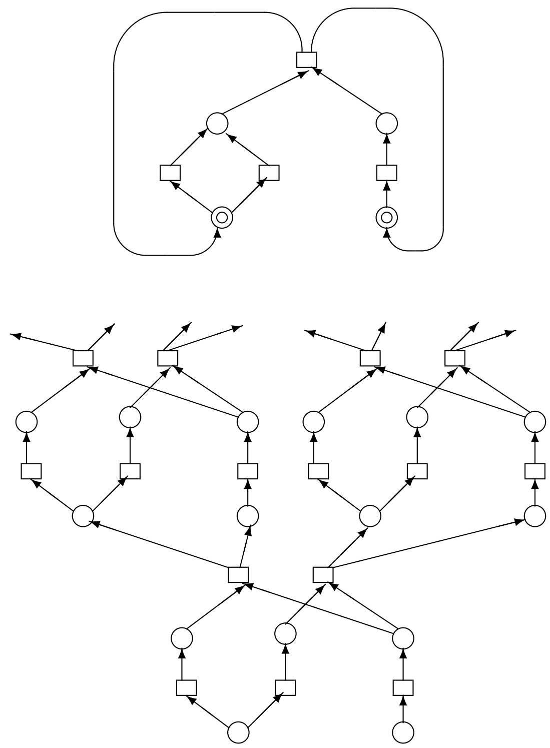
To illustrate the close relationship between Petri nets and the 'partial-order models' of occurrence nets and event structures, we sketch how a (1-safe) Petri net can be unfolded first to a net of occurrences and from there to an event structure [2]. The unfolding construction is analogous to the well-known method of unfolding a transition system to a tree, and is central to several analysis tools in the applications above. In the figure, the net on top has loops. The net below it is its *occurrence-net unfolding.* It consists of all the occurrences of conditions and events of the original net, and is infinite because of the original repetitive behaviour. The occurrences keep track of what enabled them. The simplest form of event structure, the one we shall consider here, arises by abstracting away the conditions in the occurrence net and capturing their role in relations of causal dependency and conflict on event occurrences.

The relations between the different forms of causal models are well understood [3]. Despite this and their often very successful, specialized applications, causal models lack a *comprehensive* theory which would support their systematic use in giving semantics to a broad range of programming and process languages, in particular we lack an expressive form of '*domain theory*' for causal models with rich higher-order type constructions needed by mathematical semantics.

1.1.3 From semantics

Denotational semantics and domain theory of Scott and Strachey set the standard for semantics of computation. The theory provided a global mathematical setting for sequential computation, and thereby placed programming languages in connection with each other; connected with the mathematical worlds of algebra, topology and logic; and inspired programming languages, type disciplines and methods of reasoning. Despite the many striking successes it has become very clear that many aspects of computation do not fit within the traditional framework of denotational semantics and domain theory. In particular, classical domain theory has not scaled up to the more intricate models used in interactive/distributed computation. Nor has it been as operationally informative as one could hope.

While, as Kahn was early to show, deterministic dataflow is a shining application of simple domain theory, nondeterministic dataflow is beyond its scope. The compositional semantics of nondeterministic dataflow needs a form of generalized relation which specifies the *ways* input-output pairs are realized.A compelling example comes from the early work of Brock and Ackerman who were

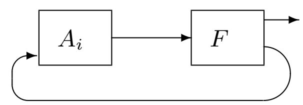


A Petri net and its occurrence-net unfolding

the first to emphasize the difficulties in giving a compositional semantics to nondeterministic dataflow, though our example is based on simplifications in the later work of Rabinovich and Trakhtenbrot, and Russell.

Nondeterministic dataflow—Brock-Ackerman anomaly

$C[A_i] =$



There are two simple nondeterministic processes A_1 and A_2, which have the same input-output relation, and yet behave differently in the common feedback context $C[-]$, illustrated above. The context consists of a fork process F (a process that copies every input to two outputs), through which the output of the automata A_i is fed back to the input channel, as shown in the figure. Process A_1 has a choice between two behaviours: either it outputs a token and stops, *or* it outputs a token, waits for a token on input and then outputs another token. Process A_2 has a similar nondeterministic behaviour: Either it outputs a token and stops, *or* it waits for an input token, then outputs two tokens. For both automata, the input-output relation relates empty input to the eventual output of one token, and non-empty input to one or two output tokens. But $C[A_1]$ can output two tokens, whereas $C[A_2]$ can only output a single token. Notice that A_1 has two ways to realize the output of a single token from empty input, while A_2 only has one. It is this extra way, not caught in a simple input-output relation, that gives A_1 the richer behaviour in the feedback context.

Over the years there have been many solutions to giving a compositional semantics to nondeterministic dataflow. But they all hinge on some form of generalized relation, to distinguish the different ways in which output is produced from input. A compositional semantics can be given using *stable spans* of event structures, an extension of Berry's stable functions to include nondeterminism [4]—see Section 6.2.1.

How are we to extend the methodology of denotational semantics to the much broader forms of computational processes we need to design, understand and analyze today? How are we to maintain clean algebraic structure and abstraction alongside the operational nature of computation?

Game semantics advanced the idea of replacing the traditional continuous functions of domain theory and denotational semantics by strategies. The reason for doing this was to obtain a representation of interaction in computation that was more faithful to operational reality. It is not always convenient or mathematically tractable to assume that the environment interacts with a computation in the form of an input argument. It is built into the view of a process as a strategy that the environment can direct the course of evolution of a process throughout its duration. Game semantics has had many dramatic successes. But it has developed from simple well-understood games, based on alternating sequences of player and opponent moves, to sometimes arcane extensions and generalizations designed to fit the demands of a succession of additional programming or process features. It is perhaps time to stand back and see how games fit within a very general model of computation, to understand better what current features of games in computer science are simply artefacts of the particular history of their development.

1.1.4 From logic

An informal understanding of games and strategies goes back at least as far as the ancient Greeks where truth was sought through debate using the dialectic method; a contention being true if there was an argument for it that could survive all counter-arguments. Formalizing this idea, logicians such as Lorenzen and Blass investigated the meaning of a logical assertion through strategies in a game built up from the assertion. These ideas were reinforced in game semantics which can provide semantics to proofs as well as programs. The study of the mathematics and computational nature of proof continues. There are several strands of motivation for games in logic. Along with automata games constitute one of the tools of logic and algorithmics; often a logical or algorithmic question can be reduced to the question of whether a particular game has a winning/optimal strategy or counterstrategy. Games are used in verification and, for example, the central equivalence of bisimulation on processes has a reading in terms of strategies.

Chapter 2

Event structures

Event structures are a fundamental model of concurrent computation and, along with their extension to stable families, provide a mathematical foundation for the course.

2.1 Event structures

Event structures are a model of computational processes. They represent a process, or system, as a set of event occurrences with relations to express how events causally depend on others, or exclude other events from occurring. In one of their simpler forms they consist of a set of events on which there is a consistency relation expressing when events can occur together in a history and a partial order of causal dependency—writing $e' \leq e$ if the occurrence of e depends on the previous occurrence of e'.

An *event structure* comprises $(E, \leq, \text{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$
\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \text{Con for all } e \in E,\\
&Y \subseteq X \in \text{Con} \implies Y \in \text{Con}, \quad \text{and}\\
&X \in \text{Con} \;\&\; e \leq e' \in X \implies X \cup \{e\} \in \text{Con}.
\end{aligned}
$$

The events are to be thought of as event occurrences without significant duration; in any history an event is to appear at most once. We say that events e, e' are *concurrent*, and write $e \; co \; e'$ if $\{e, e'\} \in \text{Con} \;\&\; e \not\leq e' \;\&\; e' \not\leq e$. Concurrent events can occur together, independently of each other. The relation of *immediate* dependency $e \rightarrowtriangle e'$ means e and e' are distinct with $e \leq e'$ and no event in between. Clearly $\leq$ is the reflexive transitive closure of $\rightarrowtriangle$.

An event structure represents a process. A configuration is the set of all events which may have occurred by some stage, or history, in the evolution of the process. According to our understanding of the consistency relation and causal dependency relations a configuration should be consistent and such that if an event appears in a configuration then so do all the events on which it causally depends.

The *configurations* of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and

Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

We shall largely work with *finite* configurations, written $\mathcal{C}(E)$. Write $\mathcal{C}^\infty(E)$ for the set of *finite and infinite* configurations of the event structure E.

The configurations of an event structure are ordered by inclusion, where $x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'. Note that an individual configuration inherits an order of causal dependency on its events from the event structure so that the history of a process is captured through a partial order of events. The finite configurations correspond to those events which have occurred by some finite stage in the evolution of the process, and so describe the possible (finite) states of the process.

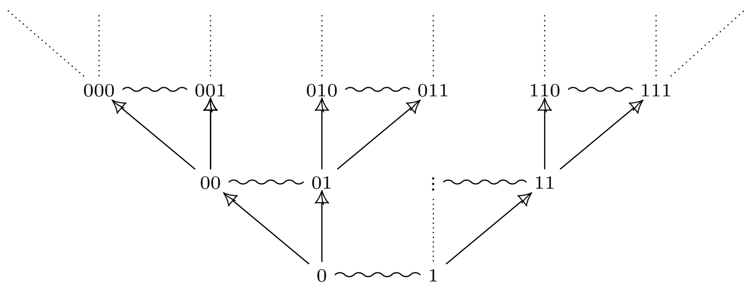
For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X. The axioms on the consistency relation ensure that the down-closure of any finite set in the consistency relation s a finite configuration, and that any event appears in a configuration: given $X \in \mathrm{Con}$ its down-closure $\{e' \in E \mid \exists e \in X.\ e' \leq e\}$ is a finite configuration; in particular, for an event e, the set $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e. We shall sometimes write $[e) =_{\mathrm{def}} \{e' \in E \mid e' < e\}$.

When the consistency relation is determined by the pairwise consistency of events we can replace it by a binary relation or, as is more usual, by a complementary binary conflict relation on events (written as # or $\smile$).

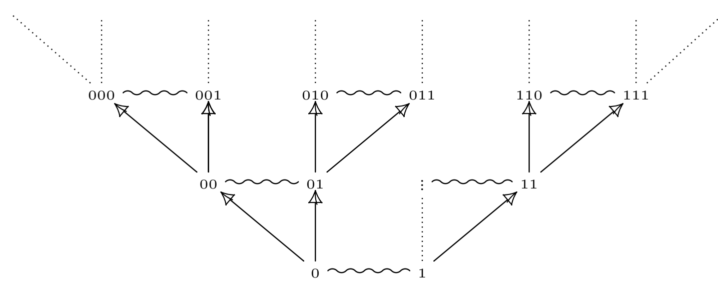
Remark on the use of "cause." In an event structure $(E, \leq, \mathrm{Con})$ the relation $e' \leq e$ means that the occurrence of e depends on the previous occurrence of the event e'; if the event e has occurred then the event e' must have occurred previously. In informal speech cause is also used in the forward-lookciaing sense of one thing arising because of another. Often when used in this way the history of events is understood beforehand. According to the history around my life, the meeting of my parents caused my birth. But the history might have been very different: in an alternative world the meeting of my parents might not have led to my birth. More formally, w.r.t. a configuration x in which an event e occurs while it seems sensible to talk about the events $[e)$ causing e, it is so only by virtue of the understood configuration x.

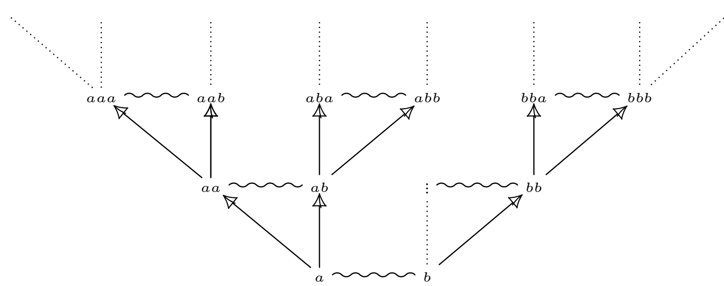
We also encounter events which in a history may have been caused in more than one way. There are generalisations of the current event structures which do this—see the chapter in [1] on "disjunctive causes." But for now we will work with the simple definition above in which an event, or really an event occurrence, e is causally dependent on a unique set of events $[e)$. Much of the mathematics we develop around these simpler forms of event structures (sometimes called prime event structures in the literature) will be reusable when we come to consider events with several causes. Roughly the simpler event structures will suffice in considering nondeterministic strategies. Where their limitations will first show up is in our treatment of probabilistic strategies.

Example 2.1. The diagram below illustrates an event structure representing streams of 0s and 1s:



Above we have indicated conflict (or inconsistency) between events by $\sim\sim\sim$. The event structure representing pairs of 0/1-streams and a/b-streams is represented by the juxtaposition of two event structures:





Exercise 2.2. *Draw the event structure of the occurrence net unfolding in the introduction.* □

2.2 Maps of event structures

Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined), then } e_1 = e_2.$$

The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. The map f respects the instantaneous nature of events: two distinct event occurrences which are consistent with each other cannot both coincide with the occurrence of a common event in the image. Partial maps of event structures compose as partial functions, with identity maps given by identity functions.

We will say the map is *total* if the function f is total. Notice that for a total map f the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective. Say a total map of event structures is *rigid* when it preserves causal dependency.

Maps preserve the concurrency relation, when defined.

Definition 2.3. Write $\mathcal{E}$ for the category of event structures with (partial) maps. Write $\mathcal{E}_t$ and $\mathcal{E}_r$ for the categories of event structures with total, respectively rigid, maps.

Exercise 2.4. *Show a map $f : A \rightharpoonup B$ of $\mathcal{E}$ is mono if the function $\mathcal{C}(A) \to \mathcal{C}(B)$ taking configuration x to its direct image fx is injective. [Recall a map $f : A \to B$ is mono iff for all maps $g, h : C \to A$ if $fg = fh$ then $g = h$.] Show the converse does not hold, that it is possible for a map to be mono but not injective on configurations.* □

Proposition 2.5. *Let E and E' be event structures. Suppose*

$$\theta_x : x \cong \theta_x x, \text{ indexed by } x \in \mathcal{C}(E),$$

is a family of bijections such that whenever $\theta_y : y \cong \theta_y y$ is in the family then its restriction $\theta_z : z \cong \theta_z z$ is also in the family, whenever $z \in \mathcal{C}(E)$ and $z \subseteq y$. Then, $\theta =_{\text{def}} \bigcup_{x \in \mathcal{C}(E)} \theta_x$ is the unique total map of event structures from E to E' such that $\theta\, x = \theta_x x$ for all $x \in \mathcal{C}(E)$.

Proof. The conditions ensure that $\theta =_{\text{def}} \bigcup_{x \in \mathcal{C}(A)} \theta_x$ is a function $\theta : A \to B$ such that the image of any finite configuration x of A under θ is a configuration of B and local injectivity holds. □

2.2.1 Partial-total factorisation

Let $(E, \leq, \text{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\text{def}} (V, \leq_V, \text{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \text{Con}_V$ iff $X \in \text{Con}$ & $X \subseteq V$.

Consider a partial map of event structures $f : E \to E'$. Let

$$V =_{\mathrm{def}} \{e \in E \mid f(e) \text{ is defined}\}\,.$$

Then f clearly factors into the composition

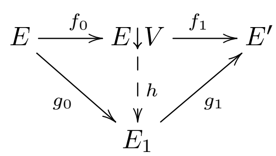
$$E \xrightarrow{f_0} E{\downarrow}V \xrightarrow{f_1} E'$$

of f_0, a partial map of event structures taking $e \in E$ to itself if $e \in V$ and undefined otherwise, and f_1, a total map of event structures acting like f on V. We call f_1 the *defined part* of the partial map f. We say a map $f : E \to E'$ is a *projection* if its defined part is an isomorphism.

The factorisation is characterised to within isomorphism by the following universal characterisation: for any factorisation

$$E \xrightarrow{g_0} E_1 \xrightarrow{g_1} E'$$

where g_0 is partial and g_1 is total there is a (necessarily total) unique map $h : E{\downarrow}V \to E_1$ such that



commutes.

2.3 Rigid maps

Recall a map f is *rigid* iff it is total and f preserves causal dependency, *i.e.*, if $e' \leq e$ in E then $f(e') \leq f(e)$ in E'.

Proposition 2.6. *A total map $f : E \to E'$ of event structures is rigid iff for all $x \in \mathcal{C}(E)$ and $y \in \mathcal{C}(E')$*

$$y \subseteq f(x) \implies \exists z \in \mathcal{C}(E).\ z \subseteq x \text{ and } fz = y\ .$$

The configuration z is necessarily unique by the local injectivity of f. (The class of maps would be unaffected if we allow all configurations in the definition above.)

Proof. "*Only if*": Total maps reflect causal dependency. So, if f preserves causal dependency, then for any configuration x of E, the bijection $f : x \to fx$ preserves and reflects causal dependency. Hence for any subconfiguration y of fx, the bijection restricts to a bijection $f : z \to y$ with z a down-closed subset of x. But then z must be a configuration of E. "*If*": Let $e \in E$. Then $[f(e)] \subseteq f[e]$. Hence there is a subconfiguration z of $[e]$ such that $fz = [f(e)]$. By local injectivity, $e \in z$, so $z = [e]$. Hence $f[e] = [f(e)]$. It follows that if $e' \leq e$ then $f(e') \leq f(e)$. ∎

A rigid map of event structures preserves the causal dependency relation "rigidly," so that the causal dependency relation on the image fx is a copy of that on a configuration x of E—in this sense f is a local isomorphism. This is not so for general maps where x may be augmented with extra causal dependency over that on fx.

Proposition 2.7. *The inclusion functor $\mathcal{E}_r \hookrightarrow \mathcal{E}_t$ has a right adjoint. The category $\mathcal{E}_t$ is isomorphic to the Kleisli category of the monad for the adjunction.*

Proof. The right adjoint's action on objects is given as follows. Let B be an event structure. For $x \in \mathcal{C}(B)$, an *augmentation* of x is a partial order (x, α) where $\forall b, b' \in x.\ b \leq_B b' \implies b\,\alpha\,b'$. We can regard such augmentations as elementary event structures in which all subsets of events are consistent. Order all augmentations by taking $(x, \alpha) \sqsubseteq (x', \alpha')$ iff $x \subseteq x'$ and the inclusion $i : x \hookrightarrow x'$ is a rigid map $i : (x, \alpha) \to (x', \alpha')$. Augmentations under $\sqsubseteq$ form a prime algebraic domain; the complete primes are precisely the augmentations with a top element. Define $aug(B)$ to be its associated event structure.

There is an obvious total map of event structures $\epsilon_B : aug(B) \to B$ taking a complete prime to the event which is its top element. It can be checked that post-composition by ϵ_B yields a bijection

$$\epsilon_B \circ \_ : \mathcal{E}_r(A, aug(B)) \cong \mathcal{E}(A, B)\ .$$

Hence aug extends to a right adjoint to the inclusion $\mathcal{E}_r \hookrightarrow \mathcal{E}_t$.

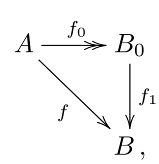
Write aug also for the monad induced by the adjunction and $Kl(aug)$ for its Kleisli category. Under the bijection of the adjunction

$$Kl(aug)(A, B) =_{\text{def}} \mathcal{E}_r(A, aug(B)) \cong \mathcal{E}(A, B)\ .$$

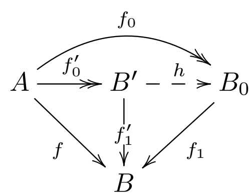
The categories $Kl(aug)$ and $\mathcal{E}$ share the same objects, and so are isomorphic. □

2.3.1 Rigid image

Rigid maps $f : A \to B$ have a useful image given by restricting the causal dependency of B to the set of events in the image of A under f and taking a finite set of events to be consistent if they are the image of a consistent set in A. More generally, a total map $f : A \to B$ has a *rigid image* given by the image of its corresponding Kleisli map, the rigid map $f : A \to aug(B)$. A total map $f : A \to B$ has a *rigid image* comprising



where f_0 is rigid epi and f_1 is a total map, with the universal property summarised in the diagram below:



for a unique rigid h; the map h is necessarily also epi. If we don't specify further we shall take the rigid image of a total map $f : A \to B$ to be a substructure of $aug(B)$. By a substructure of B we mean an event structure B_0 with events included in those of B so that the inclusion is a map.

2.3.2 Rigid embeddings and inclusions

Special forms of rigid maps appeared as *rigid embeddings* in Kahn and Plotkin's work on concrete domains. Their extension to event structures can be used in defining event structures recursively.

A total map $f : E \to E'$ is a *rigid embedding* iff it is rigid and an injective function on events for which the inverse relation f^{op} is a (partial) map of event structures $f^{\mathrm{op}} : E' \to E$. (There are several alternative equivalent definitions.)

Rigid embeddings include as a special case those in which the function f is an inclusion. These give the well-known approximation order $\trianglelefteq$ on event structures:

$$\begin{aligned}(E', \leq', \mathrm{Con}') \trianglelefteq (E, \leq, \mathrm{Con}) \iff\ & E' \subseteq E \ \& \\ & \forall e' \in E'.\ [e']' = [e'] \ \& \\ & \forall X' \subseteq E'.\ X' \in \mathrm{Con}' \iff X \in \mathrm{Con}\,.\end{aligned}$$

The order $\trianglelefteq$ forms a 'large cpo,' with bottom the empty event structure, and is useful when defining event structures recursively [5, 6, 3]. With some care in defining the precise constructions on event structures they can be ensured to be continuous w.r.t. $\trianglelefteq$; for this it suffices to check that they are $\trianglelefteq$-monotonic and continuous on event sets. Further details can be found in [5, 6].

2.3.3 Rigid families

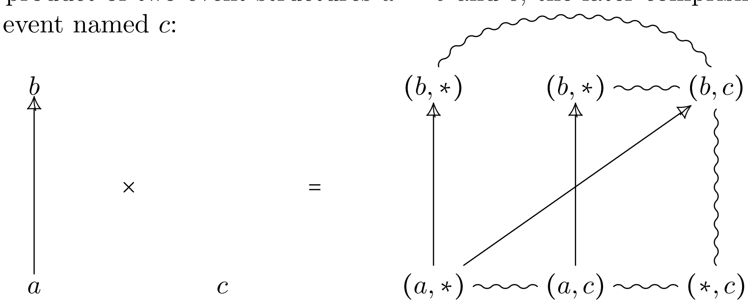
It is occasionally useful to build an event structure out of a non-empty family $\mathcal{Q}$ of finite partial orders.

For $\mathcal{Q}$ to be a rigid family we require that its is closed under rigid inclusions, or equivalently, that any down-closed subset of any element q, with order the restriction of that of q, is itself an element of $\mathcal{Q}$. (In this case rigid inclusions coincide withn rigid embeddings.)

From a rigid family $\mathcal{Q}$ we construct an event structure as follows. Its events are those partial orders in $\mathcal{Q}$ with a top element. Its causal dependency is given by rigid inclusion. We say a finite subset of partial orders with top is consistent iff all its members are rigidly included in a common member of $\mathcal{Q}$.

2.4 Products of event structures

The category of event structures has products, which essentially allow arbitrary synchronizations between their components. For example, here is an illustration of the product of two event structures $a \rightarrowtriangle b$ and c, the later comprising just a single event named c:



The original event b has split into three events, one a synchronization with c, another b occurring unsynchronized after an unsynchronized a, and the third b occurring unsynchronized after a synchronizes with c. The splittings correspond to the different histories of the event.

It can be awkward to describe operations such as products, pullbacks and synchronized parallel compositions directly on the simple event structures here, essentially because an event determines its whole causal history. One closely related and more versatile, though perhaps less intuitive and familiar, model is that of stable families. Stable families will play an important technical role in establishing and reasoning about constructions on event structures.

Chapter 3

Stable families

Stable families, their basic properties and relations to event structures are developed.[1]

3.1 Stable families

The notion of stable family extends that of finite configurations of an event structure to allow an event can occur in several incompatible ways.

Notation 3.1. *Let $\mathcal{F}$ be a family of subsets. Let $X \subseteq \mathcal{F}$. We write $X\uparrow$ for $\exists y \in \mathcal{F}.\ \forall x \in X. x \subseteq y$ and say X is* compatible. *When $x, y \in \mathcal{F}$ we write $x \uparrow y$ for $\{x, y\}\uparrow$.*

A *stable family* comprises $\mathcal{F}$, a nonempty family of finite subsets, satisfying:
Completeness: $\forall Z \subseteq \mathcal{F}.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}$;
Stability: $\forall Z \subseteq \mathcal{F}.\ Z \neq \varnothing\ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}$;
Coincidence-freeness: For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,

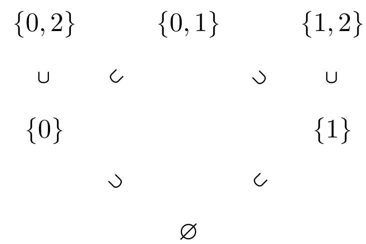
$$\exists y \in \mathcal{F}.\ y \subseteq x\ \&\ (e \in y \iff e' \notin y)\,.$$

Proposition 3.2. *The family of finite configurations of an event structure forms a stable family.*

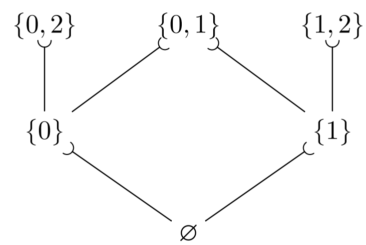
On the other hand stable families are more general than finite configurations of an event structure, as the following example shows.

[1]A useful reference for stable families is the report "Event structure semantics for CCS and related languages," a full version of the article [5], available from `www.cl.cam.ac.uk/~gw104`, though its terminology can differ from that here.

Example 3.3. Let $\mathcal{F}$ be the stable family, with events $E = \{0, 1, 2\}$,



or equivalently



where $\mathrel{-\!\!\!\subset}$ is the covering relation representing an occurrence of one event. The events 0 and 1 are concurrent, neither depends on the occurrence or non-occurrence of the other to occur. The event 2 can occur in two incompatible ways, either through event 0 having occurred or event 1 having occurred. This possibility can make stable families more flexible to work with than event structures.

A (partial) map of stable families $f : \mathcal{F} \to \mathcal{G}$ is a partial function f from the events of $\mathcal{F}$ to the events of $\mathcal{G}$ such that for all $x \in \mathcal{F}$,

$$fx \in \mathcal{G} \ \& \ (\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2)\,.$$

Maps of stable families compose as partial functions, with identity maps given by identity functions. We call a map $f : \mathcal{F} \to \mathcal{G}$ of stable families *total* when it is total as a function; the f restricts to a bijection $x \cong fx$ for all $x \in \mathcal{F}$.

Definition 3.4. Let $\mathcal{F}$ be a stable family. We use $x \mathrel{-\!\!\!\subset} y$ to mean y covers x in $\mathcal{F}$, *i.e.* $x \subset y$ in $\mathcal{F}$ with nothing in between, and $x \overset{e}{\mathrel{-\!\!\!\subset}} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{F}$ and event $e \notin x$. We sometimes use $x \overset{e}{\mathrel{-\!\!\!\subset}}$, expressing that event e is enabled at configuration x, when $x \overset{e}{\mathrel{-\!\!\!\subset}} y$ for some y.

Exercise 3.5. Let $\mathcal{F}$ be a nonempty family of sets satisfying the Completeness axiom in the definition of stable families. Show $\mathcal{F}$ is coincidence-free iff

$$\forall x, y \in \mathcal{F}.\ x \subsetneq y \implies \exists x_1, e_1.\ x \overset{e_1}{\mathrel{-\!\!\!\subset}} x_1 \subseteq y\,.$$

[Hint: For 'only if' use induction on the size of $y \setminus x$.] □

3.1.1 Stable families and event structures

Finite configurations of an event structure form a stable family. Conversely, a stable family determines an event structure:

Proposition 3.6. *Let x be a configuration of a stable family $\mathcal{F}$. For $e, e' \in x$ define*

$$e' \leq_x e \text{ iff } \forall y \in \mathcal{F}.\ y \subseteq x \ \&\ e \in y \implies e' \in y.$$

When $e \in x$ define the prime *configuration*

$$[e]_x = \bigcap \{y \in \mathcal{F} \mid y \subseteq x \ \&\ e \in y\}\ .$$

Then $\leq_x$ is a partial order and $[e]_x$ is a configuration such that

$$[e]_x = \{e' \in x \mid e' \leq_x e\}.$$

Moreover the configurations $y \subseteq x$ are exactly the down-closed subsets of $\leq_x$.

Proposition 3.7. *Let $\mathcal{F}$ be a stable family. Then,* $\mathrm{Pr}(\mathcal{F}) =_{\mathrm{def}} (P, \mathrm{Con}, \leq)$ *is an event structure where:*

$$\begin{aligned} &P = \{[e]_x \mid e \in x \ \&\ x \in \mathcal{F}\}\ ,\\ &Z \in \mathrm{Con} \text{ iff } Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F} \quad \textit{and},\\ &p \leq p' \text{ iff } p, p' \in P \ \&\ p \subseteq p'\ . \end{aligned}$$

Exercise 3.8. *Prove the two propositions 3.6 and 3.7.* □

The operation Pr is right adjoint to the "inclusion" functor, taking an event structure E to the stable family $\mathcal{C}(E)$. The unit of the adjunction $E \to \mathrm{Pr}(\mathcal{C}(E))$ takes an event e to the prime configuration $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$. The counit $top : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ takes prime configuration $[e]_x$ to e.

Definition 3.9. Let $\mathcal{F}$ be a stable family. W.r.t. $x \in \mathcal{F}$, write $[e)_x =_{\mathrm{def}} \{e' \in E \mid e' \leq_x e \ \&\ e' \neq e\}$. The relation of *immediate* dependence of event structures generalizes: with respect to $x \in \mathcal{F}$, the relation $e \rightarrowtriangle_x e'$ means $e \leq_x e'$ with $e \neq e'$ and no event in between. For $e, e' \in x \in \mathcal{F}$ we write $e\ co_x\ e'$ when neither $e \leq_x e'$ nor $e' \leq_x e$. Note the relations $\leq_x$, $\rightarrowtriangle_x$ and co_x, 'local' to a configuration x, coincide with the 'global' versions $\leq$, $\rightarrowtriangle$ and co when the stable family comprises the finite configurations of an event structure.

We shall use the following property of maps repeatedly, both for stable families and the special case of event structures. It says that their maps locally reflect causal dependency.

Proposition 3.10. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. If $f(e)$ and $f(e')$ are defined and $f(e) \leq_{fx} f(e')$ then $e \leq_x e'$.*

Proof. Let $e, e' \in x \in \mathcal{F}$. Suppose $f(e)$ and $f(e')$ are defined and $f(e) \le_{fx} f(e')$. Suppose y is a subconfiguration of x, *i.e.* $y \in \mathcal{F}$ and $y \subseteq x$, which contains e'. Then clearly fy is a subconfiguration of fx which contains $f(e')$. We have $f(e) \in fy$ as $f(e) \le_{fx} f(e')$. Hence there is $e" \in y$ such that $f(e") = f(e)$. But now $e, e" \in x$ with $f(e) = f(e")$, so $e = e"$. We deduce $e \in y$. The argument was for an arbitrary y, so $e \le_x e'$ as required. □

The next two propositions relate immediate causal dependency between events to the covering relation between configurations.

Proposition 3.11. *Let $\mathcal{F}$ be a stable family. Let $e, e' \in x \in \mathcal{F}$.*

$$\exists y, y_1 \in \mathcal{F}.\ y, y_1 \subseteq x \ \&\ y \overset{e}{\longrightarrow\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\subset} \iff e \rightarrowtriangle_x e' \ \text{ or }\ e\ co_x\ e', \qquad (i)$$

$$\text{and} \quad e \rightarrowtriangle_x e' \iff \exists y, y_1 \in \mathcal{F}.\ y, y_1 \subseteq x \ \&\ y \overset{e}{\longrightarrow\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\subset} \ \&\ \neg\, e\ co_x\ e' \qquad (ii)$$

$$\iff \exists y, y_1 \in \mathcal{F}.\ y, y_1 \subseteq x \ \&\ y \overset{e}{\longrightarrow\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\subset} \ \&\ \neg\, y \overset{e'}{\longrightarrow\!\!\subset}. \qquad (iii)$$

The proposition simplifies in the special case of event structures:

Proposition 3.12. *Let E be an event structure. Let $e, e' \in E$.*

$$\exists y, y_1 \in \mathcal{C}^\infty(E).\ y \overset{e}{\longrightarrow\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\subset} \iff e \rightarrowtriangle e' \ \text{ or }\ e\ co\ e',$$

$$\text{and} \quad e \rightarrowtriangle e' \iff \exists y, y_1 \in \mathcal{C}^\infty(E).\ y \overset{e}{\longrightarrow\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\subset} \ \&\ \neg\, e\ co\ e',$$

$$\iff \exists y, y_1 \in \mathcal{C}^\infty(E).\ y \overset{e}{\longrightarrow\!\!\subset} y_1 \overset{e'}{\longrightarrow\!\!\subset} \ \&\ \neg\, y \overset{e'}{\longrightarrow\!\!\subset}.$$

3.2 Infinite configurations

We can extend a stable family to include infinite configurations, by constructing its "ideal completion."

Definition 3.13. Let $\mathcal{F}$ be a stable family. Define $\mathcal{F}^\infty$ to comprise all $\bigcup I$ where $I \subseteq \mathcal{F}$ is an ideal (*i.e.*, I is a nonempty subset of $\mathcal{F}$ closed downwards w.r.t. $\subseteq$ in $\mathcal{F}$ and such that if $x, y \in I$ then $x \cup y \in I$).

Exercise 3.14. *For an event structure E, show $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$.* □

Exercise 3.15. *Let $\mathcal{F}$ be a stable family. Show $\mathcal{F}^\infty$ satisfies:*

Completeness: $\forall Z \subseteq \mathcal{F}^\infty.(\forall X \subseteq_{\text{fin}} Z.\ X\uparrow) \implies \bigcup Z \in \mathcal{F}^\infty$;
Stability: $\forall Z \subseteq \mathcal{F}^\infty.\ Z \neq \emptyset \ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}^\infty$;
Coincidence-freeness: *For all $x \in \mathcal{F}^\infty$, $e, e' \in x$ with $e \neq e'$,*

$$\exists y \in \mathcal{F}^\infty.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y);$$

Finiteness: *For all $x \in \mathcal{F}^\infty$,*

$$\forall e \in x \exists y \in \mathcal{F}.\ e \in y \ \&\ y \subseteq x \ \&\ y \text{ is finite}.$$

Show that $\mathcal{F}$ consists of precisely the finite sets in $\mathcal{F}^\infty$. □

Remark Above the conditions of Finiteness and Coincidence-freeness together can be replaced by the equivalent condition
Secured: if $e \in x \in \mathcal{F}$ then there exists a securing chain $e_1, \cdots, e_n = e$ in x s.t. $\{e_1, \cdots, e_i\} \in \mathcal{F}$ for all $i \leq n$.

3.3 Process constructions

3.3.1 Products

Let $\mathcal{A}$ and $\mathcal{B}$ be stable families with events A and B, respectively. Their product, the stable family $\mathcal{A} \times \mathcal{B}$, has events comprising pairs in $A \times_* B =_{\text{def}} \{(a,*) \mid a \in A\} \cup \{(a,b) \mid a \in A \ \&\ b \in B\} \cup \{(*,b) \mid b \in B\}$, the product of sets with partial functions, with (partial) projections π_1 and π_2—treating $*$ as 'undefined'—with configurations

$$\begin{aligned}
&x \in \mathcal{A} \times \mathcal{B} \text{ iff}\\
&x \text{ is a finite subset of } A \times_* B \text{ such that } \pi_1 x \in \mathcal{A} \ \&\ \pi_2 x \in \mathcal{B},\\
&\forall e, e' \in x.\ \pi_1(e) = \pi_1(e') \text{ or } \pi_2(e) = \pi_2(e') \Rightarrow e = e', \&\\
&\forall e, e' \in x.\ e \neq e' \Rightarrow \exists y \subseteq x.\ \pi_1 y \in \mathcal{A} \ \&\ \pi_2 y \in \mathcal{B} \ \&\\
&\qquad\qquad\qquad\qquad (e \in y \iff e' \notin y)\,.
\end{aligned}$$

Theorem 3.16. *For stable families $\mathcal{A}$ and $\mathcal{B}$ the construction $\mathcal{A} \times \mathcal{B}$ with projections π_1 and π_2 described above is the product in the category of stable families.*

Proof. Essentially in the report for [5]. □

Right adjoints preserve products. Consequently we obtain a product of event structures A and B by first regarding them as stable families $\mathcal{C}(A)$ and $\mathcal{C}(B)$, forming their product $\mathcal{C}(A) \times \mathcal{C}(B), \pi_1, \pi_2$, and then constructing the event structure

$$A \times B =_{\text{def}} \Pr(\mathcal{C}(A) \times \mathcal{C}(B))$$

and its projections as $\Pi_1 =_{\text{def}} \pi_1 \mathit{top}$ and $\Pi_2 =_{\text{def}} \pi_2 \mathit{top}$.

Exercise 3.17. *Let A be the event structure consisting of two distinct events $a_1 \leq a_2$ and B the event structure with a single event b. Following the method above describe the product of event structures $A \times B$.* □

Proposition 3.18. *Let $x \in \mathcal{A} \times \mathcal{B}$, a product of stable families with projections π_1 and π_2. Then, for all $y \subseteq x$,*

$$y \in \mathcal{A} \times \mathcal{B} \iff \pi_1 y \in \mathcal{A} \ \&\ \pi_2 y \in \mathcal{B}\,.$$

Proof. Straightforwardly from the definition of $\mathcal{A} \times \mathcal{B}$. □

Later we shall use the following properties of $\rightarrowtriangle$ in a product of stable families or event structures.

Lemma 3.19. *Let $x \in \mathcal{A} \times \mathcal{B}$, a product of stable families with projections π_1, π_2. Let $e, e' \in x$. If $e \rightarrowtriangle_x e'$, then*
either
(i) $\pi_1(e)$ and $\pi_1(e')$ are both defined with $\pi_1(e) \rightarrowtriangle_{\pi_1 x} \pi_1(e')$ in $\mathcal{A}$ and
if $\pi_2(e)$, $\pi_2(e')$ are defined then $\pi_2(e) \rightarrowtriangle_{\pi_2 x} \pi_2(e')$ or $\pi_2(e)\ co_{\pi_2 x}\ \pi_2(e')$ in $\mathcal{B}$,
or
(ii) $\pi_2(e)$ and $\pi_2(e')$ are both defined with $\pi_2(e) \rightarrowtriangle_{\pi_2 x} \pi_2(e')$ in $\mathcal{B}$ and
if $\pi_1(e)$, $\pi_1(e')$ are defined then $\pi_1(e) \rightarrowtriangle_{\pi_1 x} \pi_1(e')$ or $\pi_1(e)\ co_{\pi_1 x}\ \pi_1(e')$ in $\mathcal{A}$.

Proof. By Proposition 3.11(iii), $e \rightarrowtriangle_x e'$ iff (I) $y \overset{e}{\longrightarrow\!\subset} y_1 \overset{e'}{\longrightarrow\!\subset}$ and (II) $\neg\, y \overset{e'}{\longrightarrow\!\subset}$, for subconfigurations y, y_1 of x. From (I),

(a) if $\pi_1(e)$, $\pi_1(e')$ are defined then $\pi_1 y \overset{\pi_1(e)}{\longrightarrow\!\subset} \pi_1 y_1 \overset{\pi_1(e')}{\longrightarrow\!\subset}$
and

(b) if $\pi_2(e)$, $\pi_2(e')$ are defined then $\pi_2 y \overset{\pi_2(e)}{\longrightarrow\!\subset} \pi_2 y_2 \overset{\pi_2(e')}{\longrightarrow\!\subset}$.

Suppose both ($\pi_1(e')$ defined $\Rightarrow \pi_1 y \overset{\pi_1 e'}{\longrightarrow\!\subset}$) and ($\pi_2(e')$ defined $\Rightarrow \pi_2 y \overset{\pi_2 e'}{\longrightarrow\!\subset}$). Then $y \cup \{e'\} \subseteq x$ with $\pi_1(y \cup \{e'\}) \in \mathcal{A}$ and $\pi_2(y \cup \{e'\}) \in \mathcal{B}$. So, by Proposition 3.18, $y \cup \{e'\} \in \mathcal{A} \times \mathcal{B}$—contradicting (II). Hence, either $\neg\, \pi_1 y \overset{\pi_1 e'}{\longrightarrow\!\subset}$, with $\pi_1 e'$ defined, or $\neg\, \pi_2 y \overset{\pi_2 e'}{\longrightarrow\!\subset}$, with $\pi_2 e'$ defined.

Assume the case $\neg\, \pi_1 y \overset{\pi_1 e'}{\longrightarrow\!\subset}$, with $\pi_1 e'$ defined. Supposing $\pi_1(e)$ is undefined, from (I) we obtain the contradictory $\pi_1 y = \pi_1 y_1 \overset{\pi_1 e'}{\longrightarrow\!\subset}$. Hence, in this case, both $\pi_1 e$ and $\pi_1 e'$ are defined with $\pi_1 y \overset{\pi_1(e)}{\longrightarrow\!\subset} \pi_1 y_1 \overset{\pi_1(e')}{\longrightarrow\!\subset}$ and $\neg\, \pi_1 y \overset{\pi_1 e'}{\longrightarrow\!\subset}$. So $\pi_1(e) \rightarrowtriangle_{\pi_1 x} \pi_1(e')$ in $\mathcal{A}$, by Proposition 3.11(iii). Meanwhile from (b), this time by Proposition 3.11(i), if $\pi_2(e)$, $\pi_2(e')$ are defined then $\pi_2(e) \rightarrowtriangle_{\pi_2 x} \pi_2(e')$ or $\pi_2(e)\ co_{\pi_2 x}\ \pi_2(e')$ in $\mathcal{B}$. Hence (i), above.

Similarly, the case $\neg\, \pi_2 y \overset{\pi_2 e'}{\longrightarrow\!\subset}$, with $\pi_2 e'$ defined, yields (ii).

□

Corollary 3.20. *Let $A \times B$, Π_1, Π_2 be a product of event structures. If $p \rightarrowtriangle p'$ in $A \times B$, then*
either
(i) $\Pi_1(p)$ and $\Pi_1(p')$ are both defined with $\Pi_1(p) \rightarrowtriangle \Pi_1(p')$ in A and
if $\Pi_2(p)$, $\Pi_2(p')$ are defined then $\Pi_2(p) \rightarrowtriangle \Pi_2(p')$ or $\Pi_2(p)\ co\ \Pi_2(p')$ in B,
or
(ii) $\Pi_2(p)$ and $\Pi_2(p')$ are both defined with $\Pi_2(p) \rightarrowtriangle \Pi_2(p')$ in B and
if $\Pi_1(p)$, $\Pi_1(p')$ are defined then $\Pi_1(p) \rightarrowtriangle \Pi_1(p')$ or $\Pi_1(p)\ co\ \Pi_1(p')$ in A.

Proof. Directly by Lemma 3.19, because $p \rightarrowtriangle p'$ in $A \times B$ implies $top(p) \rightarrowtriangle_{p'} top(p')$ in $\mathcal{C}(A) \times \mathcal{C}(B)$. □

The converse to Lemma 3.19, above, is false. A more explicit, case-by-case, form of the above Lemma 3.19 is helpful:

Lemma 3.21. *Suppose $e \multimap_x e'$ in a product of stable families $\mathcal{A} \times \mathcal{B}, \pi_1, \pi_2$.*
(i) If $e = (a, *)$ then $e' = (a', b)$ or $e' = (a', *)$ with $a \multimap_{\pi_1 x} a'$ in $\mathcal{A}$.
(ii) If $e' = (a', *)$ then $e = (a, b)$ or $e = (a, *)$ with $a \multimap_{\pi_1 x} a'$ in $\mathcal{A}$.
(iii) If $e = (a, b)$ and $e' = (a', b')$ then $a \multimap_{\pi_1 x} a'$ in $\mathcal{A}$ or $b \multimap_{\pi_2 x} b'$ in $\mathcal{B}$. Furthermore both ($a \multimap_{\pi_1 x} a'$ or $a\ co_{\pi_1 x}\ a'$) and ($b \multimap_{\pi_2 x} b'$ or $b\ co_{\pi_2 x}\ b'$).

The obvious analogues of (i) and (ii) hold for $e = (*, b)$ and $e' = (*, b')$.

Proof. A restatement of Lemma 3.19, writing $a = \pi_1(e)$, $b = \pi_2(e)$, $a' = \pi_1(e')$ and $b = \pi_2(e')$ when these results of projections are defined. □

Exercise 3.22. Let $z \in \mathcal{A} \times \mathcal{B}$, the product of stable families. For any chain

$$(a, *) \multimap_z e_1 \multimap_z \cdots \multimap_z e_m = (*, b)$$

show there is $e_i = (a_i, b_i)$ for some events a_i of $\mathcal{A}$ and b_i of $\mathcal{B}$.

Corollary 3.23. *Let $f : A \to A'$ and $g : B \to B'$ be rigid maps of event structures. Then the map $\langle f, g \rangle : A \times B \to A' \times B'$ is rigid.*

Proof. Write Π_1, Π_2 and Π'_1, Π'_2 for the projections of $A \times B$ and $A' \times B'$ respectively. It is easy to check that the totality of f and g above implies $\langle f, g \rangle$ is total. To show that their rigidity implies $\langle f, g \rangle$ is rigid we use Corollary 3.20 above. Assuming $p \multimap p'$ in $A \times B$ the corollary implies $\Pi_1(p) \multimap \Pi_1(p')$ or $\Pi_2(p) \multimap \Pi_2(p')$. From the rigidity of f and g, we obtain $f\Pi_1(p) \multimap f\Pi_1(p')$ or $g\Pi_2(p) \multimap g\Pi_2(p')$. But $\Pi'_1 \langle f, g \rangle(p') = f\Pi_1(p')$ and $\Pi'_2 \langle f, g \rangle(p') = f\Pi_2(p')$ whence as $\langle f, g \rangle$ is a map so reflects causal dependency locally we deduce $\langle f, g \rangle(p) \le \langle f, g \rangle(p')$ (or in fact $\langle f, g \rangle(p) \multimap \langle f, g \rangle(p')$), showing $\langle f, g \rangle$ is rigid. □

3.3.2 Restriction

The *restriction* of $\mathcal{F}$ to a subset of events R is the stable family $\mathcal{F} \upharpoonright R =_{\mathrm{def}} \{x \in \mathcal{F} \mid x \subseteq R\}$. Defining $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, to have events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency induced by E, we obtain $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$.

Proposition 3.24. *Let $\mathcal{F}$ be a stable family and R a subset of its events. Then,* $\Pr(\mathcal{F} \upharpoonright R) = \Pr(\mathcal{F}) \upharpoonright top^{-1} R$.

We remark that we can regard restriction as arising as an equaliser. *E.g.* for an event structure E write $|E|$ for the event structure comprising the events of E but with discrete causal dependency and all subsets consistent. W.r.t. a subset R of events, the inclusion map $E \upharpoonright R \hookrightarrow E$ is the equaliser of the two maps $I : E \to |E|$, acting as identity on events, and $U : E \to |E|$, acting as identity on events in R and undefined elsewhere.

3.3.3 Synchronized compositions

Synchronized parallel compositions are obtained as restrictions of products to those events which are allowed to synchronize or occur asynchronously. For example, the synchronized composition of Milner's CCS on stable families $\mathcal{A}$ and $\mathcal{B}$ (with labelled events) is defined as $\mathcal{A} \times \mathcal{B} \upharpoonright R$ where R comprises events which are pairs $(a, *)$, $(*, b)$ and (a, b), where in the latter case the events a of $\mathcal{A}$ and b of $\mathcal{B}$ carry complementary labels. Similarly, synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. By Proposition 3.24, we can equivalently form a synchronized composition of event structures by forming the synchronized composition of their stable families of configurations, and then obtaining the resulting event structure—this has the advantage of eliminating superfluous events earlier.

Products of stable families within the subcategory of total maps can be obtained by restricting the product (w.r.t. partial maps). Construct

$$\mathcal{A} \times_t \mathcal{B} = \mathcal{A} \times \mathcal{B} \upharpoonright A \times B$$

where we restrict to the cartesian product of the sets of events of $\mathcal{A}$ and $\mathcal{B}$, called A and B respectively; projection maps are obtained from the projection functions from the cartesian product. Products of stable families within the subcategory of total maps have a particularly simple characterisation:

Proposition 3.25. *Finite configurations of a product $\mathcal{A} \times_t \mathcal{B}$ of stable families with total maps are* secured bijections $\theta : x \cong y$ *between configurations $x \in \mathcal{A}$ and $y \in \mathcal{B}$, such that the transitive relation generated on θ by taking $(a, b) \le (a', b')$ if $a \le_x a'$ or $b \le_y b'$ is a partial order.*

Proof. Let $z \in \mathcal{A} \times_t \mathcal{B}$. By Proposition3.10 the projections π_1 and π_2 locally reflect causal dependency. Hence the partial order $\le_z$ satisfies: $(a, b) \le_z (a', b')$ if $a \le_x a$ or $b \le_y b'$, for all $(a, b), (a', b') \in z$. Thus the transitive relation on z generated by taking $(a, b) \le (a', b')$ if $a \le_x a'$ or $b \le_y b'$ is certainly a partial order; failure of antisymmetry for the relation generated would imply its failure for $\le_z$, a contradiction. To see that $\le_z$ is precisely the transitive relation generated in this way, let θ be the elementary event structure comprising events the set z with causal dependency the least transitive relation $\le$ for which $(a, b) \le (a', b')$ if $a \le_x a'$ or $b \le_y b'$. Let Θ be its stable family of configurations with $r_1 : \Theta \to \mathcal{A}$ and $r_2 : \Theta \to \mathcal{B}$ the obvious projection maps. By the universal properties of the product $\mathcal{A} \times_t \mathcal{B}$, π_1, π_2 there is a unique map $h : \Theta \to \mathcal{A} \times_t \mathcal{B}$ s.t. $r_1 = \pi_1 h$ and $r_2 = \pi_2 h$. As a function on the underlying sets of events $h : \theta \to z$ acts as the identity on events and reflects causal dependency. Hence $\le_z \subseteq \le_p$. It follows that $\le_z$ and $\le_p$ coincide, so that $\le_z$ is a secured bijection.

Conversely, suppose θ is a secured bijection between $x \in \mathcal{A}$ and $y \in \mathcal{B}$ with generated partial order $\le$. Regard $\theta, \le$ as an elementary event structure with stable family of configurations Θ. From the way $\le$ is generated, there are projection maps $r_1 : \Theta \to \mathcal{A}$ and $r_2 : \Theta \to \mathcal{B}$. Hence by universality, there is a unique map $h : \Theta \to \mathcal{A} \times_t \mathcal{B}$ s.t. $r_1 = \pi_1 h$ and $r_2 = \pi_2 h$. But then h must act as the identity function, ensuring $\theta \in \mathcal{A} \times_t \mathcal{B}$. □

3.3.4 Pullbacks

The construction of pullbacks can be viewed as a special case of synchronized composition. Once we have products of event structures pullbacks are obtained by restricting products to the appropriate equalizing set. Pullbacks of event structures can also be constructed via pullbacks of stable families, in a similar manner to the way we have constructed products of event structures. We obtain pullbacks of stable families as restrictions of products. Suppose $f_1 : \mathcal{F}_1 \to \mathcal{G}$ and $f_2 : \mathcal{F}_2 \to \mathcal{G}$ are maps of stable families. Let E_1, E_2 and C be the sets of events of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$, respectively. The set $P =_{\mathrm{def}} \{(e_1, e_2) \mid f(e_1) = f(e_2)\}$ with projections π_1, π_2 to the left and right, forms the pullback, in the category of sets, of the functions $f_1 : E_1 \to C$, $f_2 : E_2 \to C$. We obtain the pullback in stable families of f_1, f_2 as the stable family $\mathcal{P}$, consisting of those subsets of P which are also configurations of the product $\mathcal{F}_1 \times \mathcal{F}_2$—its associated maps are the projections π_1, π_2 from the events of $\mathcal{P}$. When f_1 and f_2 are total maps we obtain the pullback in the subcategory of stable families with total maps.

As a corollary of Proposition 3.25 we obtain a simple characterization of pullbacks of total maps within stable families:

Lemma 3.26. *Let $\mathcal{P}, \pi_1, \pi_2$ form a pullback of total maps $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ in the category of stable families. Configurations of $\mathcal{P}$ are precisely those composite bijections $\theta : x \cong fx = gy \cong y$ between configurations $x \in \mathcal{A}$ and $y \in \mathcal{B}$ s.t. $fx = gy$ for which the transitive relation generated on θ by taking $(a, b) \leq (a', b')$ if $a \leq_x a'$ or $b \leq_y b'$ is a partial order.*

For future reference we give the detailed construction of pullbacks of total maps in stable families. Let $f : \mathcal{A} \to \mathcal{C}$ and $g : \mathcal{B} \to \mathcal{C}$ be total maps of stable families. Assume $\mathcal{A}$ and $\mathcal{B}$ have underlying sets A and B. Define $D =_{\mathrm{def}} \{(a, b) \in A \times B \mid f(a) = g(b)\}$ with projections π_1 and π_2 to the left and right components. Define a family of configurations of the *pullback* to consist of

$$
\begin{aligned}
& x \in \mathcal{D} \text{ iff} \\
& x \text{ is a finite subset of } D \text{ such that } \pi_1 x \in \mathcal{A} \ \&\ \pi_2 x \in \mathcal{B}, \\
& \forall e, e' \in x.\ e \neq e' \Rightarrow \exists y \subseteq x.\ \pi_1 y \in \mathcal{A} \ \&\ \pi_2 y \in \mathcal{B} \ \& \\
& \qquad\qquad\qquad\qquad\qquad\qquad (e \in y \iff e' \notin y)\,.
\end{aligned}
$$

The extra local injectivity property we needed in the definition of product is not necessary here; it follows from the definition of D and that σ_1 and σ_2 are locally injective.

We obtain the pullback of event structures by first forming the pullback in stable families of their families of configurations and then applying Pr.

As a corollary of Lemma 3.26 we obtain a useful way to understand configurations of the pullback of total maps on event structures.

Proposition 3.27. *When $f : A \to C$ and $g : B \to C$ are total, maps of event* *structures, in their pullback* P, Π_1, Π_2

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\Pi_1} & \lrcorner & ^{\Pi_2}\searrow & \\ A & & & & B \\ & _{f}\searrow & & \swarrow_{g} & \\ & & C\,. & & \end{array}$$

the finite configurations of P *correspond to composite bijections*

$$\theta : x \cong fx = gy \cong y$$

between finite configurations x *of* A *and* y *of* B *such that* $fx = gy$*, for which the transitive relation generated on* θ *by* $(a,b) \le (a',b')$ *if* $a \le_A a'$ *or* $b \le_B b'$ *forms a partial order.*

As a consequence the pullback of rigid maps, respectively rigid epi maps, across total maps are rigid, respectively rigid epi.

Proposition 3.28. *Let* P, Π_1, Π_2 *be a pullback of total maps* $f : A \to C$ *and* $g : B \to C$ *in the category of event structures. If* f *is rigid so is* Π_2*. If* f *is rigid and epi so is* Π_2*.*

Proof. Use Proposition 3.27 to construct the appropriate configurations of the pullback of event structures; the rigidity of f ensures their existence. □

3.3.5 Projection

As we have seen, event structures support a simple form of hiding associated with the partial-total factorisation of a partial map. Let $(E, \le, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \le_V, \mathrm{Con}_V)$, where $v \le_V v'$ iff $v \le v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.

Proposition 3.29. *Let* $f : E \to E'$ *be a total map of event structures. Let* $V \subseteq E$ *and* $V' \subseteq E'$ *be such that*

$$\forall e \in E.\ e \in V \iff f(e) \in V'\,.$$

Then f *restricts to a total map* $f \restriction V : E \downarrow V \to E' \downarrow V'$*. Moreover, if* f *is rigid then so is* $f \restriction V$*.*

3.3.6 Recursion

Both stable families and event structures support recursive definitions via the 'large cpo' based on the substructure relation $\trianglelefteq$ [5, 6]. For two stable families $\mathcal{F}$ and $\mathcal{G}$ with events F and G respectively,

$$\mathcal{F} \trianglelefteq \mathcal{G} \text{ iff } F \subseteq G \ \&\ \forall x \subseteq_{\mathrm{fin}} F.\ x \in \mathcal{F} \iff x \in \mathcal{G}\,.$$

Chapter 4

Games and strategies

Very general nondeterministic concurrent games and strategies are presented. The intention is to formalize distributed games in which both Player (or a team of players) and Opponent (or a team of opponents) can interact in highly distributed fashion, without, for instance, enforcing that their moves alternate. Strategies, those nondeterministic plays which compose well with copy-cat strategies, are characterized.[1]

4.1 Event structures with polarities

We shall represent both a game and a strategy in a game as an event structure with polarity, comprising an event structure together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity + or − to its events E. The events correspond to (occurrences of) moves. The two polarities +/− express the dichotomy: Player/Opponent; Process/Environment; Prover/Disprover; or Ally/Enemy. Maps of event structures with polarity are maps of event structures which preserve polarity.

4.2 Operations

4.2.1 Dual

The *dual*, $E^\perp$, of an event structure with polarity E comprises a copy of the event structure E but with a reversal of polarities. It obviously extends to a functor. Write $\overline{e} \in E^\perp$ for the event complementary to $e \in E$ and *vice versa*.

4.2.2 Simple parallel composition

This operation simply juxtaposes two event structures with polarity. Let $(A, \leq_A, \mathrm{Con}_A, pol_A)$ and $(B, \leq_B, \mathrm{Con}_B, pol_B)$ be event structures with polarity. The

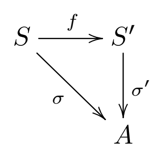
[1]This key chapter is the result of joint work with Silvain Rideau [7].

events of $A\|B$ are $(\{1\}\times A)\cup(\{2\}\times B)$, their polarities unchanged, with: the only relations of causal dependency given by $(1,a) \le (1,a')$ iff $a \le_A a'$ and $(2,b) \le (2,b')$ iff $b \le_B b'$; a subset of events C is consistent in $A\|B$ iff $\{a \mid (1,a) \in C\} \in \mathrm{Con}_A$ and $\{b \mid (2,b) \in C\} \in \mathrm{Con}_B$. The operation extends to a functor—put the two maps in parallel. The empty event structure with polarity $\varnothing$ is the unit w.r.t. $\|$.

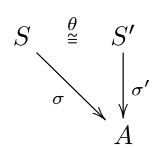
4.3 Pre-strategies

Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy* in A is a total map $\sigma : S \to A$ from an event structure with polarity S. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy* (and *winning strategy* in Section 8.1).

A map from a pre-strategy $\sigma : S \to A$ to a pre-strategy $\sigma' : S' \to A$ is a map $f : S \to S'$ such that

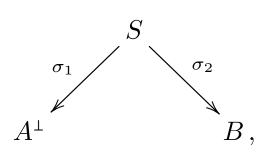


commutes. Accordingly, we regard two pre-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ as essentially the same when they are isomorphic, and write $\sigma \cong \sigma'$, *i.e.* when there is an isomorphism of event structures $\theta : S \cong S'$ such that

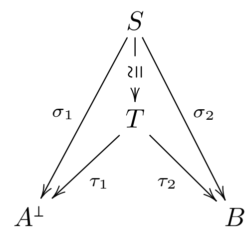


commutes.

Let A and B be event structures with polarity. Following Joyal [8], a pre-strategy from A to B is a pre-strategy in $A^\perp\|B$, so a total map $\sigma : S \to A^\perp\|B$. It thus determines a span



of event structures with polarity where σ_1, σ_2 are *partial* maps. In fact, a pre-strategy from A to B corresponds to such spans where for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. Two pre-strategies σ and τ from A to B are isomorphic, $\sigma \cong \tau$, when their spans are isomorphic, *i.e.*



commutes. We write $\sigma : A \twoheadrightarrow B$ to express that σ is a pre-strategy from A to B. Note a pre-strategy in a game A coincides with a pre-strategy from the empty game $\sigma : \varnothing \twoheadrightarrow A$.

4.3.1 Concurrent copy-cat

Identities on games are given by copy-cat strategies—strategies for Player based on copying the latest moves made by Opponent.

Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$. It describes a concurrent, or distributed, strategy based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity.

For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component, *i.e.*

$$\overline{(1,a)} = (2,\overline{a}) \text{ and } \overline{(2,a)} = (1,\overline{a})\,.$$

Proposition 4.1. *Let A be an event structure with polarity. There is an event structure with polarity $\mathbb{C}\mathbb{C}_A$ having the same events and polarity as $A^\perp \| A$ but with causal dependency $\le_{\mathbb{C}\mathbb{C}_A}$ given as the transitive closure of the relation*

$$\le_{A^\perp \| A} \cup \{(\overline{c}, c) \mid c \in A^\perp \| A \ \&\ pol_{A^\perp \| A}(c) = +\}\,.$$

and finite subsets of $\mathbb{C}\mathbb{C}_A$ consistent if their down-closure w.r.t. $\le_{\mathbb{C}\mathbb{C}_A}$ are consistent in $A^\perp \| A$. Moreover,
(i) $c \rightarrowtriangle c'$ in $\mathbb{C}\mathbb{C}_A$ iff

$$c \rightarrowtriangle c' \text{ in } A^\perp \| A \text{ or } pol_{A^\perp \| A}(c') = + \ \&\ \overline{c} = c'\,;$$

(ii) $x \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ iff

$$x \in \mathcal{C}(A^\perp \| A) \ \&\ \forall c \in x.\ pol_{A^\perp \| A}(c) = + \implies \overline{c} \in x\,.$$

Proof. It can first be checked that defining

$$\begin{aligned} c \le_{\mathbb{C}\mathbb{C}_A} c' \text{ iff } & (i)\ c \le_{A^\perp \| A} c' \text{ or} \\ & (ii)\ \exists c_0 \in A^\perp \| A.\ pol_{A^\perp \| A}(c_0) = + \ \& \\ & \qquad c \le_{A^\perp \| A} \overline{c_0} \ \&\ c_0 \le_{A^\perp \| A} c'\,, \end{aligned}$$

yields a partial order. Note that

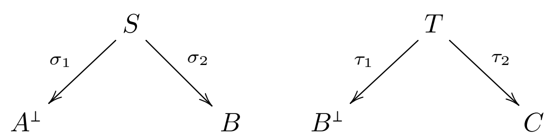
$$c \le_{A^\perp \| A} d \;\text{ iff }\; \overline{c} \le_{A^\perp \| A} \overline{d}\,,$$

used in verifying transitivity and antisymmetry. The relation $\le_{\mathbb{C}\mathbb{C}_A}$ is clearly the transitive closure of $\le_{A^\perp \| A}$ together with all extra causal dependencies $(\overline{c}, c)$ where $pol_{A^\perp \| A}(c) = +$. The remaining properties required for $\mathbb{C}\mathbb{C}_A$ to be an event structure follow routinely.
(i) From the above characterization of $\le_{\mathbb{C}\mathbb{C}_A}$.
(ii) From $\mathbb{C}\mathbb{C}_A$ and $A^\perp \| A$ sharing the same consistency relation and the extra causal dependency adjoined to $\mathbb{C}\mathbb{C}_A$. □

Based on Proposition 4.1, define the *copy-cat* pre-strategy from A to A to be the pre-strategy $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ where $\mathbb{C}\mathbb{C}_A$ comprises the event structure with polarity $A^\perp \| A$ together with extra causal dependencies $\overline{c} \le_{\mathbb{C}\mathbb{C}_A} c$ for all events c with $pol_{A^\perp \| A}(c) = +$, and γ_A is the identity on the set of events common to both $\mathbb{C}\mathbb{C}_A$ and $A^\perp \| A$.

4.3.2 Composing pre-strategies

Consider two pre-strategies $\sigma : A \mathrel{+\!\!\!\!\rightarrow} B$ and $\tau : B \mathrel{+\!\!\!\!\rightarrow} C$ as spans:



We show how to define their composition $\tau \odot \sigma : A \mathrel{+\!\!\!\!\rightarrow} C$. If we ignore polarities the partial maps of event structures σ_2 and τ_1 have a common codomain, the underlying event structure of B and $B^\perp$. The composition $\tau \odot \sigma$ will be constructed as a synchronized composition of S and T, in which output events of S synchronize with input events of T, followed by an operation of hiding 'internal' synchronization events. Only those events s from S and t from T for which $\sigma_2(s) = \overline{\tau_1(t)}$ synchronize; note that then s and t must have opposite polarities as this is so for their images $\sigma_2(s)$ in B and $\tau_1(t)$ in $B^\perp$. The event resulting from the synchronization of s and t has indeterminate polarity and will be hidden in the composition $\tau \odot \sigma$.

Formally, we use the construction of synchronized composition and projection of Section 3.3.3. Via projection we hide all those events with undefined polarity.

We first define the composition of the families of configurations of S and T as a synchronized composition of stable families. We form the product of stable families $\mathcal{C}(S) \times \mathcal{C}(T)$ with projections π_1 and π_2, and then form a restriction:

$$\mathcal{C}(T) \circledast \mathcal{C}(S) =_{\text{def}} \mathcal{C}(S) \times \mathcal{C}(T) \upharpoonright R$$

where

$$\begin{aligned} R \;=\; & \{(s,*) \mid s \in S \;\&\; \sigma_1(s) \text{ is defined}\} \cup \\ & \{(s,t) \mid s \in S \;\&\; t \in T \;\&\; \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\} \cup \\ & \{(*,t) \mid t \in T \;\&\; \tau_2(t) \text{ is defined}\}\,. \end{aligned}$$

The stable family $\mathcal{C}(T) \circledast \mathcal{C}(S)$ is the synchronized composition of the stable families $\mathcal{C}(S)$ and $\mathcal{C}(T)$ in which synchronizations are between events of S and T which project, under σ_2 and τ_1 respectively, to complementary events in B and $B^\perp$. The stable family $\mathcal{C}(T) \circledast \mathcal{C}(S)$ represents all the configurations of the composition of pre-strategies, including internal events arising from synchronizations. We obtain the synchronized composition as an event structure by forming $\mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S))$, in which events are the primes of $\mathcal{C}(T) \circledast \mathcal{C}(S)$. This synchronized composition still has internal events.

To obtain the composition of pre-strategies we hide the internal events due to synchronizations. The event structure of the composition of pre-strategies is defined to be

$$T{\odot}S =_{\mathrm{def}} \mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S)) \downarrow V\,,$$

the projection onto "visible" events,

$$\begin{aligned} V = & \{p \in \mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S)) \mid \exists s \in S.\ \mathit{top}(p) = (s,*)\} \cup \\ & \{p \in \mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S)) \mid \exists t \in T.\ \mathit{top}(p) = (*,t)\}\,. \end{aligned}$$

Finally, the composition $\tau{\odot}\sigma$ is defined by the span

$$\begin{array}{ccc} & T{\odot}S & \\ {\scriptstyle \upsilon_1}\swarrow & & \searrow{\scriptstyle \upsilon_2} \\ A^\perp & & C \end{array}$$

where υ_1 and υ_2 are maps of event structures, which on events p of $T{\odot}S$ act so $\upsilon_1(p) = \sigma_1(s)$ when $\mathit{top}(p) = (s,*)$ and $\upsilon_2(p) = \tau_2(t)$ when $\mathit{top}(p) = (*,t)$, and are undefined elsewhere.

Proposition 4.2. *Above, υ_1 and υ_2 are partial maps of event structures with polarity, which together define a pre-strategy $\upsilon : A \twoheadrightarrow C$. For $x \in \mathcal{C}(T{\odot}S)$,*

$$\upsilon_1 x = \sigma_1 \pi_1 \bigcup x \ \textit{ and } \ \upsilon_2 x = \tau_2 \pi_2 \bigcup x\,.$$

Proof. Consider the two maps of event structures

$$\begin{aligned} u_1 : \mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S)) \xrightarrow{\Pi_1} S \xrightarrow{\sigma_1} A^\perp\,, \\ u_2 : \mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S)) \xrightarrow{\Pi_2} T \xrightarrow{\tau_2} C\,, \end{aligned}$$

where Π_1, Π_2 are (restrictions of) projections of the product of event structures. *E.g.* for $p \in \mathrm{Pr}(\mathcal{C}(T) \circledast \mathcal{C}(S))$, $\Pi_1(p) = s$ precisely when $\mathit{top}(p) = (s,*)$, so $\sigma_1(s)$

is defined, or when $top(p) = (s, t)$, so $\sigma_1(s)$ is undefined. The partial functions υ_1 and υ_2 are restrictions of the two maps u_1 and u_2 to the projection set V. But V consists exactly of those events in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$ where u_1 or u_2 is defined. It follows that υ_1 and υ_2 are maps of event structures.

Clearly one and only one of υ_1, υ_2 are defined on any event in $T \odot S$ so they form a pre-strategy. Their effect on $x \in \mathcal{C}(T \odot S)$ follows directly from their definition. □

Proposition 4.3. *Let $\sigma : A \mathrel{\rightarrow\!\!\!\!\!+} B$, $\tau : B \mathrel{\rightarrow\!\!\!\!\!+} C$ and $\upsilon : C \mathrel{\rightarrow\!\!\!\!\!+} D$ be pre-strategies. The two compositions $\upsilon \odot (\tau \odot \sigma)$ and $(\upsilon \odot \tau) \odot \sigma$ are isomorphic.*

Proof. The natural isomorphism $S \times (T \times U) \cong (S \times T) \times U$, associated with the product of event structures S, T, U, restricts to the required isomorphism of spans as the synchronizations involved in successive compositions are disjoint. □

4.3.3 Composition via pullback

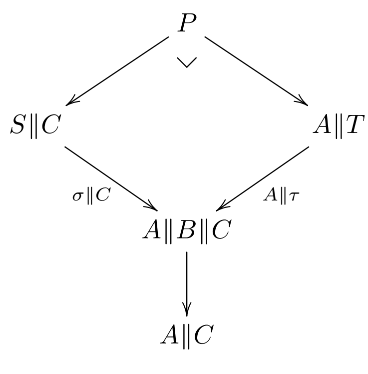
We can alternatively present the composition of pre-strategies via pullbacks.[2] For this section assume that the correspondence $a \leftrightarrow \overline{a}$ between the events of A and its dual $A^\perp$ is the identity, so A and $A^\perp$ share the same events, though assign opposite polarities to them. Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in $\mathcal{E}$ (or $\mathcal{E}_t$, as the maps along which we are pulling back are total)

$$
\begin{array}{ccccc}
 & & P & & \\
 & \swarrow & & \searrow & \\
S \| C & & & & A \| T \\
 & \searrow{\scriptstyle \sigma \| C} & & {\scriptstyle A \| \tau}\swarrow & \\
 & & A \| B \| C\,. & &
\end{array}
$$

There is an obvious partial map of event structures $A \| B \| C \to A \| C$ undefined on B and acting as identity on A and C. The partial map from P to $A \| C$ given

[2] I'm grateful to Nathan Bowler for the observations of this section.

by following the diagram (either way round the pullback square)



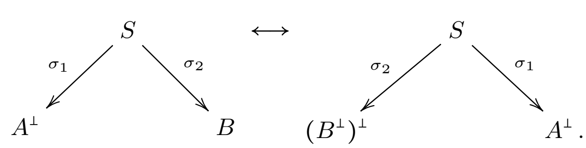
factors through the projection of P to V, those events at which the partial map is defined:

$$P \to P \downarrow V \to A\|C\,.$$

The resulting total map $\upsilon : P \downarrow V \to A\|C$ gives us the composition $\tau \odot \sigma : P \downarrow V \to A^\perp\|C$ once we reinstate polarities.

4.3.4 Duality

A pre-strategy $\sigma : A \rightarrow\!\!\!\!+ B$ corresponds to a dual pre-strategy $\sigma^\perp : B^\perp \rightarrow\!\!\!\!+ A^\perp$. This duality arises from the correspondence



It is easy to check that the dual of copy-cat, $\gamma_A^\perp$, is isomorphic, as a span, to the copy-cat of the dual, $\gamma_{A^\perp}$, for A an event structure with polarity. It is also straightforward, though more involved, to show that the dual of a composition of pre-strategies $(\tau \odot \sigma)^\perp$ is isomorphic as a span to the composition $\sigma^\perp \odot \tau^\perp$. Duality, as usual, will save us work.

4.4 Strategies

This section is devoted to the main result of this chapter: that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient in order for copy-cat to behave as identity w.r.t. the composition of pre-strategies. It becomes compelling to define a *(nondeterministic) concurrent strategy*, in general, as a pre-strategy which is receptive and innocent.

4.4.1 Necessity of receptivity and innocence

The properties of *receptivity* and *innocence* of a pre-strategy, described below, will play a central role.

Receptivity. Say a pre-strategy $\sigma : S \to A$ is *receptive* when $\sigma x \overset{a}{\longrightarrow\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\subset}$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$. Receptivity ensures that no Opponent move which is possible is disallowed.

Innocence. Say a pre-strategy σ is *innocent* when it is both +-innocent and −-innocent:

+-Innocence: If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

−-Innocence: If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

The definition of a pre-strategy $\sigma : S \to A$ ensures that the moves of Player and Opponent respect the causal constraints of the game A. Innocence restricts Player further. Locally, within a configuration, Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$. Thus innocence gives Player the freedom to await Opponent moves before making their move, but prevents Player having any influence on the moves of Opponent beyond those stipulated in the game A; more surprisingly, innocence also disallows any immediate causality of the form $\oplus \rightarrowtriangle \oplus$, purely between Player moves, not already stipulated in the game A.

Two important consequences of −-innocence:

Lemma 4.4. *Let $\sigma : S \to A$ be a pre-strategy. Suppose, for $s, s' \in S$, that*

$$[s) \uparrow [s') \ \&\ pol_S(s) = pol_S(s') = - \ \&\ \sigma(s) = \sigma(s')\,.$$

(i) If σ is −-innocent, then $[s) = [s')$.
(ii) If σ is receptive and −-innocent, then $s = s'$.
[$x \uparrow y$ expresses the compatibility of $x, y \in \mathcal{C}(S)$.]

Proof. (i) Assume the property above holds of $s, s' \in S$. Assume σ is −-innocent. Suppose $s_1 \rightarrowtriangle s$. Then by −-innocence, $\sigma(s_1) \rightarrowtriangle \sigma(s)$. As $\sigma(s') = \sigma(s)$ and σ is a map of event structures there is $s_2 < s'$ such that $\sigma(s_2) = \sigma(s_1)$. But s_1, s_2 both belong to the configuration $[s) \cup [s')$ so $s_1 = s_2$, as σ is a map, and $s_1 < s'$. Symmetrically, if $s_1 \rightarrowtriangle s'$ then $s_1 < s$. It follows that $[s) = [s')$. (ii) Now both $[s) \overset{s}{\longrightarrow\!\!\subset}$ and $[s) \overset{s'}{\longrightarrow\!\!\subset}$ with $\sigma(s) = \sigma(s')$ where both s, s' have −ve polarity. If, further, σ is receptive, $s = s'$. □

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^- x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{-\}$, *i.e.* the configuration x' extends the configuration x solely by events of −ve polarity. In the presence of −-innocence, receptivity strengthens to the following useful *strong-receptivity* property:

Lemma 4.5. *Let $\sigma : S \to A$ be a −-innocent pre-strategy. The pre-strategy σ is* receptive *iff whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a* unique *$x' \in \mathcal{C}(S)$ so that*

$x \subseteq x'$ & $\sigma x' = y$. Diagrammatically,

$$\begin{array}{ccc} x & \cdots\subseteq\cdots & x' \\ {\scriptstyle\sigma}\big\downarrow & & \vdots{\scriptstyle\sigma} \\ \sigma x & \subseteq^- & y\,. \end{array}$$

[It will necessarily be the case that $x \subseteq^- x'$.]

Proof. *"if":* Clear. *"Only if":* Assuming $\sigma x \subseteq^- y$ we can form a covering chain

$$\sigma x \stackrel{a_1}{-\!\!\!\subset} y_1 \cdots \stackrel{a_n}{-\!\!\!\subset} y_n = y\,.$$

By repeated use of receptivity we obtain the existence of x' where $x \subseteq x'$ and $\sigma x' = y$. To show the uniqueness of x' suppose $x \subseteq z, z'$ and $\sigma z = \sigma z' = y$. Suppose that $z \neq z'$. Then, without loss of generality, suppose there is a $\leq_S$-minimal $s' \in z'$ with $s' \notin z$. Then $[s') \subseteq z$. Now $\sigma(s') \in y$ so there is $s \in z$ for which $\sigma(s) = \sigma(s')$. We have $[s), [s') \subseteq z$ so $[s) \uparrow [s')$. By Lemma 4.4(ii) we deduce $s = s'$ so $s' \in z$, a contradiction. Hence, $z = z'$. □

It is useful to define innocence and receptivity on partial maps of event structures with polarity.

Definition 4.6. Let $f : S \to A$ be a partial map of event structures with polarity. Say f is *receptive* when

$$f(x) \stackrel{a}{-\!\!\!\subset} \;\&\; pol_A(a) = - \implies \exists! s \in S.\ x \stackrel{s}{-\!\!\!\subset} \;\&\; f(s) = a$$

for all $x \in \mathcal{C}(S)$, $a \in A$.

Say f is *innocent* when it is both +-innocent and −-innocent, *i.e.*

$$\begin{aligned} s \rightarrowtriangle s' \;\&\; pol(s) = + \;\&\; f(s) \text{ is defined} &\implies \\ &f(s') \text{ is defined } \;\&\; f(s) \rightarrowtriangle f(s')\,, \\ s \rightarrowtriangle s' \;\&\; pol(s') = - \;\&\; f(s') \text{ is defined} &\implies \\ &f(s) \text{ is defined } \;\&\; f(s) \rightarrowtriangle f(s')\,. \end{aligned}$$

Proposition 4.7. *A pre-strategy $\sigma : A \mathrel{-\!\!\!+\!\!\!\!\to} B$ is receptive, respectively +/−-innocent, iff both the partial maps σ_1 and σ_2 of its span are receptive, respectively +/−-innocent.*

Proposition 4.8. *For $\sigma : A \mathrel{-\!\!\!+\!\!\!\!\to} B$ a pre-strategy, σ_1 is receptive, respectively +/−-innocent, iff $(\sigma^\perp)_2$ is receptive, respectively +/−-innocent; σ is receptive and innocent iff $\sigma^\perp$ is receptive and innocent.*

The next lemma will play a major role in importing receptivity and innocence to compositions of pre-strategies.

Lemma 4.9. *For pre-strategies $\sigma : A \mathrel{-\!\!\!+\!\!\!\!\to} B$ and $\tau : B \mathrel{-\!\!\!+\!\!\!\!\to} C$, if σ_1 is receptive, respectively +/−-innocent, then $(\tau \odot \sigma)_1$ is receptive, respectively +/−-innocent.*

Proof. Abbreviate $\tau \odot \sigma$ to υ.
Receptivity: We show the receptivity of υ_1 assuming that σ_1 is receptive. Let $x \in \mathcal{C}(T \odot S)$ such that $\upsilon_1 x \overset{a}{\longrightarrow\!\subset}$ in $\mathcal{C}(A^\perp)$ with $pol_{A^\perp}(a) = -$. By Proposition 4.2, $\sigma_1 \pi_1 \bigcup x \overset{a}{\longrightarrow\!\subset}$ with $\pi_1 \bigcup x \in \mathcal{C}(S)$. As σ_1 is receptive there is a unique $s \in S$ such that $\pi_1 \bigcup x \overset{s}{\longrightarrow\!\subset}$ in S and $\sigma_1(s) = a$. It follows that $\bigcup x \overset{(s,*)}{\longrightarrow\!\subset} z$, for some z, in $\mathcal{C}(T) \circledast \mathcal{C}(S)$. Defining $p =_{\text{def}} [(s,*)]_z$ we obtain $x \overset{p}{\longrightarrow\!\subset}$ and $\upsilon_1(p) = a$, with p the unique such event.
Innocence: Assume that σ_1 is innocent. To show the +-innocence of υ_1 we first establish a property of the $\rightarrowtriangle$-relation in the event structure $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, the synchronized composition of event structures S and T, before projection to V:

> *If $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$ with $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ defined, then $e' \in V$ and $\upsilon_1(e')$ is defined.*

Assume $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ is defined. From the definition of $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, the event e is a prime configuration of $\mathcal{C}(T) \circledast \mathcal{C}(S)$ where $top(e)$ must have the form $(s,*)$, for some event s of S where $\sigma_1(s)$ is defined. By Lemma 3.21, $top(e')$ has the form $(s',*)$ or (s',t) with $s \rightarrowtriangle s'$ in S. Now, as $s \rightarrowtriangle s'$ and $pol(s) = +$, from the +-innocence of σ_1, we obtain $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp \| A$. Whence $\sigma_1(s')$ is defined ensuring $top(e') = (s',*)$. It follows that $e' \in V$ and $\upsilon_1(e')$ is defined.

Now suppose $e \rightarrowtriangle e'$ in $T \odot S$. Then either
(i) $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$, or
(ii) $e \rightarrowtriangle e_1 < e'$ in $\Pr(\mathcal{C}(T) \circledast \mathcal{C}(S))$ for some 'invisible' event $e_1 \notin V$.

But the above argument shows that case (ii) cannot occur when $pol(e) = +$ and $\upsilon_1(e)$ is defined. It follows that whenever $e \rightarrowtriangle e'$ in $T \odot S$ with $pol(e) = +$ and $\upsilon_1(e)$ defined, then $\upsilon_1(e')$ is defined and $\upsilon_1(e) \rightarrowtriangle \upsilon_1(e')$, as required.

The argument showing −-innocence of υ_1 assuming that of σ_1 is similar. □

Corollary 4.10. *For pre-strategies $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\rightarrow B$ and $\tau : B \rightarrow\!\!\!\!\!+\!\!\!\!\rightarrow C$, if τ_2 is receptive, respectively +/−-innocent, then $(\tau \odot \sigma)_2$ is receptive, respectively +/−-innocent.*

Proof. By duality using Lemma 4.9: if τ_2 is receptive, respectively +/−-innocent, then $(\tau^\perp)_1$ is receptive, respectively +/−-innocent, and hence $(\sigma^\perp \odot \tau^\perp)_1 = ((\tau \odot \sigma)^\perp)_1 = (\tau \odot \sigma)_2$ is receptive, respectively +/−-innocent. □

Lemma 4.11. *For an event structure with polarity A, the pre-strategy copy-cat $\gamma_A : A \rightarrow\!\!\!\!\!+\!\!\!\!\rightarrow A$ is receptive and innocent.*

Proof. Receptive: Suppose $x \in \mathcal{C}(\mathbb{C}_A)$ such that $\gamma_A x \overset{c}{\longrightarrow\!\subset}$ in $\mathcal{C}(A^\perp \| A)$ where $pol_{A^\perp \| A}(c) = -$. Now $\gamma_A x = x$ and $x' =_{\text{def}} x \cup \{c\} \in \mathcal{C}(A^\perp \| A)$. Proposition 4.1(ii) characterizes those configurations of $A^\perp \| A$ which are also configurations of $\mathbb{C}_A$: the characterization applies to x and to its extension $x' = x \cup \{c\}$ because of the

−ve polarity of c. Hence $x' \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ and $x \overset{c}{\longrightarrow\!\!\subset} x'$ in $\mathcal{C}(\mathbb{C}\mathbb{C}_A)$, and clearly c is unique so $\gamma_A(c) = c$.

−-*Innocent:* Suppose $c \twoheadrightarrow c'$ in $\mathbb{C}\mathbb{C}_A$ and $pol(c') = -$. By Proposition 4.1(i), $c \twoheadrightarrow c'$ in $A^\perp \| A$. The argument for +-innocence is similar. □

Theorem 4.12. *Let $\sigma : A \rightarrow\!\!\!\!\!+\,\, B$ be a pre-strategy from A to B. If $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$, then σ is receptive and innocent.*

Let $\sigma : A \rightarrow\!\!\!\!\!+\,\, B$ and $\tau : B \rightarrow\!\!\!\!\!+\,\, C$ be pre-strategies which are both receptive and innocent. Then their composition $\tau \odot \sigma : A \rightarrow\!\!\!\!\!+\,\, C$ is receptive and innocent.

Proof. We know the copy-cat pre-strategies γ_A and γ_B are receptive and innocent—Lemma 4.11. Assume $\sigma \odot \gamma_A \cong \sigma$ and $\gamma_B \odot \sigma \cong \sigma$. By Lemma 4.9, $(\sigma \odot \gamma_A)_1$ is receptive and innocent so σ_1 is receptive and innocent. From its dual, Corollary 4.10, $(\gamma_B \odot \sigma)_2$ so σ_2 is receptive and innocent. Hence σ is receptive and innocent.

Assume that $\sigma : A \rightarrow\!\!\!\!\!+\,\, B$ and $\tau : B \rightarrow\!\!\!\!\!+\,\, C$ are receptive and innocent. The fact that σ is receptive and innocent ensures that $(\tau \odot \sigma)_1$ is receptive and innocent, that τ is receptive and innocent that $(\tau \odot \sigma)_2$ is too. Combining, we obtain that $\tau \odot \sigma$ is receptive and innocent. □

In other words, if a pre-strategy is to compose well with copy-cat, in the sense that copy-cat behaves as an identity w.r.t. composition, the pre-strategy must be receptive and innocent. Copy-cat behaving as identity is a hallmark of game-based semantics, so any sensible definition of concurrent strategy will have to ensure receptivity and innocence.

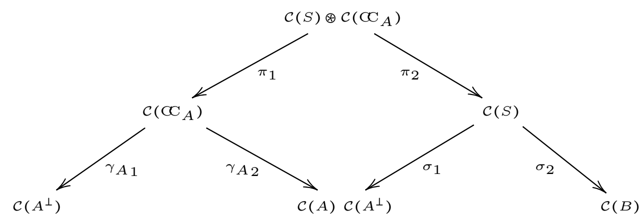
4.4.2 Sufficiency of receptivity and innocence

In fact, as we will now see, not only are the conditions of receptivity and innocence on pre-strategies necessary to ensure that copy-cat acts as identity. They are also sufficient.

Technically, this section establishes that for a pre-strategy $\sigma : A \rightarrow\!\!\!\!\!+\,\, B$ which is receptive and innocent both the compositions $\sigma \odot \gamma_A$ and $\gamma_B \odot \sigma$ are isomorphic to σ. We shall concentrate on the isomorphism from $\sigma \odot \gamma_A$ to σ. The isomorphism from $\gamma_B \odot \sigma$ to σ follows by duality.

Recall, from Section 4.3.2, the construction of the pre-strategy $\sigma \odot \gamma_A$ as a total map $S \odot \mathbb{C}\mathbb{C}_A \to A^\perp \| B$. The event structure $S \odot \mathbb{C}\mathbb{C}_A$ is built from the synchronized composition of stable families $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}\mathbb{C}_A)$, a restriction of the product of stable families to events

$$
\begin{aligned}
&\{(c, *) \mid c \in \mathbb{C}\mathbb{C}_A \ \&\ \gamma_{A1}(c) \text{ is defined}\} \cup \\
&\{(c, s) \mid c \in \mathbb{C}\mathbb{C}_A \ \&\ s \in S \ \&\ \gamma_{A2}(c) = \overline{\sigma_1(s)}\} \cup \\
&\{(*, s) \mid s \in S \ \&\ \sigma_2(t) \text{ is defined}\} :
\end{aligned}
$$



Finally $S \odot \mathbb{C}_A$ is obtained from the prime configurations of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$ whose maximum events are defined under $\gamma_{A1}\pi_1$ or $\sigma_2\pi_2$.

We will first present the putative isomorphism from $\sigma \odot \gamma_A$ to σ as a total map of event structures $\theta : S \odot \mathbb{C}_A \to S$. The definition of θ depends crucially on the lemmas below. They involve special configurations of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$, *viz.* those of the form $\bigcup x$, where x is a configuration of $S \odot \mathbb{C}_A$.

Lemma 4.13. *For $x \in \mathcal{C}(S \odot \mathbb{C}_A)$,*

$$(c, s) \in \bigcup x \implies (\bar{c}, *) \in \bigcup x\,.$$

Proof. The case when $pol(c) = +$ follows directly because then $\bar{c} \rightarrowtriangle c$ in $\mathbb{C}_A$ so $(\bar{c}, *) \rightarrowtriangle_{\bigcup x} (c, s)$.
Suppose the lemma fails in the case when $pol(c) = -$, so there is a $\leq_{\bigcup x}$-maximal $(c, s) \in \bigcup x$ such that

$$pol(c) = - \ \&\ (\bar{c}, *) \notin \bigcup x\,. \qquad (\dagger)$$

The event (c, s) cannot be maximal in $\bigcup x$ as its maximal events take the form $(c', *)$ or $(*, s')$. There must be $e \in \bigcup x$ for which

$$(c, s) \rightarrowtriangle_{\bigcup x} e\,.$$

Consider the possible forms of e:
Case $e = (c', s')$: Then, by Lemma 3.21, either $c \rightarrowtriangle c'$ in $\mathbb{C}_A$ or $s \rightarrowtriangle s'$ in S. However if $s \rightarrowtriangle s'$ then, as $pol(s) = +$ by innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp$, so $\gamma_{A2}(c) \rightarrowtriangle \gamma_{A2}(c')$ in A; but then $c \rightarrowtriangle c'$ in $\mathbb{C}_A$. Either way, $c \rightarrowtriangle c'$ in $\mathbb{C}_A$.

Suppose $pol(c') = +$. Then,

$$(c, s) \rightarrowtriangle_{\bigcup x} (\bar{c}, *) \rightarrowtriangle_{\bigcup x} (\overline{c'}, *) \rightarrowtriangle_{\bigcup x} (c', s')\,.$$

But this contradicts $(c, s) \rightarrowtriangle_{\bigcup x} (c', s')$.

Suppose $pol(c') = -$. Because (c, s) is maximal such that $(\dagger)$, $(\overline{c'}, *) \in \bigcup x$. But $(\bar{c}, *) \rightarrowtriangle_{\bigcup x} (\overline{c'}, *)$ whence $(\bar{c}, *) \in \bigcup x$, contradicting $(\dagger)$.
Case $e = (*, s')$: Now $(c, s) \rightarrowtriangle_{\bigcup x} (*, s')$. By Lemma 3.21, $s \rightarrowtriangle s'$ in S with $pol(s) = +$. By innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ and in particular $\sigma_1(s')$ is defined, which forbids $(*, s')$ as an event of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$.
Case $e = (c', *)$: Now $(c, s) \rightarrowtriangle_{\bigcup x} (c', *)$. By Lemma 3.21, $c \rightarrowtriangle c'$ in $\mathbb{C}_A$. Because (c, s) and $(c', *)$ are events of $\mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$ we must have $\gamma_2(c)$ and $\gamma_1(c')$ are defined—they are in different components of $\mathbb{C}_A$. By Proposition 4.1, $c' = \bar{c}$, contradicting $(\dagger)$.

In all cases we obtain a contradiction—hence the lemma. □

Lemma 4.14. *For* $x \in \mathcal{C}(S \odot \mathbb{C}\mathbb{C}_A)$,

$$\sigma_1 \pi_2 \bigcup x \subseteq^- \gamma_{A1} \pi_1 \bigcup x \,.$$

Proof. As a direct corollary of Lemma 4.13, we obtain:

$$\sigma_1 \pi_2 \bigcup x \subseteq \gamma_{A1} \pi_1 \bigcup x \,.$$

The current lemma will follow provided all events of +ve polarity in $\gamma_{A1}\pi_1 \bigcup x$ are in $\sigma_1\pi_2 \bigcup x$. However, $(\overline{c}, s) \rightarrowtriangle_{\bigcup x} (c, *)$, for some $s \in S$, when $pol(c) = +$. □

Lemma 4.15. *For* $x \in \mathcal{C}(S \odot \mathbb{C}\mathbb{C}_A)$,

$$\sigma \pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A \, x \,.$$

Proof.

$$\begin{aligned} \sigma \pi_2 \bigcup x &= \{1\} \times \sigma_1 \pi_2 \bigcup x \ \cup \ \{2\} \times \sigma_2 \pi_2 \bigcup x \\ &\subseteq^- \{1\} \times \gamma_{A1} \pi_1 \bigcup x \ \cup \ \{2\} \times \sigma_2 \pi_2 \bigcup x \,, \quad \text{by Lemma 4.14} \\ &= \sigma \odot \gamma_A \, x \,, \quad \text{by Proposition 4.2.} \end{aligned}$$

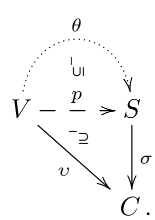
□

Lemma 4.15 is the key to defining a map $\theta : S \odot \mathbb{C}\mathbb{C}_A \to S$ via the following map-lifting property of receptive maps:

Lemma 4.16. *Let* $\sigma : S \to C$ *be a total map of event structures with polarity which is receptive and* $-$*-innocent. Let* $p : \mathcal{C}(V) \to \mathcal{C}(S)$ *be a monotonic function, i.e. such that* $p(x) \subseteq p(y)$ *whenever* $x \subseteq y$ *in* $\mathcal{C}(V)$*. Let* $\upsilon : V \to C$ *be a total map of event structures with polarity such that*

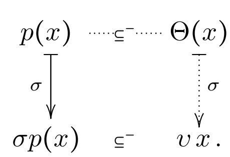
$$\forall x \in \mathcal{C}(V).\ \sigma p(x) \subseteq^- \upsilon \, x \,.$$

Then, there is a unique total map of event structures with polarity $\theta : V \to S$ *such that* $\forall x \in \mathcal{C}(V).\ p(x) \subseteq^- \theta \, x$ *and* $\upsilon = \sigma\theta$:



[We use a broken arrow to signify that p is not a map of event structures.]

Proof. Let $x \in \mathcal{C}(V)$. Then $\sigma p(x) \subseteq^- \upsilon \, x$. Define $\Theta(x)$ to be the unique configuration of $\mathcal{C}(S)$, determined by the receptivity of σ, such that



Define θ_x to be the composite bijection

$$\theta_x : x \quad\cong\quad \upsilon x \quad\cong\quad \Theta(x)$$

where the bijection $x \cong \upsilon x$ is that determined locally by the total map of event structures υ, and the bijection $\upsilon x \cong \Theta(x)$ is the inverse of the bijection $\sigma \upharpoonright \Theta(x) : \Theta(x) \cong \upsilon\, x$ determined locally by the total map σ.

Now, let $y \in \mathcal{C}(V)$ with $x \subseteq y$. We claim that θ_x is the restriction of θ_y. This will follow once we have shown that $\Theta(x) \subseteq \Theta(y)$. Then, treating the inclusions as inclusion maps, both squares in the diagram below will commute:

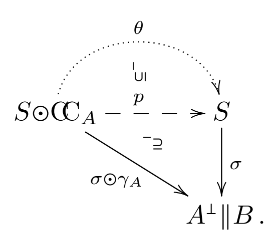
$$\begin{array}{ccccc} \theta_y : y & \cong & \upsilon\, y & \cong & \Theta(y) \\ \cup\!\!| & & \cup\!\!| & & \cup\!\!| \\ \theta_x : x & \cong & \upsilon\, x & \cong & \Theta(x) \end{array}$$

This will make the composite rectangle commute, *i.e.* make θ_x the restriction of θ_y.

To show $\Theta(x) \subseteq \Theta(y)$ we suppose otherwise. Then there is an event $s \in \Theta(x)$ of minimum depth w.r.t. $\leq_S$ such that $s \notin \Theta(y)$. Note that $pol(s) = -$, as otherwise $s \in p(x) \subseteq p(y) \subseteq \Theta(y)$. As $\sigma(s) \in \upsilon\, x \subseteq \upsilon\, y$ there is $s' \in \Theta(y)$ such that $\sigma(s') = \sigma(s)$. From the minimality of s, both $[s), [s') \subseteq \Theta(y)$ ensuring the compatibility of $[s)$ and $[s')$. By Lemma 4.4(ii), $s = s'$ and $s \in \Theta(y)$—a contradiction.

By Proposition 2.5, the family θ_x, $x \in \mathcal{C}(V)$, determines the unique total map $\theta : V \to S$ such that $\theta\, x = \Theta(x)$. By construction, $p(x) \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(V)$, and $\upsilon = \sigma\theta$. This property in itself ensures that $\theta\, x = \Theta(x)$ so determines θ uniquely. □

In Lemma 4.16, instantiate $p : \mathcal{C}(S \odot \mathbb{C}\!\!\mathbb{C}_A) \to \mathcal{C}(S)$ to the function $p(x) = \pi_2 \bigcup x$ for $x \in \mathcal{C}(S \odot \mathbb{C}\!\!\mathbb{C}_A)$, the map σ to the pre-strategy $\sigma : S \to A^\perp \| B$ and υ to the pre-strategy $\sigma \odot \gamma_A$. By Lemma 4.15, $\sigma \pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A\, x$, so the conditions of Lemma 4.16 are met and we obtain a total map $\theta : S \odot \mathbb{C}\!\!\mathbb{C}_A \to S$ such that $\pi_2 \bigcup x \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(S \odot \mathbb{C}\!\!\mathbb{C}_A)$, and $\sigma\theta = \sigma \odot \gamma_A$:



The next lemma is used in showing θ is an isomorphism.

Lemma 4.17. *(i) Let $z \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}\!\!\mathbb{C}_A)$. If $e \leq_z e'$ and $\pi_2(e)$ and $\pi_2(e')$ are defined, then $\pi_2(e) \leq_S \pi_2(e')$. (ii) The map π_2 is surjective on configurations.*

Proof. (i) It suffices to show when

$$e \rightarrowtriangle_z e_1 \rightarrowtriangle_z \cdots \rightarrowtriangle_z e_{n-1} \rightarrowtriangle_z e'$$

with $\pi_2(e)$ and $\pi_2(e')$ defined and all $\pi_2(e_i)$, $1 \le i \le n-1$, undefined, that $\pi_2(e) \le_S \pi_2(e')$.
Case $n = 1$, so $e \rightarrowtriangle_z e'$: Use Lemma 3.21. If either e or e' has the form $(*, s)$ then the other event must have the form $(*, s')$ or (c', s') with $s \rightarrowtriangle s'$ in S. In the remaining case $e = (c, s)$ and $e' = (c', s')$ with either (1) $c \rightarrowtriangle c'$ in $\mathbb{C}_A$, and $\gamma_{A2}(c) \rightarrowtriangle \gamma_{A2}(c')$ in A, or (2) $s \rightarrowtriangle s'$ in S. If (1), $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp$ where $s, s' \in \pi_2 z$. By Proposition 3.10, $s \le_S s'$. In either case (1) or (2), $\pi_2(e) \le_S \pi_2(e')$.
Case $n > 1$: Each e_i has the form $(c_i, *)$, for $1 \le i \le n-1$. By Lemma 3.21, events e and e' must have the form (c, s) and (c', s') with $c \rightarrowtriangle c_1$ and $c_{n-1} \rightarrowtriangle c'$ in $\mathbb{C}_A$. As $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, $c_1 = \overline{c}$ and similarly $c_{n-1} = \overline{c'}$. Again by Lemma 3.21, $c_i \rightarrowtriangle c_{i+1}$ in $\mathbb{C}_A$ for $1 \le i \le i-2$. Consequently $\gamma_{A2}(c) \le_A \gamma_{A2}(c')$. Now, $s, s' \in \pi_2 z$ with $\sigma_1(s) \le_{A^\perp} \sigma_1(s')$. By Proposition 3.10, $s \le_S s'$, as required.
(ii) Let $y \in \mathcal{C}(S)$. Then $\sigma_1 y \in \mathcal{C}(A^\perp)$ and by the clear surjectivity of γ_{A2} on configurations there exists $w \in \mathcal{C}(\mathbb{C}_A)$ such that $\gamma_{A2} w = \sigma_1 y$. Now let

$$\begin{aligned} z = &\{(c, *) \mid c \in w \ \&\ \gamma_{A1}(c) \text{ is defined}\} \\ &\cup \{(c, s) \mid c \in w \ \&\ s \in y \ \&\ \gamma_{A2}(c) = \sigma_1(s)\} \\ &\cup \{(*, s) \mid s \in y \ \&\ \sigma_2(s) \text{ is defined}\}. \end{aligned}$$

Then, from the definition of the product of stable families—3.3.1, it can be checked that $z \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{C}_A)$. By construction, $\pi_2 z = y$. Hence π_2 is surjective on configurations. □

Theorem 4.18. *$\theta : \sigma \odot \gamma_A \cong \sigma$, an isomorphism of pre-strategies.*

Proof. We show θ is an isomorphism of event structures by showing θ is rigid and both surjective and injective on configurations (Lemma 3.3 of [9]). The rest is routine.
Rigid: It suffices to show $p \rightarrowtriangle p'$ in $S \odot \mathbb{C}_A$ implies $\theta(p) \le_S \theta(p')$. Suppose $p \rightarrowtriangle p'$ in $S \odot \mathbb{C}_A$ with $top(p) = e$ and $top(p') = e'$. Take $x \in \mathcal{C}(S \odot \mathbb{C}_A)$ containing p' so p too. Then

$$e \rightarrowtriangle_{\bigcup x} e_1 \rightarrowtriangle_{\bigcup x} \cdots \rightarrowtriangle_{\bigcup x} e_{n-1} \rightarrowtriangle_{\bigcup x} e'$$

where $e, e' \in V_0$ and $e_i \notin V_0$ for $1 \le i \le n-1$. (V_0 consists of 'visible' events of the form $(c, *)$ with $\gamma_{A1}(c)$ defined, or $(*, s)$, with $\sigma_2(s)$ defined.)
Case $n = 1$, so $e \rightarrowtriangle_{\bigcup x} e'$: By Lemma 3.21, either (i) $e = (*, s)$ and $e' = (*, s')$ with $s \rightarrowtriangle s'$ in S, or (ii) $e = (c, *)$ and $e' = (c', *)$ with $c \rightarrowtriangle c'$ in $\mathbb{C}_A$.
If (i), we observe, via $\sigma\theta = \sigma \odot \gamma_A$, that $s \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s)$, so $\theta(p) = s$ by the local injectivity of σ. Similarly, $\theta(p') = s'$, so $\theta(p) \le_S \theta(p')$.
If (ii), we obtain $\theta(p), \theta(p') \in \theta x$ with $\sigma_1 \theta(p) = \gamma_{A1}(c)$, $\sigma_1 \theta(p') = \gamma_{A1}(c')$ and $\gamma_{A1}(c) \rightarrowtriangle \gamma_{A1}(c')$ in $A^\perp$. By Proposition 3.10, $\theta(p) \le_S \theta(p')$.

Case $n > 1$: Note $e_i = (c_i, s_i)$ for $1 \le i \le n-1$, and that $s_1 \le_S s_{n-1}$ by Lemma 4.17(i). Consider the case in which $e = (c, *)$ and $e' = (c', *)$—the other cases are similar. By Lemma 3.21, $c \rightarrowtriangle c_1$ and $c_{n-1} \rightarrowtriangle c'$ in $\mathbb{CC}_A$. But $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, so $c_1 = \overline{c}$, and similarly $c_{n-1} = \overline{c'}$. We remark that $\theta(p) = s_1$, by the local injectivity of σ, as both $s_1 \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s_1)$. Similarly $\theta(p') = s_{n-1}$, whence $\theta(p) \le_S \theta(p')$.
Surjective: Let $y \in \mathcal{C}(S)$. By Lemma 4.17(ii), there is $z \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{CC}_A)$ such that $\pi_2 z = y$. Let

$$z' = z \cup \{(c, *) \mid pol(c) = + \ \& \ \exists s \in S.\ (\overline{c}, s) \in z\}\,.$$

It is straightforward to check $z' \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{CC}_A)$. Now let

$$z'' = z' \smallsetminus \{(c, *) \mid pol(c) = - \ \& \ \forall s \in S.\ (\overline{c}, s) \notin z'\}\,.$$

Then $z'' \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{CC}_A)$ by the following argument. The set z'' is certainly consistent, so it suffices to show

$$pol(c) = - \ \& \ (c, *) \le_{z'} e \in z'' \implies \exists s \in S.\ (\overline{c}, s) \in z'\,,$$

for all $c \in \mathbb{CC}_A$ and $e \in z''$. This we do by induction on the number of events between $(c, *)$ and e. Suppose

$$pol(c) = - \ \& \ (c, *) \rightarrowtriangle_{z'} e_1 \le_{z'} e \in z'\,.$$

In the case where $e_1 = (c_1, s_1)$, we deduce $c \rightarrowtriangle c_1$ in $\mathbb{CC}_A$ and as $\gamma_{A1}(c)$ is defined while $\gamma_{A2}(c_1)$ is defined, we must have $c_1 = \overline{c}$, as required. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = -$, by induction, we obtain $(\overline{c_1}, s_1) \in z'$ for some $s_1 \in S$. Also $c \rightarrowtriangle c_1$, so $\overline{c} \rightarrowtriangle \overline{c_1}$ in $\mathbb{CC}_A$. As z' is a configuration we must have $(\overline{c}, s) \le_{z'} (\overline{c_1}, s_1)$, for some $s \in S$, so $(\overline{c}, s) \in z'$. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = +$, we have $c \rightarrowtriangle c_1$ in $\mathbb{CC}_A$. Moreover, $(\overline{c}_1, s) \in z'$, for some $s \in S$, as z' is a configuration and $\overline{c_1} \rightarrowtriangle c_1$ in $\mathbb{CC}_A$. Again, from the fact that z' is a configuration, there must be $(\overline{c}, s) \in z'$ for some $s \in S$. We have exhausted all cases and conclude $z'' \in \mathcal{C}(S) \circledast \mathcal{C}(\mathbb{CC}_A)$ with $\theta z'' = \pi_2 z = y$, as required to show θ is surjective on configurations.
Injective: Abbreviate $\sigma \odot \gamma_A$ to υ. Assume $\theta x = \theta y$, where $x, y \in \mathcal{C}(S \odot \mathbb{CC}_A)$. Via the commutativity $\upsilon = \sigma\theta$, we observe

$$\upsilon x = \sigma\theta\, x = \sigma\theta\, y = \upsilon y\,.$$

Recall by Proposition 4.2, that $\upsilon_1 x = \gamma_{A1}\pi_1 \bigcup x = \pi_1 \bigcup x$. It follows that

$$(c, *) \in \bigcup x \iff c \in \upsilon_1 x \iff c \in \upsilon_1 y \iff (c, *) \in \bigcup y\,.$$

Observe

$$(*, s) \in \bigcup x \iff \sigma_2(s) \text{ is defined } \& \ s \in \theta x :$$

"$\Rightarrow$" by the local injectivity of σ_2, as $p =_{\text{def}} [(*, s)]_{\bigcup x}$ yields $\theta(p) \in \theta x$ and $s \in \pi_2 \bigcup x \subseteq \theta x$ with $\sigma_2(\theta(p)) = \sigma_2(s)$, so $\theta(p) = s$; "$\Leftarrow$" as $\sigma_2(s)$ defined and

$s \in \theta x$ entails $s = \theta(p)$ for some $p \in x$, necessarily with $top(p) = (*, s)$. Hence

$$\begin{aligned}(*, s) \in \bigcup x &\iff \sigma_2(s) \text{ is defined } \& \ s \in \theta x\\ &\iff \sigma_2(s) \text{ is defined } \& \ s \in \theta y\\ &\iff (*, s) \in \bigcup y\,.\end{aligned}$$

Assuming $(c, s) \in \bigcup x$ we now show $(c, s) \in \bigcup y$. (The converse holds by symmetry.) There is $p \in x$, such that $(c, s) \in p$. If $top(p) = (*, s')$ (also in $\bigcup y$ as it is visible) then as π_2 is rigid, $s \le s'$ and we must have $(c', s) \in \bigcup y$. Otherwise, $top(p) = (d, *)$ and we can suppose (by taking p minimal) that $(c, s) \le_{\bigcup x} (d', s') \rightarrowtriangle_{\bigcup x} (d, *)$. But then $\theta(p) = s' \in \theta x = \theta y$. Also $s \le_S s'$, by the rigidity of π_2, and, as we have seen before, $d' = \overline{d}$ with d' −ve. Hence s' is +ve and as θy is a −ve extension of $\pi_2 \bigcup y$ we must have $s' \in \pi_2 \bigcup y$. Hence there is $(*, s')$ or (c'', s') in $\bigcup y$, and as $s \le_S s'$ there is some $(c', s) \in \bigcup y$. In both cases, $\gamma_{A2}(c') = \overline{\sigma_1(s)} = \gamma_{A2}(c)$, so $c' = c$, and thus $(c, s) \in \bigcup y$.

We conclude $\bigcup x = \bigcup y$, so $x = y$, as required for injectivity. □

4.5 Concurrent strategies

Define a *strategy* to be a pre-strategy which is receptive and innocent. We obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \rightarrow\!\!\!\!\!\!+ B$ and the 2-cells are maps of pre-strategies. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition). The isomorphisms expressing associativity and the identity of copy-cat are those of Proposition 4.3 and Theorem 4.18 with its dual.

We remark for future use that composition of strategies respects less general notions of 2-cell. The horizontal composition of rigid 2-cells is rigid. The essential ingredients in showing this are that the product and pullback of event structures preserve rigid maps when regarded as functor (from Corollary 3.23) and that under appropriate conditions hiding as formalized through projection preserves rigid maps (Proposition 3.29).

4.5.1 Alternative characterizations

Via saturation conditions

An alternative description of concurrent strategies exhibits the correspondence between innocence and earlier "saturation conditions," *reflecting* specific independence, in [10, 11, 12]:

Proposition 4.19. *A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that*

(i) $\sigma x \stackrel{a}{\longrightarrow\!\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \stackrel{s}{\longrightarrow\!\!\!\subset}$ & $\sigma(s) = a$, *for all* $x \in \mathcal{C}(S)$, $a \in A$.

(ii)(+) If $x \stackrel{e}{-\!\!\!\subset} x_1 \stackrel{e'}{-\!\!\!\subset}$ & $pol_S(e) = +$ *in* $\mathcal{C}(S)$ *and* $\sigma x \stackrel{\sigma(e')}{-\!\!\!\subset}$ *in* $\mathcal{C}(A)$*, then* $x \stackrel{e'}{-\!\!\!\subset}$ *in* $\mathcal{C}(S)$*.*

(ii)(−) If $x \stackrel{e}{-\!\!\!\subset} x_1 \stackrel{e'}{-\!\!\!\subset}$ & $pol_S(e') = -$ *in* $\mathcal{C}(S)$ *and* $\sigma x \stackrel{\sigma(e')}{-\!\!\!\subset}$ *in* $\mathcal{C}(A)$*, then* $x \stackrel{e'}{-\!\!\!\subset}$ *in* $\mathcal{C}(S)$*.*

Proof. Note that if $x \stackrel{e}{-\!\!\!\subset} x_1 \stackrel{e'}{-\!\!\!\subset}$ then either $e\ co\ e'$ or $e \rightarrowtriangle e'$. Condition (ii) is a contrapositive reformulation of innocence. □

Via lifting conditions

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^+ x'$ to mean $x \subseteq x'$ and $pol(x' \smallsetminus x) \subseteq \{+\}$, *i.e.* the configuration x' extends the configuration x solely by events of +ve polarity. With this notation in place we can give an attractive characterization of concurrent strategies:

Proposition 4.20. *A strategy in a game A comprises a total map of event structures with polarity* $\sigma : S \to A$ *such that*
(i) whenever $y \subseteq^+ \sigma x$ *in* $\mathcal{C}(A)$ *there is a (necessarily unique)* $x' \in \mathcal{C}(S)$ *so that* $x' \subseteq x$ & $\sigma x' = y$ *, i.e.*

$$\begin{array}{ccc} x' & \subseteq & x \\ \sigma \downarrow & & \downarrow \sigma \\ y & \subseteq^+ & \sigma x \,, \end{array}$$

and
(ii) whenever $\sigma x \subseteq^- y$ *in* $\mathcal{C}(A)$ *there is a unique* $x' \in \mathcal{C}(S)$ *so that* $x \subseteq x'$ & $\sigma x' = y$ *, i.e.*

$$\begin{array}{ccc} x & \subseteq & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma x & \subseteq^- & y \,. \end{array}$$

Proof. Let $\sigma : S \to A$ be a total map of event structures with polarity. It is claimed that σ is a strategy iff (i) and (ii).

"Only if": Lemma 4.5 directly implies (ii). To establish (i) it suffices to show the seemingly weaker property (i)$'$ that

$$y \stackrel{a}{-\!\!\!\subset} \sigma x \ \&\ pol(a) = + \implies \exists x' \in \mathcal{C}(S).\, x' -\!\!\!\subset x \ \&\ \sigma x' = y$$

for $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$. Then (i), with $y \subseteq^+ \sigma x$, follows by considering a covering chain $y -\!\!\!\subset \cdots -\!\!\!\subset \sigma x$. (The uniqueness of x is a direct consequence of σ being a total map of event structures.) To show (i)$'$, suppose $y \stackrel{a}{-\!\!\!\subset} \sigma x$ with a +ve. Then $\sigma(s) = a$ for some unique $s \in x$ with s +ve. Supposing s were not $\le$-maximal in x, then $s \rightarrowtriangle s'$ for some $s' \in x$. By +-innocence $a = \sigma(s) \rightarrowtriangle \sigma(s') \in \sigma x$

implying a is not $\le$-maximal in σx. This contradicts $y \overset{a}{-\!\!\!\subset} \sigma x$. Hence s is $\le$-maximal and $x' =_{\mathrm{def}} x \setminus \{s\} \in \mathcal{C}(S)$ with $x' -\!\!\!\subset x$ and $\sigma x' = y$.

"If": Assume σ satisfies (i) and (ii). Clearly σ is receptive by (ii). We establish innocence via Proposition 4.19.

Suppose $x \overset{s}{-\!\!\!\subset} x_1 \overset{s'}{-\!\!\!\subset} x'$ and $pol(s) = +$ with $\sigma x \overset{\sigma(s')}{-\!\!\!\subset} y_2$. Then $y_2 \overset{\sigma(s)}{-\!\!\!\subset} \sigma x'$ with $pol(\sigma(s)) = +$. From (i) we obtain a unique $x_2 \in \mathcal{C}(S)$ such that $x_2 \subseteq x'$ and $\sigma x_2 = y_2$. As σ is a total map of event structures, we obtain $x_2 \overset{s}{-\!\!\!\subset} x'$ and subsequently $x \overset{s'}{-\!\!\!\subset} x_2$, as required by Proposition 4.19(ii)+.

Suppose $x \overset{s}{-\!\!\!\subset} x_1 \overset{s'}{-\!\!\!\subset} x'$ and $pol(s') = -$ with $\sigma x \overset{\sigma(s')}{-\!\!\!\subset} y_2$. The case where $pol(s) = +$ is covered by the previous argument: we obtain $x \overset{s'}{-\!\!\!\subset} x_2$, as required by Proposition 4.19(ii)−. Suppose $pol(s) = -$. We have

$$\sigma x \overset{\sigma(s')}{-\!\!\!\subset} y_2 \overset{\sigma(s)}{-\!\!\!\subset} \sigma x' .$$

As σ is already known to be receptive, we obtain

$$x \overset{e'}{-\!\!\!\subset} x_2 \overset{e}{-\!\!\!\subset} x'' \ \&\ \sigma x_2 = y_2 \ \&\ \sigma x'' = \sigma x' .$$

From the uniqueness part of (ii) we deduce $x'' = x'$. As σ is a total map of event structures, $e = s$ and $e' = s'$ ensuring $x \overset{s'}{-\!\!\!\subset}$, as required by Proposition 4.19(ii)−.
□

As its proof makes clear, condition (i) in Proposition 4.20 can be replaced by: for all $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$,

$$y \overset{+}{-\!\!\!\subset} \sigma x \implies \exists x' \in \mathcal{C}(S).\, x' -\!\!\!\subset x \ \&\ \sigma x' = y\,, \quad i.e.$$

$$\begin{array}{ccc} x' & \cdots -\!\!\!\subset \cdots & x \\ {\scriptstyle \sigma}\,\vdots & & \downarrow{\scriptstyle \sigma} \\ y & \overset{+}{-\!\!\!\subset} & \sigma x\,, \end{array}$$

where the relation $\overset{+}{-\!\!\!\subset}$ signifies the covering relation induced by an event of +ve polarity.

The proposition above generalises to the situation in which configurations may be infinite, but first a lemma extending receptivity to possibly infinite configurations.

Lemma 4.21. *Let $\sigma : S \to A$ be receptive and −-innocent. Then,*

$$\sigma x \overset{a}{-\!\!\!\subset} \ \&\ pol_A(a) = - \Rightarrow \exists! s \in S.\ x \overset{s}{-\!\!\!\subset} \ \&\ \sigma(s) = a\,,$$

for all $x \in \mathcal{C}^\infty(S)$, $a \in A$.

Proof. Suppose $\sigma x \stackrel{a}{\longrightarrow}\!\!\subset$ and $pol_A(a) = -$. Then there is $x_0 \in \mathcal{C}(S)$ with $x_0 \subseteq x$ and $\sigma x_0 \stackrel{a}{\longrightarrow}\!\!\subset$. By receptivity, there is a unique $s \in S$ such that $x_0 \stackrel{s}{\longrightarrow}\!\!\subset$ & $\sigma(s) = a$. In fact, $x \cup \{s\} \in \mathcal{C}^\infty(S)$. Suppose otherwise. Then there is $x_1 \in \mathcal{C}(S)$ with $x_0 \subseteq x_1 \subseteq x$ for which $x_1 \cup \{s\} \notin \mathcal{C}(S)$. But $\sigma x_1 \stackrel{a}{\longrightarrow}\!\!\subset$ so there is a unique $s_1 \in S$ such that $x_1 \stackrel{s_1}{\longrightarrow}\!\!\subset$ & $\sigma(s_1) = a$. Both $[s)$ and $[s_1)$ are included in x_1 so $s = s_1$ by Lemma 4.4—a contradiction. Now that $x \cup \{s\} \in \mathcal{C}^\infty(S)$ we have $x \stackrel{s}{\longrightarrow}\!\!\subset$ and $\sigma(s) = a$. Uniqueness of s follows by Lemma 4.4: if also $x \stackrel{s'}{\longrightarrow}\!\!\subset$ and $\sigma(s') = a$ then $[s) \uparrow [s')$. □

Corollary 4.22. *A strategy in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that*
(i) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}^\infty(A)$ there is a (necessarily unique) $x' \in \mathcal{C}^\infty(S)$ so that $x' \subseteq x$ & $\sigma x' = y$, i.e.

$$\begin{array}{ccc} x' & \cdots\subseteq\cdots & x \\ {\scriptstyle\sigma}\,\vdots & & \downarrow{\scriptstyle\sigma} \\ y & \subseteq^+ & \sigma x\,, \end{array}$$

and
(ii) whenever $\sigma x \subseteq^- y$ in $\mathcal{C}^\infty(A)$ there is a unique $x' \in \mathcal{C}^\infty(S)$ so that $x \subseteq x'$ & $\sigma x' = y$, i.e.

$$\begin{array}{ccc} x & \cdots\subseteq\cdots & x' \\ {\scriptstyle\sigma}\downarrow & & \vdots\,{\scriptstyle\sigma} \\ \sigma x & \subseteq^- & y\,. \end{array}$$

Proof. Let $\sigma : S \to A$ be a total map of event structures with polarity. It is claimed that σ is a strategy iff (i) and (ii). The *"If"* case is obvious by Proposition 4.20. *"Only if"*:
(i) Take $x' =_{\text{def}} \{s \in x \mid \sigma(s) \notin (\sigma x) \smallsetminus y\}$. Suppose $s' \rightarrowtriangle s$ in x. Then

$$\sigma(s') \in (\sigma x) \smallsetminus y \implies \sigma(s) \in (\sigma x) \smallsetminus y$$

by +-innocence. Hence its contrapositive, *viz.*

$$\sigma(s) \notin (\sigma x) \smallsetminus y \implies \sigma(s') \notin (\sigma x) \smallsetminus y\,,$$

so that $s \in x'$ implies $s' \in x'$. Thus, being down-closed and consistent, $x' \in \mathcal{C}^\infty(S)$ with $\sigma x' = y$ from the definition of x'.
(ii) Let $x' \supseteq x$ be a $\subseteq$-maximal $x' \in \mathcal{C}^\infty(S)$ for which $\sigma x' \subseteq y$—this exists by Zorn's lemma. Then, $\sigma x \subseteq^- \sigma x' \subseteq^- y$. Supposing $\sigma x' \subsetneq^- y$ there is $a \in A$ with $pol_A(a) = -$ such that $\sigma x' \stackrel{a}{\longrightarrow}\!\!\subset y_1 \subsetneq^- y$. But, by Lemma 4.21, there is $s \in S$ for which $x' \stackrel{s}{\longrightarrow}\!\!\subset$ and $\sigma(s) = a$, contradicting the $\subseteq$-maximality of x'. Hence $\sigma x' = y$. Uniqueness of x' follows as in the proof of Lemma 4.5. □

Via +-moves

A strategy is determined by its +-moves. More precisely, a strategy $\sigma : S \to A$ determines a monotone function $d : \mathcal{C}(S^+) \to \mathcal{C}(A)$ given by $d(x) = \sigma[x]_S$ for $x \in \mathcal{C}(S^+)$. The event structure S^+ is the projection of S to its purely +-ve moves. Intuitively, d specifies the position in the game at which Player moves occur. The function d determines the original strategy σ via the universal property described in the proposition below.

Proposition 4.23. *Let $\sigma : S \to A$ be a receptive −-innocent pre-strategy. Define $q : S \to S^+$ be the partial map of event structures with polarity mapping S to its projection S^+ comprising only the +ve events of S, so $q\,y = y^+$ for $y \in \mathcal{C}(S)$. Define the function $d : \mathcal{C}(S^+) \to \mathcal{C}(A)$ to act as $d(x) = \sigma[x]_S$ for $x \in \mathcal{C}(S^+)$. Then, $d(qy) \subseteq^- \sigma y$ for all $y \in \mathcal{C}(S)$, i.e.*

$$\begin{array}{ccc} S & \xrightarrow{\;q\;} & S^+ \\ {\scriptstyle \sigma}\big\downarrow & {}^{-}\!\supseteq \;\; \swarrow {\scriptstyle d} & \\ A\,. & & \end{array} \tag{1}$$

[The dotted line indicates that d is not a map of event structures.]
Suppose $f : U \to A$ is a total map and $g : U \to S^+$ a partial map of event structures with polarity such that $d(gy) \subseteq^- fy$ for all $y \in \mathcal{C}(U)$, i.e.

$$\begin{array}{ccc} U & \xrightarrow{\;g\;} & S^+ \\ {\scriptstyle f}\big\downarrow & {}^{-}\!\supseteq \;\; \swarrow {\scriptstyle d} & \\ A\,. & & \end{array} \tag{2}$$

Then, there is a unique total map of event structures with polarity $\theta : U \to S$ such that $f = \sigma\theta$ and $g = q\theta$,

$$\begin{array}{ccccc} & & \overset{g}{\frown\!\!\!\longrightarrow} & & \\ U & \overset{\theta}{\dashrightarrow} & S & \xrightarrow[\;q\;]{} & S^+ \\ & {\scriptstyle f}\searrow & {\scriptstyle \sigma}\big\downarrow & {}^{-}\!\supseteq \;\; \swarrow {\scriptstyle d} & \\ & & A\,. & & \end{array} \tag{3}$$

Proof. We first check (1). Letting $y \in \mathcal{C}(S)$,

$$d(q\,y) = d(y^+) = \sigma[y^+]_S \subseteq^- y\,.$$

Suppose (2). Define $p : \mathcal{C}(U) \to \mathcal{C}(S)$ by taking

$$p(z) =_{\mathrm{def}} [g\,z]_S\,.$$

Clearly p is monotonic and

$$\sigma p(z) = \sigma[g\,z]_S = d(gz) \subseteq^- f\,z$$

for all $z \in \mathcal{C}(U)$. By Lemma 4.16, there is a unique total map of event structures with polarity $\theta : U \to S$ such that

$$f = \sigma\theta \quad \text{and} \quad \forall z \in \mathcal{C}(U).\ p(z) \subseteq^- \theta\, z\,.$$

From the latter, $[g\, z]_S \subseteq^- \theta z$ from which $g\, z = (g\, z)^+ = (\theta\, z)^+$, so $g\, z = q\theta\, z$, for all $z \in \mathcal{C}(U)$. Hence we have the commuting diagram (3). Noting

$$\forall z \in \mathcal{C}(U).\ \ g\, z = (\theta\, z)^+ \iff [g\, z]_S \subseteq^- \theta z\,,$$

we see that θ is the unique map making (3) commute. □

It follows that a strategy σ is determined up to isomorphism by its 'position function' d specifying at what state of the game Player moves are made. The position functions d which arise from strategies have been characterized by Alex Katovsky and GW [13].

4.6 Rigid-image strategies

It can be useful to replace a strategy by its rigid image in its game. As is to be expected something can be lost in the process. Precisely what is related to notions of equivalence between strategies. For now suffice it to say, that while 'may' behaviour is preserved, 'must' behaviour need not be. What is gained is that we can replace the bicategory of games by a category; a rigid-image strategy can be identified with its rigid image, a substructure of the game so we have canonical representatives of isomorphism classes of rigid-image strategies. Rigid images are important for equivalences on strategies. For several important behavioural equivalences, a representative of an equivallence class of strategies can be found in their sharing a common rigid image and some additional structure (probability or stopping configurations, for instance).

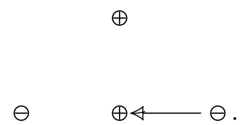
A strategy $\sigma : S \to A$ factors through its rigid image

$$S \xrightarrow{f} S_0 \xrightarrow{\sigma_0} A$$

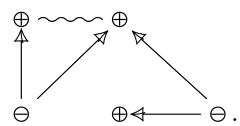
where f is rigid surjective and $\sigma_0 : S_0 \to A$ is itself a strategy. In a *rigid-image* strategy such as $\sigma_0 : S_0 \to A$ the rigid image S_0 is bounded to be a substructure of $aug(A)$. This provides us with a characterisation of rigid-image strategies. A rigid-image strategy in a game A is an innocent, receptive substructure S_0 of $aug(A)$ in the sense that there is a rigid inclusion $i_0 : S_0 \hookrightarrow aug(A)$ for which the composition $\epsilon_A \circ i_0$ is innocent and i_0 is receptive. In other words S_0 is a down-closed subset of $aug(A)$ which is closed under possible Opponent moves and comprises only innocent augmentations of A.

The following example shows that the composition of the rigid images of two strategies is not necessarily a rigid image, both for composition of strategies with and without hiding.

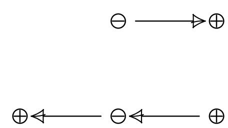
Example 4.24. Let B be the game



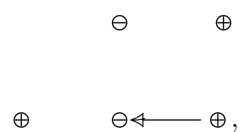
Let C be the game consisting of a single Player move $\oplus$. Let $\sigma : S \to B$ be the strategy sending S equal to



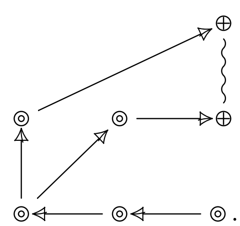
to B in the obvious way indicated by the layout. Let $\tau : T \to B^\perp \| C$ be the strategy sending T equal to



to $B^\perp \| C$, which we can draw as



in the obvious way. Their composition, before hiding, is given by $T \circledast S$:



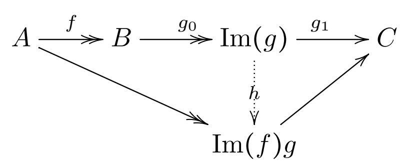
Both σ and τ are rigid-image strategies yet there composition both before and after hiding is not. Before hiding the two Player moves in $T \circledast S$ over the common move in C go to a common image. After hiding $T \odot S$ looks like

withn both moves going to the common sole move in C; while distinct they clearly go to a common event in the rigid image. □

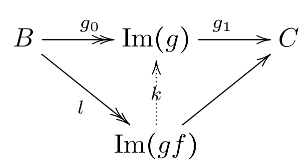
So the compositions, with and without hiding, $\tau_0 \odot \sigma_0$ and $\tau_0 \circledast \sigma_0$ of the rigid images of two strategies σ and τ is not necessarily a rigid-image strategies, we are forced to take the rigid image of the result. However once we do, the operation of forming the rigid image of a strategy respects composition, both with and without hiding: letting $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be strategies, $(\tau \odot \sigma)_0 = (\tau_0 \odot \sigma_0)_0$ and $(\tau \circledast \sigma)_0 = (\tau_0 \circledast \sigma_0)_0$, as we shall now show in the following.

Proposition 4.25. *Let $f : A \to B$ and $g : B \to C$ be maps of event structures. Assume that f is rigid and epi. Then, the rigid image of g equals the rigid image of $g \circ f$.*

Proof. Write the rigid image of g as $\mathrm{Im}(g)$ and the rigid image of gf as $\mathrm{Im}(gf)$. From the universal property associated with the rigid image of gf there is a unique (necessarily rigid epi) map $h : \mathrm{Im}(g) \to \mathrm{Im}(gf)$ such that



commutes. Write $l =_{\mathrm{def}} hg_0$. Then l is rigid epi being the composition of such. From the universal property associated with the rigid image of g there is a unique (necessarily rigid epi) map $k : \mathrm{Im}(g)f \to \mathrm{Im}(g)$ such that



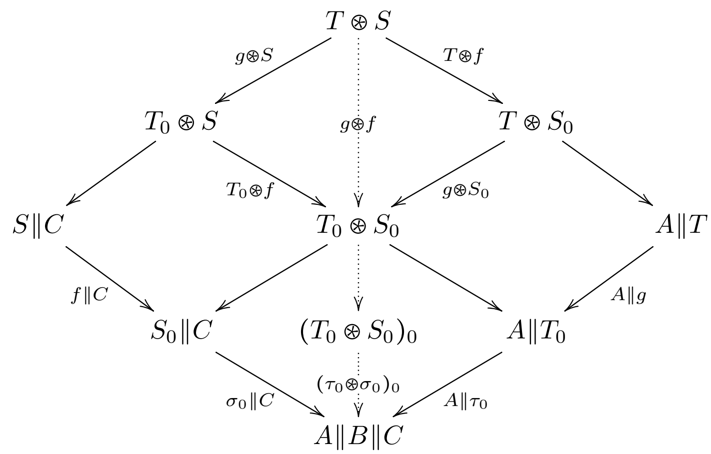
commutes. By uniqueness of the universal property of the rigid-image of g we obtain $kh = \mathrm{id}_{\mathrm{Im}(g)}$. By uniqueness of the universal property of the rigid-image of gf we obtain $hk = \mathrm{id}_{\mathrm{Im}(gf)}$. Hence the rigid images are isomorphic. Because they are chosen to be substructures of $aug(C)$ they are equal. $\square$

Corollary 4.26. *If two strategies are connected by a 2-cell which is rigid epi, then they share the same rigid image..*

Lemma 4.27. *Let $\sigma : S \xrightarrow{f} S_0 \xrightarrow{\sigma_0} A^\perp \| B$ and $\tau : T \xrightarrow{g} T_0 \xrightarrow{\tau_0} B^\perp \| C$ be the rigid image factorisations of strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$. Then,*

(i) $(\tau_0 \circledast \sigma_0)_0 = (\tau \circledast \sigma)_0$ *and (ii)* $(\tau_0 \odot \sigma_0)_0 = (\tau \odot \sigma)_0$.

Proof. (i) Consider the following compound pullback square in which all the squares are pullbacks—we are ignoring polarites.



In the diagram we have inserted the rigid-image factorisation of the map $T_0 \circledast S_0 \to A\|B\|C$. Notice that in the uppermost square all the maps are rigid epi being the pullbacks of such maps. Consequently $g \circledast f$ is rigid epi. Now applying Corollary 4.26 we deduce that the rigid image of the map $T \circledast S$ coincides with that of $T_0 \circledast S_0$ in $A\|B\|C$ and is therefore $(T_0 \circledast S_0)_0$. This ensures that

$$(\tau_0 \circledast \sigma_0)_0 = (\tau \circledast \sigma)_0\,.$$

(ii) We can also deduce

$$(\tau_0 \odot \sigma_0)_0 = (\tau \odot \sigma)_0\,.$$

Recall we obtain $\tau \odot \sigma$ as the defined part of the partial map

$$T \circledast S \xrightarrow{\tau \circledast \sigma} A\|B\|C \longrightarrow A\|C$$

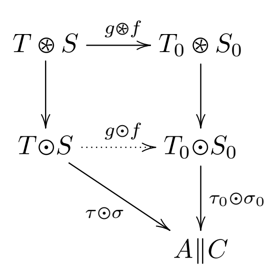
and similarly $\tau_0 \odot \sigma_0$ as the defined part of the partial map

$$T_0 \circledast S_0 \xrightarrow{\tau_0 \circledast \sigma_0} A\|B\|C \longrightarrow A\|C$$

—in both cases the map $A\|B\|C \to A\|C$ is that eliding B. From the diagram in (i) we see

$$\tau \circledast \sigma = (\tau_0 \circledast \sigma_0) \circ (g \circledast f)\,.$$

In the commuting diagram



we have filled in the total map $g{\odot}f$ given by the universal property of partial-total factorisation. As in (i) above $g\circledast f$ is rigid epi. It follows that the map $g{\odot}f$ is also rigid epi: the map $g{\odot}f$ preserves causal dependency because $g\circledast f$ does; it is epi because the composite map $T\circledast S \xrightarrow{g\circledast f} T_0\circledast S_0 \longrightarrow T_0{\odot}S_0$ is epi—the latter projection map is epi. Now by Corollary 4.26 we deduce that $\tau_0{\odot}\sigma_0$ and $\tau{\odot}\sigma$ share the same rigid image in $A\|C$. Consequently $(\tau_0{\odot}\sigma_0)_0 = (\tau{\odot}\sigma)_0$. □

Let $\mathbf{Games}_0$ be the order-enriched category of rigid-image strategies defined as follows. Its objects are games. Its maps are rigid-image strategies. Its 2-cells are rigid 2-cells between strategies which are necessarily rigid inclusions as they are between rigid images. Under composition composable strategies σ and τ are taken to $(\tau{\odot}\sigma)_0$. Recall that in a copycat strategy $\gamma_A : \mathbb{C}_A \to A^\perp\|A$ the underlying function of the map γ_A acts as the identity on events; this ensures that copycat strategies are rigid-image.

The operation of taking the rigid image of a strategy yields a functor from $\mathbf{Games}_r$, the bicategory of strategies with with rigid 2-cells, to $\mathbf{Games}_0$. From the results above composition is preserved. A rigid 2-cell $f:\sigma\Rightarrow\tau$ is sent to a rigid inclusion between their rigid images: by taking its image, any rigid 2-cell between strategies factors into a 2-cell which is a rigid epi, followed by 2-cells which is a rigid inclusion; strategies connected by a rigid epi share the same rigid image, while rigid inclusions are preserved in taking the rigid image.

Chapter 5

Deterministic strategies

This chapter concentrates on the important special case of *deterministic* concurrent strategies and their properties. They are shown to coincide with Melliès and Mimram's *receptive ingenuous strategies*.

5.1 Definition

We say an event structure with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid pol(s') = - \ \&\ \exists s \in X.\ s' \le s\}$. In other words, S is deterministic iff any finite set of moves is consistent when it causally depends only on a consistent set of opponent moves. Say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

Lemma 5.1. *An event structure with polarity S is deterministic iff*

$$\forall s, s' \in S, x \in \mathcal{C}(S).\ \ x \overset{s}{\longrightarrow\!\!\subset}\ \&\ x \overset{s'}{\longrightarrow\!\!\subset}\ \&\ pol(s) = + \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: Assume S is deterministic, $x \overset{s}{\longrightarrow\!\!\subset}$, $x \overset{s'}{\longrightarrow\!\!\subset}$ and $pol(s) = +$. Take $X =_{\text{def}} x \cup \{s, s'\}$. Then $Neg[X] \subseteq x \cup \{s\}$ so $Neg[X] \in \text{Con}_S$. As S is deterministic, $X \in \text{Con}_S$ and being down-closed $X = x \cup \{s, s'\} \in \mathcal{C}(S)$.
"If": Assume S satisfies the property stated above in the proposition. Let $X \subseteq_{\text{fin}} S$ with $Neg[X] \in \text{Con}_S$. Then the down-closure $[Neg[X]] \in \mathcal{C}(S)$. Clearly $[Neg[X]] \subseteq [X]$ where all events in $[X] \setminus [Neg[X]]$ are necessarily +ve. Suppose, to obtain a contradiction, that $X \notin \text{Con}_S$. Then there is a maximal $z \in \mathcal{C}(S)$ such that

$$[Neg[X]] \subseteq z \subseteq [X]$$

and some $e \in [X] \setminus z$, necessarily +ve, for which $[e) \subseteq z$. Take a covering chain

$$[e) \overset{s_1}{\longrightarrow\!\!\subset} z_1 \overset{s_2}{\longrightarrow\!\!\subset} \cdots \overset{s_k}{\longrightarrow\!\!\subset} z_k = z\,.$$

As $[e) \stackrel{e}{\longrightarrow\!\!\!\!\subset} [e]$ with e +ve, by repeated use of the property of the lemma—illustrated below—we obtain $z \stackrel{e}{\longrightarrow\!\!\!\!\subset} z'$ in $\mathcal{C}(S)$ with $[Neg[X]] \subseteq z' \subseteq [X]$, which contradicts the maximality of z.

$$
\begin{array}{ccccccccc}
[e] & \stackrel{s_1}{\longrightarrow\!\!\!\!\subset} & z'_1 & \stackrel{s_2}{\longrightarrow\!\!\!\!\subset} & \cdots & \stackrel{s_k}{\longrightarrow\!\!\!\!\subset} & z'_k & = & z' \\
{\scriptstyle e}\big\uparrow & & {\scriptstyle e}\big\uparrow & & \cdots & & {\scriptstyle e}\big\uparrow & & \\
[e) & \stackrel{s_1}{\longrightarrow\!\!\!\!\subset} & z_1 & \stackrel{s_2}{\longrightarrow\!\!\!\!\subset} & \cdots & \stackrel{s_k}{\longrightarrow\!\!\!\!\subset} & z_k & = & z
\end{array}
$$

□

So, above, an event structure with polarity can fail to be deterministic in two ways, either with $pol(s) = pol(s') = +$ or with $pol(s) = +$ & $pol(s') = -$. In general for an event structure with polarity A the copy-cat strategy can fail to be deterministic in either way, illustrated in the examples below.

Example 5.2. (i) Take A to consist of two +ve events and one −ve event, with any two but not all three events consistent. The construction of $\mathbb{C}\mathbb{C}_A$ is pictured:

$$
\begin{array}{cccc}
 & \ominus \rightarrowtriangle \oplus & \\
A^\perp & \ominus \rightarrowtriangle \oplus & A \\
 & \oplus \leftarrowtriangle \ominus &
\end{array}
$$

Here γ_A is not deterministic: take x to be the set of all three −ve events in $\mathbb{C}\mathbb{C}_A$ and s, s' to be the two +ve events in the A component.
(ii) Take A to consist of two events, one +ve and one −ve event, inconsistent with each other. The construction $\mathbb{C}\mathbb{C}_A$:

$$
\begin{array}{ccc}
A^\perp & \ominus \rightarrowtriangle \oplus & A \\
 & \oplus \leftarrowtriangle \ominus &
\end{array}
$$

To see $\mathbb{C}\mathbb{C}_A$ is not deterministic, take x to be the singleton set consisting *e.g.* of the −ve event on the left and s, s' to be the +ve and −ve events on the right.

5.2 The bicategory of deterministic strategies

We first characterize those games for which copy-cat is deterministic; they only allow immediate conflict between events of the same polarity; there can be no races between Player and Opponent moves.

Lemma 5.3. *Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A satisfies*

$$\forall x \in \mathcal{C}(A).\ x \stackrel{a}{\longrightarrow\!\!\!\!\subset}\ \&\ x \stackrel{a'}{\longrightarrow\!\!\!\!\subset}\ \&\ pol(a) = +\ \&\ pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(race-free)}$$

Proof. *"Only if":* Suppose $x \in \mathcal{C}(A)$ with $x \stackrel{a}{\longrightarrow\!\!\!\!\subset}$ and $x \stackrel{a'}{\longrightarrow\!\!\!\!\subset}$ where $pol(a) = +$ and $pol(a') = -$. Construct $y =_{\text{def}} \{(1, \overline{b}) \mid b \in x\} \cup \{(1, \overline{a})\} \cup \{(2, b) \mid b \in x\}$. Then

$y \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$ with $y \overset{(2,a)}{\relbar\!\!\!\subset}$ and $y \overset{(2,a')}{\relbar\!\!\!\subset}$, by Proposition 4.1(ii). Assuming $\mathbb{C}\mathbb{C}_A$ is deterministic, we obtain $y \cup \{(2,a),(2,a')\} \in \mathcal{C}(\mathbb{C}\mathbb{C}_A)$, so $y \cup \{(2,a),(2,a')\} \in \mathcal{C}(A^\perp \| A)$. This entails $x \cup \{a, a'\} \in \mathcal{C}(A)$, as required to show (**race-free**).
"If": Assume A satisfies (**race-free**). It suffices to show for $X \subseteq_{\mathrm{fin}} \mathbb{C}\mathbb{C}_A$, with X down-closed, that $Neg[X] \in \mathrm{Con}_{\mathbb{C}\mathbb{C}_A}$ implies $X \in \mathrm{Con}_{\mathbb{C}\mathbb{C}_A}$. Recall $Z \in \mathrm{Con}_{\mathbb{C}\mathbb{C}_A}$ iff $Z \in \mathrm{Con}_{A^\perp \| A}$.

Let $X \subseteq_{\mathrm{fin}} \mathbb{C}\mathbb{C}_A$ with X down-closed. Assume $Neg[X] \in \mathrm{Con}_{\mathbb{C}\mathbb{C}_A}$. Observe

(i) $\{c \mid c \in X \ \&\ pol(c) = -\} \subseteq Neg[X]$ and

(ii) $\{\overline{c} \mid c \in X \ \&\ pol(c) = +\} \subseteq Neg[X]$ as by Proposition 4.1, X being down-closed must contain $\overline{c}$ if it contains c with $pol(c) = +$.

Consider $X_2 =_{\mathrm{def}} \{a \mid (2,a) \in X\}$. Then X_2 is a finite down-closed subset of A. From (i),

$$X_2^- =_{\mathrm{def}} \{a \in X_2 \mid pol(a) = -\} \in \mathrm{Con}_A\,.$$

From (ii),

$$X_2^+ =_{\mathrm{def}} \{a \in X_2 \mid pol(a) = +\} \in \mathrm{Con}_A\,.$$

We show (**race-free**) implies $X_2 \in \mathrm{Con}_A$.

Define $z^- =_{\mathrm{def}} [X_2^-]$ and $z^+ =_{\mathrm{def}} [X_2^+]$. Being down-closures of consistent sets, $z^-, z^+ \in \mathcal{C}(A)$. We show $z^- \uparrow z^+$ in $\mathcal{C}(A)$. First note $z^- \cap z^+ \in \mathcal{C}(A)$. If $a \in z^- \smallsetminus z^- \cap z^+$ then $pol(a) = -$; otherwise, if $pol(a) = +$ then $a \in z^+$ a well as $a \in z^-$ making $a \in z^- \cap z^+$, a contradiction. Similarly, if $a \in z^+ \smallsetminus z^- \cap z^+$ then $pol(a) = +$. We can form covering chains

$$z^- \cap z^+ \overset{p_1}{\relbar\!\!\!\subset} x_1 \overset{p_2}{\relbar\!\!\!\subset} \cdots \overset{p_k}{\relbar\!\!\!\subset} x_k = z^- \quad \text{and} \quad z^- \cap z^+ \overset{n_1}{\relbar\!\!\!\subset} y_1 \overset{n_2}{\relbar\!\!\!\subset} \cdots \overset{n_l}{\relbar\!\!\!\subset} y_l = z^+$$

where each p_i is +ve and each n_j is −ve.

Consequently, by repeated use of (**race-free**), we obtain $x_k \cup y_l \in \mathcal{C}(A)$, *i.e.* $z^+ \cup z^- \in \mathcal{C}(A)$, as is illustrated below. But $X_2 \subseteq z^+ \cup z^-$, so $X_2 \in \mathrm{Con}_A$. A similar argument shows $X_1 =_{\mathrm{def}} \{a \in A^\perp \mid (1,a) \in X\} \in \mathrm{Con}_{A^\perp}$. It follows that $X \in \mathrm{Con}_{A^\perp \| A}$, so $X \in \mathrm{Con}_{\mathbb{C}\mathbb{C}_A}$ as required.

$$\begin{array}{ccccccccc}
y_l & \overset{p_1}{\relbar\!\!\!\subset} & x_1 \cup y_l & \overset{p_2}{\relbar\!\!\!\subset} & x_2 \cup y_l & \overset{p_3}{\relbar\!\!\!\subset} & \cdots & \overset{p_k}{\relbar\!\!\!\subset} & x_k \cup y_l \\
{\scriptstyle n_l}\Big| & & {\scriptstyle n_l}\Big| & & {\scriptstyle n_l}\Big| & & & & {\scriptstyle n_l}\Big| \\
\vdots & & \vdots & & \vdots & \cdots & \cdots & \cdots & \vdots \\
{\scriptstyle n_2}\Big| & & {\scriptstyle n_2}\Big| & & {\scriptstyle n_2}\Big| & & & & {\scriptstyle n_2}\Big| \\
y_1 & \overset{p_1}{\relbar\!\!\!\subset} & x_1 \cup y_1 & \overset{p_2}{\relbar\!\!\!\subset} & x_2 \cup y_1 & \overset{p_3}{\relbar\!\!\!\subset} & \cdots & \overset{p_k}{\relbar\!\!\!\subset} & x_k \cup y_1 \\
{\scriptstyle n_1}\Big| & & {\scriptstyle n_1}\Big| & & {\scriptstyle n_1}\Big| & & & & {\scriptstyle n_1}\Big| \\
z^- \cap z^+ & \overset{p_1}{\relbar\!\!\!\subset} & x_1 & \overset{p_2}{\relbar\!\!\!\subset} & x_2 & \overset{p_3}{\relbar\!\!\!\subset} & \cdots & \overset{p_k}{\relbar\!\!\!\subset} & x_k
\end{array}$$

□

Proposition 5.4. *Let A be an event structure with polarity. Then, A satisfies* (**race-free**) *iff*

$$\forall x, x_1, x_2 \in \mathcal{C}(A).\ x \subseteq^{+} x_1 \ \&\ x \subseteq^{-} x_2 \implies x_1 \cup x_2 \in \mathcal{C}(A)\,.$$

Proof. *"If"* is obvious. *"Only if"*: by repeated use of (**race-free**) as in the proof of Lemma 5.3. □

Via the next lemma, when games satisfy (**race-free**) we can simplify the condition for a strategy to be deterministic.

Lemma 5.5. *Let $\sigma : S \to A$ be a strategy. Suppose $x \overset{s}{\longrightarrow\!\!\subset} y$ & $x \overset{s'}{\longrightarrow\!\!\subset} y'$ & $pol_S(s) = -$. Then, $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A) \implies y \uparrow y'$ in $\mathcal{C}(S)$. A fortiori, if A satisfies* (**race-free**) *then so does S.*

Proof. Assume $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A)$, so $\sigma y' \overset{\sigma(s)}{\longrightarrow\!\!\subset} \sigma y \cup \sigma y'$ in $\mathcal{C}(A)$. As $\sigma(s)$ is −ve, by receptivity, there is a unique $s'' \in S$, necessarily −ve, such that $\sigma(s'') = \sigma(s)$ and $y' \overset{s''}{\longrightarrow\!\!\subset} x \cup \{s', s''\}$ in $\mathcal{C}(S)$. In particular, $x \cup \{s', s''\} \in \mathcal{C}(S)$. By −-innocence, we cannot have $s' \rightarrowtriangle s''$, so $x \cup \{s''\} \in \mathcal{C}(S)$. But now $x \overset{s}{\longrightarrow\!\!\subset}$ and $x \overset{s''}{\longrightarrow\!\!\subset}$ with $\sigma(s) = \sigma(s'')$ and both s, s'' −ve and hence $s'' = s$ by the uniqueness part of receptivity. We conclude that $x \cup \{s', s\} \in \mathcal{C}(S)$ so $y \uparrow y'$. □

Corollary 5.6. *Assume A satisfies* (**race-free**) *of Lemma 5.3. A strategy $\sigma : S \to A$ is deterministic iff it is* weakly-deterministic*, i.e. for all +ve events $s, s' \in S$ and configurations $x \in \mathcal{C}(S)$,*

$$x \overset{s}{\longrightarrow\!\!\subset} \ \&\ x \overset{s'}{\longrightarrow\!\!\subset} \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: clear. *"If"*: Let $x \overset{s}{\longrightarrow\!\!\subset}$ and $x \overset{s'}{\longrightarrow\!\!\subset}$ where $pol_S(s) = +$. For S to be deterministic we require $x \cup \{s, s'\} \in \mathcal{C}(S)$. The above assumption ensures this when $pol_S(s') = +$. Otherwise $pol_S(s') = -$ with $\sigma x \overset{\sigma(s)}{\longrightarrow\!\!\subset}$ and $\sigma x \overset{\sigma(s')}{\longrightarrow\!\!\subset}$. As A satisfies (**race-free**), $\sigma x \cup \sigma(s), \sigma(s') \in \mathcal{C}(A)$. Now by Lemma 5.5, $x \cup \{s, s'\} \in \mathcal{C}(S)$. □

Lemma 5.7. *The composition $\tau \odot \sigma$ of deterministic strategies σ and τ is deterministic.*

Proof. Let $\sigma : S \to A^{\perp} \| B$ and $\tau : T \to B^{\perp} \| C$ be deterministic strategies. The composition $T \odot S$ is constructed as $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S)) \downarrow V$, a synchronized composition of event structures S and T projected to visible events $e \in V$ where $top(e)$ has the form $(s, *)$ or $(*, t)$.

We first note a fact about the effect of internal, or "invisible," events not in V on configurations of $\mathcal{C}(T) \odot \mathcal{C}(S)$. If

$$z \overset{(s,t)}{\longrightarrow\!\!\subset} w \ \&\ z \overset{(s',t')}{\longrightarrow\!\!\subset} w' \ \&\ w \not\uparrow w' \tag{1}$$

within $\mathcal{C}(T)\odot\mathcal{C}(S)$, then either

$$\pi_1 z \overset{s}{\longrightarrow\!\!\subset} \pi_1 w \;\&\; \pi_1 z \overset{s'}{\longrightarrow\!\!\subset} \pi_1 w' \;\&\; \pi_1 w \not\uparrow \pi_1 w' \,, \tag{2}$$

within $\mathcal{C}(S)$, or

$$\pi_2 z \overset{t}{\longrightarrow\!\!\subset} \pi_2 w \;\&\; \pi_2 z \overset{t'}{\longrightarrow\!\!\subset} \pi_2 w' \;\&\; \pi_2 w \not\uparrow \pi_2 w' \,, \tag{3}$$

within $\mathcal{C}(T)$. Assume (1). If $t = t'$ then $\sigma(s) = \overline{\tau(t)} = \overline{\tau(t')} = \sigma(s')$ and we obtain (2) as σ is a map of event structures. Similarly if $s = s'$ then (3). Supposing $s \neq s'$ and $t \neq t'$ then if both (2) and (3) failed we could construct a configuration $z' =_{\mathrm{def}} z \cup \{(s,t),(s',t)\}$ of $\mathcal{C}(T)\odot\mathcal{C}(S)$, contradicting (1); it is easy to check that z' is a configuration of the product $\mathcal{C}(S)\times\mathcal{C}(T)$ and its events are clearly within the restriction used in defining the synchronized composition.

We now show the impossibility of (2) and (3), and so (1). Assume (2) (case (3) is similar). One of s or s' being +ve would contradict S being deterministic. Suppose otherwise, that both s and s' are −ve. Then, because σ is a strategy, by Lemma 5.5, we have

$$\sigma_2\pi_1 w \not\uparrow \sigma_2\pi_1 w'$$

in $\mathcal{C}(B)$. Also, then both t and t' are +ve ensuring $\pi_2 w \uparrow \pi_2 w'$ in $\mathcal{C}(T)$, as T is deterministic. This entails

$$\tau_1\pi_2 w \uparrow \tau_1\pi_2 w'$$

in $\mathcal{C}(B^\perp)$. But $\sigma_2\pi_1 w$ and $\tau_1\pi_2 w$, respectively $\sigma_2\pi_1 w'$ and $\tau_1\pi_2 w'$, are the same configurations on the common event structure underlying B and $B^\perp$, of which we have obtained contradictory statements of compatibility.

As (1) is impossible, it follows that

$$z \overset{(s,t)}{\longrightarrow\!\!\subset} w \;\&\; z \overset{(s',t')}{\longrightarrow\!\!\subset} w' \implies w \uparrow w' \tag{4}$$

within $\mathcal{C}(T)\odot\mathcal{C}(S)$.

Finally, we can show that $\tau\odot\sigma$ is deterministic. Suppose $x \overset{p}{\longrightarrow\!\!\subset} y$ and $x \overset{p'}{\longrightarrow\!\!\subset} y'$ in $\mathcal{C}(T\odot S)$ with $pol(p) = +$. Then,

$$\bigcup x \overset{e_1}{\longrightarrow\!\!\subset} z_1 \overset{e_2}{\longrightarrow\!\!\subset} \cdots \overset{e_k}{\longrightarrow\!\!\subset} z_k = \bigcup y \quad \text{and} \quad \bigcup x \overset{e'_1}{\longrightarrow\!\!\subset} z'_1 \overset{e'_2}{\longrightarrow\!\!\subset} \cdots \overset{e'_l}{\longrightarrow\!\!\subset} z'_l = \bigcup y'$$

in $\mathcal{C}(T)\odot\,\mathcal{C}(S)$, where $e_k = top(p)$ and $e'_l = top(p')$, and the events e_i and e'_j otherwise have the form $e_i = (s_i, t_i)$, when $1 \le i < k$, and $e'_j = (s'_j, t'_j)$, when $1 \le j < l$. By repeated use of (4) we obtain $z_{k-1} \uparrow z'_{l-1}$. (The argument is like that ending the proof of Lemma 5.3, though with the minor difference that now we may have $e_i = e'_j$.) We obtain $w =_{\mathrm{def}} z_{k-1} \cup z'_{l-1} \in \mathcal{C}(T)\odot\mathcal{C}(S)$ with $w \overset{e_k}{\longrightarrow\!\!\subset}$ and $w \overset{e'_l}{\longrightarrow\!\!\subset}$ and $pol(e_k) = +$.

Now, $w \cup \{e_k, e'_l\} \in \mathcal{C}(T)\odot\mathcal{C}(S)$ provided $w \cup \{e_k, e'_l\} \in \mathcal{C}(S)\times\mathcal{C}(T)$. Inspect the definition of configurations of the product of stable families in Section 3.3.1.

If e_k and e'_l have the form $(s,*)$ and $(s',*)$ respectively, then determinacy of S ensures that the projection $\pi_1 w \cup \{s,s'\} \in \mathcal{C}(S)$ whence $w \cup \{e_k, e'_l\}$ meets the conditions needed to be in $\mathcal{C}(S) \times \mathcal{C}(T)$. Similarly, $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$ if e_k and e'_l have the form $(*,t)$ and $(*,t')$. Otherwise one of e_k and e'_l has the form $(s,*)$ and the other $(*,t)$. In this case again an inspection of the definition of configurations of the product yields $w \cup \{e_k, e'_l\} \in \mathcal{C}(S) \times \mathcal{C}(T)$. Forming the set of primes of $w \cup \{e_k, e'_l\}$ in V we obtain $x \cup \{p, p'\} \in \mathcal{C}(T \odot S)$.

This establishes that $T \odot S$ is deterministic. □

We thus obtain a sub-bicategory **DGames** of **Games**; its objects satisfy (**race-free**) of Lemma 5.3 and its maps are deterministic strategies.

5.3 A category of deterministic strategies

In fact, **DGames** is equivalent to an order-enriched category via the following lemma. It says weakly-deterministic strategies in a game A are essentially certain subfamilies of configurations $\mathcal{C}(A)$, for which we give a characterization in the case of deterministic strategies. Recall, from Corollary 5.6, a weakly-deterministic strategy $\sigma : S \to A$ is a a strategy in which for all +ve events $s, s' \in S$ and configurations $x \in \mathcal{C}(S)$,

$$x \xrightarrow{s}\!\subset \ \& \ x \xrightarrow{s'}\!\subset \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Lemma 5.8. *Let $\sigma : S \to A$ be a weakly-deterministic strategy. Then,*

$$\sigma x \subseteq \sigma y \implies x \subseteq y$$

for all $x, y \in \mathcal{C}(S)$. In particular, a weakly-deterministic strategy σ is injective on configurations, i.e., $\sigma x = \sigma y$ implies $x = y$, for all $x, y \in \mathcal{C}(S)$ (so is mono as a map of event structures).

Proof. Let $\sigma : S \to A$ be a weakly-deterministic strategy. We show

$$x \supseteq z \longrightarrow\!\subset y \ \& \ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, by induction on $|x \setminus z|$.

Suppose $x \supseteq z \xrightarrow{e}\!\subset y$ and $\sigma y \subseteq \sigma x$. There are x_1 and event $e_1 \in S$ such that $z \xrightarrow{e_1}\!\subset x_1 \subseteq x$. If $\sigma(e_1) = \sigma(e)$ then e_1 and e have the same polarity; if −ve, $e_1 = e$ by receptivity; if +ve, $e_1 = e$ because σ is weakly-deterministic, using its local injectivity. Either way $y \subseteq x$. Suppose $\sigma(e_1) \neq \sigma(e)$. We show in all cases $y \cup \{e_1\} \subseteq x$, so $y \subseteq x$.

Case $pol(e_1) = pol(e) = +$: As σ is weakly-deterministic, e_1 and e are concurrent giving $x_1 \xrightarrow{e}\!\subset y \cup \{e_1\}$. By induction we obtain $y \cup \{e_1\} \subseteq x$.

Case $pol(e) = -$ or $pol(e_1) = -$: From Lemma 5.5, we deduce that e_1 and e are concurrent yielding $x_1 \xrightarrow{e}\!\subset y \cup \{e_1\}$, and by induction $y \cup \{e_1\} \subseteq x$.

Another, simpler induction on $|y \setminus z|$ now yields

$$x \supseteq z \subseteq y \ \& \ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, from which the result follows (taking z to be, for instance, $\varnothing$ or $x \cap y$). Injectivity of σ as a function on configurations is now obvious. □

A deterministic strategy $\sigma : S \to A$ determines, as the image of the configurations $\mathcal{C}(S)$, a subfamily $F =_{\text{def}} \sigma\mathcal{C}(S)$ of configurations of $\mathcal{C}(A)$, satisfying:

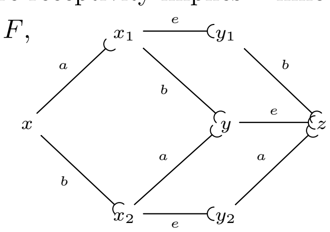
reachability: $\varnothing \in F$ and if $x \in F$ there is a covering chain $\varnothing \overset{a_1}{\longrightarrow\!\!\!\subset} x_1 \overset{a_2}{\longrightarrow\!\!\!\subset} \cdots \overset{a_k}{\longrightarrow\!\!\!\subset} x_k = x$ within F;

determinacy: If $x \overset{a}{\longrightarrow\!\!\!\subset}$ and $x \overset{a'}{\longrightarrow\!\!\!\subset}$ in F with $pol_A(a) = +$, then $x \cup \{a, a'\} \in F$;

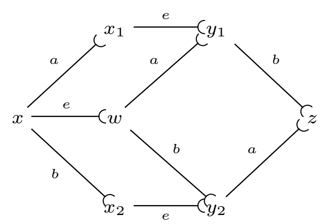
receptivity: If $x \in F$ and $x \overset{a}{\longrightarrow\!\!\!\subset}$ in $\mathcal{C}(A)$ and $pol_A(a) = -$, then $x \cup \{a\} \in F$;

+-innocence: If $x \overset{a}{\longrightarrow\!\!\!\subset} x_1 \overset{a'}{\longrightarrow\!\!\!\subset}$ & $pol_A(a) = +$ in F and $x \overset{a'}{\longrightarrow\!\!\!\subset}$ in $\mathcal{C}(A)$, then $x \overset{a'}{\longrightarrow\!\!\!\subset}$ in F (here receptivity implies −-innocence);

cube: In F,



implies



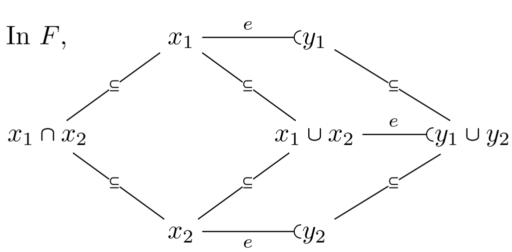
Theorem 5.9. *A subfamily $F \subseteq \mathcal{C}(A)$ satisfies the axioms above iff there is a deterministic strategy $\sigma : S \to A$ such that $F = \sigma\mathcal{C}(S)$, the image of $\mathcal{C}(S)$ under σ.*

Proof. (Sketch) It is routine to check that F, the image $\sigma\mathcal{C}(S)$ of a deterministic strategy, satisfies the axioms. Conversely, suppose a subfamily $F \subseteq \mathcal{C}(A)$ satisfies the axioms. We show F is a stable family. First note that from the axioms of determinacy and receptivity we can deduce:

$$\text{if } x \overset{a}{\longrightarrow\!\!\!\subset} \text{ and } x \overset{a'}{\longrightarrow\!\!\!\subset} \text{ in } F \text{ with } x \cup \{a, a'\} \in \mathcal{C}(A), \text{ then } x \cup \{a, a'\} \in F .$$

By repeated use of this property, using their reachability, if $x, y \in F$ and $x \uparrow y$ in $\mathcal{C}(A)$ then $x \cup y \in F$; the proof also yields a covering chain from x to $x \cup y$ and from y to $x \cup y$. (In particular, if $x \subseteq y$ in F, then there is a covering chain from x to y —a fact we shall use shortly.) Thus, if $x \uparrow y$ in F then $x \cup y \in F$. As also $\varnothing \in F$, we obtain Completeness, required of a stable family. Coincidence-freeness is a direct consequence of reachability. Repeated use of the cube axiom yields

Cube: In F,



implies

$$x_1 \cap x_2 \overset{e}{\longrightarrow\!\!\!\subset} .$$

We use *Cube* to show stability. Assume $v \uparrow w$ in F. Let $z \in F$ be maximal such that $z \subseteq v, w$. We show $z = v \cap w$. Suppose not. Then, forming covering chains in F,

$$z \overset{c_1}{\longrightarrow\!\!\!\subset} v_1 \overset{c_2}{\longrightarrow\!\!\!\subset} \cdots \overset{c_k}{\longrightarrow\!\!\!\subset} v_k = v \quad \text{and} \quad z \overset{d_1}{\longrightarrow\!\!\!\subset} w_1 \overset{d_2}{\longrightarrow\!\!\!\subset} \cdots \overset{d_l}{\longrightarrow\!\!\!\subset} w_l = w ,$$

there are c_i and d_j such that $c_i = d_j$, where we may assume c_i is the earliest event to be repeated as some d_j. Write $e =_{\text{def}} c_i = d_j$. Now, $v_{i-1} \cap w_{j-1} = z$. Also, being bounded above $v_{i-1} \cup w_{j-1} \in F$ and $v_i \cup w_j \in F$. We have an instance of *Cube*: take $x_1 = v_{i-1}$, $x_2 = w_{j-1}$, $y_1 = v_i$ and $y_2 = w_j$. Hence $z \stackrel{e}{\longrightarrow\!\!\subset}$ and $z \cup \{e\} \subseteq x, y$—contradicting the maximality of z. Therefore $z = v \cap w$, as required for stability.

Now we can form an event structure $S =_{\text{def}} \Pr(F)$. The inclusion $F \subseteq \mathcal{C}(A)$ induces a total map $\sigma : S \to A$ for which $F = \sigma\mathcal{C}(S)$. Note that $-$-innocence (*viz.* if $x \stackrel{a}{\longrightarrow\!\!\subset} x_1 \stackrel{a'}{\longrightarrow\!\!\subset}$ & $pol_A(a') = -$ in F and $x \stackrel{a'}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$, then $x \stackrel{a'}{\longrightarrow\!\!\subset}$ in F) is a direct consequence of receptivity. That S is deterministic follows from determinacy, that σ is a strategy from the axioms of receptivity and $+$-innocence. □

We can thus identify deterministic strategies from A to B with subfamilies of $\mathcal{C}(A^\perp \| B)$ satisfying the axioms above. Through this identification we obtain an order-enriched category of deterministic strategies (presented as subfamilies) equivalent to **DGames**; the order-enrichment is via the inclusion of subfamilies. As the proof of Theorem 5.9 above makes clear, in the characterization of those subfamilies F corresponding to deterministic families, the cube axiom can be replaced by

stability: if $v \uparrow w$ in F, then $v \cap w \in F$.

Chapter 6

Games people play

We briefly and incompletely examine special cases of nondeterministic concurrent games in the literature.

6.1 Categories for games

We remark that event structures with polarity appear to provide a rich environment in which to explore structural properties of games and strategies. There are adjunctions

$$\begin{array}{ccccccc}
\mathcal{PA}_r & \underset{\hookrightarrow}{\overset{\longleftarrow}{\top}} & \mathcal{PF}_r & \underset{\hookrightarrow}{\overset{\longleftarrow}{\top}} & \mathcal{PE}_r & \underset{\hookrightarrow}{\overset{\longleftarrow}{\top}} & \mathcal{PE}_t \\
\downarrow \vdash \uparrow & & \downarrow \vdash \uparrow & & & & \\
\mathcal{PA}_r^{\#} & \underset{\hookrightarrow}{\overset{\longleftarrow}{\top}} & \mathcal{PF}_r^{\#} & & & &
\end{array}$$

relating $\mathcal{PE}_t$, the category of event structures with polarity with total maps, to subcategories $\mathcal{PE}_r$, with rigid maps, $\mathcal{PF}_r$ of forest-like (or filiform) event structures with rigid maps, and $\mathcal{PA}_r$, its full subcategory where polarities alternate along a branch; in $\mathcal{PF}_r^{\#}$ and $\mathcal{PA}_r^{\#}$ distinct branches are inconsistent. We shall mainly be considering games in $\mathcal{PE}_t$. Lamarche games and those of sequential algorithms belong to $\mathcal{PA}_r$ [14]. Conway games inhabit $\mathcal{PF}_r^{\#}$, in fact a coreflective subcategory of $\mathcal{PE}_t$ as the inclusion is now full; Conway's 'sum' is obtained by applying the right adjoint to the $\|$-composition of Conway games in $\mathcal{PE}_t$. Further refinements are possible. The 'simple games' of [15, 16] belong to $\mathcal{PA}_r^{-\#}$, the coreflective subcategory of $\mathcal{PA}_r^{\#}$ comprising "polarized" games, starting with moves of Opponent. The 'tensor' of simple games is recovered by applying the right adjoint of $\mathcal{PA}_r^{-\#} \hookrightarrow \mathcal{PE}_t$ to their $\|$-composition in $\mathcal{PE}_t$. Generally, the right adjoints, got by composition, from $\mathcal{PE}_t$ to the other categories fail to conserve immediate causal dependency. Such facts led Melliès *et al.* to the insight that uses of pointers in game semantics can be an artifact of working with models of games which do not take account of the independence of moves [17, 12].

6.2 Related work—early results

6.2.1 Stable spans, profunctors and stable functions

The sub-bicategory of **Games** where the events of games are purely +ve is equivalent to the bicategory of stable spans [9]. In this case, strategies correspond to *stable spans*:

$$\begin{array}{ccc} & S & \\ {\scriptstyle \sigma_1}\swarrow & & \searrow{\scriptstyle \sigma_2} \\ A^{\perp} & & B \end{array} \quad\longleftrightarrow\quad \begin{array}{ccc} & S^{+} & \\ {\scriptstyle \sigma_1^{-}}\swarrow & & \searrow{\scriptstyle \sigma_2^{+}} \\ A & & B \end{array},$$

where S^{+} is the projection of S to its +ve events; σ_2^{+} is the restriction of σ_2 to S^{+}, necessarily a rigid map by innocence; σ_2^{-} is a *demand map* taking $x \in \mathcal{C}(S^{+})$ to $\sigma_1^{-}(x) = \sigma_1[x]$; here $[x]$ is the down-closure of x in S. Composition of stable spans coincides with composition of their associated profunctors—see [18, 19, 4]. If we further restrict strategies to be deterministic (and, strictly, event structures to be countable) we obtain a bicategory equivalent to Berry's *dI-domains and stable functions* [4].

6.2.2 Ingenuous strategies

Via Theorem 5.9, deterministic concurrent strategies coincide with the *receptive ingenuous strategies* of Melliès and Mimram [12].

6.2.3 Closure operators

In [20], deterministic strategies are presented as closure operators. A deterministic strategy $\sigma : S \to A$ determines a closure operator φ on possibly infinite configurations $\mathcal{C}^{\infty}(S)$: for $x \in \mathcal{C}^{\infty}(S)$,

$$\varphi(x) = x \cup \{ s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x \}\,.$$

Clearly φ preserves intersections of configurations and is continuous. The closure operator φ on $\mathcal{C}^{\infty}(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}^{\infty}(A)$. This in turn determines a closure operator $\varphi_p^{\top}$ on $\mathcal{C}^{\infty}(A)^{\top}$, where configurations are extended with a top $\top$, *cf.* [20]: take $y \in \mathcal{C}^{\infty}(A)^{\top}$ to the least, fixed point of φ_p above y, if such exists, and $\top$ otherwise.

6.2.4 Simple games

"Simple games" [15, 16] arise when we restrict **Games** to objects and deterministic strategies in $\mathcal{PA}_r^{-\#}$, described in Section 6.1.

6.2.5 Extensions

Games, such as those of [21, 22], allowing copying are being systematized through the use of monads and comonads [16], work now feasible on event structures with symmetry [9]. Nondeterministic strategies can potentially support probability as probabilistic or stochastic event structures [23] to become probabilistic or stochastic strategies.

Chapter 7

Strategies as profunctors

This chapter relates strategies to profunctors, a generalization of relations from sets to categories, and composition on strategies to composition of profunctors. Profunctors themselves provide a rich framework in which to generalize domain theory in a way that is arguably closer to that initiated by Dana Scott than game semantics [24, 25].

7.1 The Scott order in games

Let A be an event structure with polarity. The $\subseteq$-order on its finite configurations is obtained as compositions of two more fundamental orders $(\subseteq^+ \cup \subseteq^-)^+$. For $x, y \in \mathcal{C}^\infty(A)$,

$$x \subseteq^- y \text{ iff } x \subseteq y \ \&\ pol_A(y \setminus x) \subseteq \{-\}\,, \text{ and}$$
$$x \subseteq^+ y \text{ iff } x \subseteq y \ \&\ pol_A(y \setminus x) \subseteq \{+\}\,.$$

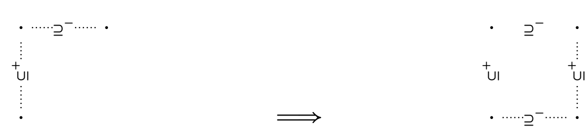
We use $\supseteq^-$ as the converse order to $\subseteq^-$. Define a new order, the *Scott* order, between configurations $x, y \in \mathcal{C}^\infty(A)$, by

$$x \sqsubseteq_A y \iff \exists z \in \mathcal{C}^\infty(A).\ x \supseteq^- z \subseteq^+ y\,.$$

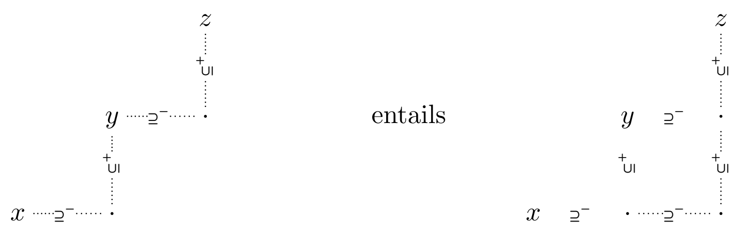
It is an easy exercise to show that when such a z exists it is necessarily $x \cap y$.

Proposition 7.1. *Let A be an event structure with polarity.*
(i) If $x \subseteq^+ w \supseteq^- y$ in $\mathcal{C}^\infty(A)$, then $x \supseteq^- x \cap y \subseteq^+ y$ in $\mathcal{C}^\infty(A)$.
(ii) $(\mathcal{C}^\infty(A), \sqsubseteq_A)$ is a partial order.

Proof. (i) Assume $x \subseteq^+ w \supseteq^- y$ in $\mathcal{C}^\infty(A)$. Clearly $x \supseteq x \cap y$. Suppose $a \in x$ and $pol_A(a) = +$. Then $a \in w$, and because only −ve events are lost from w in $w \supseteq^- y$ we obtain $a \in y$, so $a \in x \cap y$. It follows that $x \supseteq^- x \cap y$, as required. Similarly, $x \cap y \subseteq^+ y$. Summed up diagrammatically:



(ii) Clearly $\sqsubseteq$ is reflexive. Supposing $x \sqsubseteq y$, *i.e.* $x \supseteq^- z \subseteq^+ y$ in $\mathcal{C}^\infty(A)$ we see that the +ve events of x are included in y, and the −ve events of y are included in x. Hence if $x \sqsubseteq y$ and $y \sqsubseteq x$ in $\mathcal{C}^\infty(A)$ then x and y have the same +ve and −ve events and so are equal. Transitivity follows from (i):



□

Exercise 7.2. Show $(\mathcal{C}^\infty(A), \sqsubseteq_A)$ is a complete partial order: any ω-chain

$$x_0 \sqsubseteq_A x_1 \sqsubseteq_A \cdots \sqsubseteq_A x_n \sqsubseteq_A \cdots$$

has a least upper bound

$$\bigsqcup_{n\in\omega} x_n = (\bigcap_{n\in\omega} x_n)^- \cup (\bigcup_{n\in\omega} x_n)^+ .$$

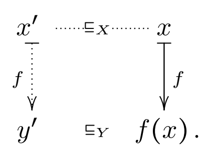
7.2 Strategies as presheaves

Let A be an event structure with polarity. A strategy in A determines a discrete fibration so a presheaf over the order of finite configurations $(\mathcal{C}(A), \sqsubseteq_A)$. In this chapter we only need discrete fibrations over partial orders.

Definition 7.3. A *discrete fibration* over a partial order $(Y, \sqsubseteq_Y)$ is a partial order $(X, \sqsubseteq_X)$ and an order-preserving function $f : X \to Y$ such that

$$\forall x \in X, y' \in Y.\ y' \sqsubseteq_Y f(x) \implies \exists! x' \sqsubseteq_X x.\ f(x') = y' ,$$

as illustrated



Proposition 7.4. *Let $\sigma : S \to A$ be a pre-strategy in game A. The map σ“ taking a finite configuration $x \in \mathcal{C}(S)$ to $\sigma x \in \mathcal{C}(A)$ is a discrete fibration from $(\mathcal{C}(S), \sqsubseteq_S)$ to $(\mathcal{C}(A), \sqsubseteq_A)$ iff σ is a strategy.*

Proof. A direct corollary of Proposition 4.20. □

As discrete fibrations correspond to presheaves, an alternative reading of Proposition 7.4 is that a pre-strategy $\sigma : S \to A$ is a strategy iff σ“ determines a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)$—the presheaf being the functor $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \to \mathbf{Set}$ which sends y to the fibre $\{x \in \mathcal{C}(S) \mid \sigma x = y\}$ and instances $y' \sqsubseteq_A y$ to functions from the fibre over y to the fibre over y' determined by the fibration.

7.3 Strategies as profunctors

A strategy

$$\sigma : A \twoheadrightarrow\!\!\!\!\!\!\!\!+\;\; B$$

determines a discrete fibration over

$$(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) \,.$$

But

$$\begin{aligned}(\mathcal{C}(A^\perp \| B), \sqsubseteq_{A^\perp \| B}) &\cong (\mathcal{C}(A^\perp), \sqsubseteq_{A^\perp}) \times (\mathcal{C}(B), \sqsubseteq_B) &(1)\\ &\cong (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)\,. &(2)\end{aligned}$$

The first step (1) relies on the correspondence

$$x \leftrightarrow (\{a \mid (1,a) \in x\},\ \{b \mid (2,b) \in x\})$$

between a configuration of $A^\perp \| B$ and a pair, with left component a configuration of $A^\perp$ and right component a configuration of B. In the last step (2) we are using the correspondence between configurations of $A^\perp$ and A induced by the correspondence $a \leftrightarrow \overline{a}$ between their events: a configuration x of $A^\perp$ corresponds to a configuration $\overline{x} =_{\mathrm{def}} \{\overline{a} \mid a \in x\}$ of A. Because $A^\perp$ reverses the roles of $+$ and $-$ in A, the order $x \sqsubseteq_{A^\perp} y$ in $\mathcal{C}(A^\perp)$,

$$\begin{array}{ccc} & & y \\ & \sqsubseteq & \cup\!\!\shortmid^{+} \\ x & \supseteq^{-} & x \cap y\,, \end{array}$$

corresponds to the order $\overline{y} \sqsubseteq_A \overline{x}$, *i.e.* $\overline{x} \sqsubseteq_A^{\mathrm{op}} y$, in $\mathcal{C}(A)$,

$$\begin{array}{ccc} & & \overline{y} \\ & \supseteq & \cup\!\!\shortmid^{-} \\ \overline{x} & \supseteq^{+} & \overline{x} \cap \overline{y}\,. \end{array}$$

It follows that a strategy

$$\sigma : S \to A^\perp \| B$$

determines a discrete fibration

$$\sigma``: (\mathcal{C}(S), \sqsubseteq_S) \to (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$$

where

$$\sigma``(x) = (\overline{\sigma_1 x},\ \sigma_2 x)\,,$$

for $x \in \mathcal{C}(S)$. The fibration can be vewed as a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$—it assigns the set

$$\{x \in \mathcal{C}(S) \mid \overline{\sigma_1 x} = v \;\&\; \sigma_2 x = z\}$$

to the pair $(v,z) \in \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(B)$. One way to define a *profunctor* from $(\mathcal{C}(A), \sqsubseteq_A)$ to $(\mathcal{C}(B), \sqsubseteq_B)$ is as a discrete fibration over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$. Hence the strategy σ determines a profunctor[1]

$$\sigma\text{``} : (\mathcal{C}(A), \sqsubseteq_A) \twoheadrightarrow (\mathcal{C}(B), \sqsubseteq_B)\,.$$

7.4 Composition of strategies and profunctors

The operation from strategies σ to profunctors σ“ preserves identities:

Lemma 7.5. *Let A be an event structure with polarity. For $x \in \mathcal{C}^\infty(A^\perp \| A)$,*

$$x \in \mathcal{C}^\infty(\mathbb{C}_A) \ \ \textit{iff} \ \ x_2 \sqsubseteq_A \overline{x}_1\,,$$

where $x_1 = \{a \in A^\perp \mid (1,a) \in x\}$ and $x_2 = \{a \in A \mid (2,a) \in x\}$.

Proof. Let $x \in \mathcal{C}^\infty(A^\perp \| A)$. From the dependency within copy-cat of the +ve events $a \in A$ on corresponding −ve events $\overline{a} \in A^\perp$, and *vice versa*, as expressed in Proposition 4.1, we deduce: $x \in \mathcal{C}^\infty(CC_A)$ iff

$$(i)\ \ \overline{x}_1^+ \supseteq x_2^+ \qquad \text{and} \qquad (ii)\ \ \overline{x}_1^- \subseteq x_2^-\,,$$

where $z^+ = \{a \in z \mid pol_A(a) = +\}$ and $z^- = \{a \in z \mid pol_A(a) = -\}$ for $z \in \mathcal{C}^\infty(A)$.

It remains to argue that (i) and (ii) iff $x_2 \supseteq^- \overline{x}_1 \cap x_2 \subseteq^+ \overline{x}_1$. *“Only if”:* Assume (i) and (ii). Clearly, $\overline{x}_1 \cap x_2 \subseteq \overline{x}_1$. Suppose $a \in \overline{x}_1$ with $pol_A(a) = -$. By (ii), $a \in x_2$. Consequently, $x_1 \cap x_2 \subseteq^+ \overline{x}_1$. Similarly, (i) entails $x_2 \supseteq^- \overline{x}_1 \cap x_2$. *“If”:* To show (i), let $a \in x_2^+$. Then as $x_2 \supseteq^- \overline{x}_1 \cap x_2$ ensures only −ve events are lost in moving from x_2 to $\overline{x}_1 \cap x_2$, we see $a \in \overline{x}_1 \cap x_2$, so $a \in \overline{x}_1^+$. The proof of (ii) is similar. □

Corollary 7.6. *Let A be an event structure with polarity. The profunctor γ_A“ of the copy-cat strategy γ_A is an identity profunctor on $(\mathcal{C}(A), \sqsubseteq_A)$.*

Proof. The profunctor $\gamma_A\text{``} : (\mathcal{C}(A), \sqsubseteq_A) \twoheadrightarrow (\mathcal{C}(A), \sqsubseteq_A)$ sends $x \in \mathcal{C}(\mathbb{C}_A)$ to $(\overline{x}_1, x_2) \in (\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(A), \sqsubseteq_A)$ precisely when $x_2 \sqsubseteq_A \overline{x}_1$. It is thus an identity on $(\mathcal{C}(A), \sqsubseteq_A)$. □

We now relate the composition of strategies to the standard composition of profunctors. Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be strategies, so $\sigma : A \twoheadrightarrow B$ and $\tau : B \twoheadrightarrow C$. Abbreviating, for instance, $(\mathcal{C}(A), \sqsubseteq_A)$ to $\mathcal{C}(A)$, strategies σ and τ give rise to profunctors $\sigma\text{``} : \mathcal{C}(A) \twoheadrightarrow \mathcal{C}(B)$ and $\tau\text{``} : \mathcal{C}(B) \twoheadrightarrow \mathcal{C}(C)$. Their composition is the profunctor $\tau\text{``} \circ \sigma\text{``} : \mathcal{C}(A) \twoheadrightarrow \mathcal{C}(C)$ built as a discrete

[1] Most often a profunctor from $(\mathcal{C}(A), \sqsubseteq_A)$ to $(\mathcal{C}(B), \sqsubseteq_B)$ is defined as a functor $(\mathcal{C}(A), \sqsubseteq_A) \times (\mathcal{C}(B), \sqsubseteq_B)^{\mathrm{op}} \to \mathbf{Set}$, *i.e.*, as a presheaf over $(\mathcal{C}(A), \sqsubseteq_A)^{\mathrm{op}} \times (\mathcal{C}(B), \sqsubseteq_B)$, and as such corresponds to a discrete fibration.

fibration from the discrete fibrations $\sigma``: \mathcal{C}(S) \to \mathcal{C}(A)^{\mathrm{op}} \times \mathcal{C}(B)$ and $\tau``: \mathcal{C}(T) \to \mathcal{C}(B)^{\mathrm{op}} \times \mathcal{C}(C)$.

First, we define the set of *matching pairs*,

$$M =_{\mathrm{def}} \{(x,y) \in \mathcal{C}(S) \times \mathcal{C}(T) \mid \sigma_2 x = \overline{\tau_1 y}\},$$

on which we define $\sim$ as the least equivalence relation for which

$$\begin{aligned}(x,y) \sim (x',y') \;\text{ if } \quad & x \sqsubseteq_S x' \;\&\; y' \sqsubseteq_T y \;\& \\ & \sigma_1 x = \sigma_1 x' \;\&\; \tau_2 y' = \tau_2 y\,.\end{aligned}$$

Define an order on equivalence classes $M/\sim$ by:

$$\begin{aligned} m \sqsubseteq m' \;\text{ iff }\; & m = \{(x,y)\}_\sim \;\&\; m' = \{(x',y')\}_\sim \;\& \\ & x \sqsubseteq_S x' \;\&\; y \sqsubseteq_T y' \;\& \\ & \sigma_2 x = \sigma_2 x' \;\&\; \tau_1 y = \tau_1 y'\,,\end{aligned}$$

for some matching pairs $(x,y),(x',y')$—so then $\sigma_2 x = \sigma_2 x' = \overline{\tau_1 y} = \overline{\tau_1 y'}$.

Exercise 7.7. *Show that $\sqsubseteq$ above is transitive, so a partial order on $M/\sim$. Verify that $\tau`` \circ \sigma``$ is a discrete fibration.* □

Lemma 7.8. *On matching pairs, define*

$$(x,y) \sim_1 (x',y') \;\; \textit{iff} \;\; \exists s \in S, t \in T.\; x \overset{s}{\longrightarrow\!\!\!\subset} x' \;\&\; y \overset{t}{\longrightarrow\!\!\!\subset} y' \;\&\; \sigma_2(s) = \overline{\tau_1(t)}\,.$$

The smallest equivalence relation including $\sim_1$ coincides with the relation $\sim$.

Proof. From their definitions, $\sim_1$ is included in $\sim$. To prove the converse, it suffices to show that matching pairs (x,y), (x',y') satisfying

$$\begin{aligned} & x \sqsubseteq_S x' \;\&\; y' \sqsubseteq_T y \;\& \\ & \sigma_1 x = \sigma_1 x' \;\&\; \tau_2 y' = \tau_2 y\,,\end{aligned}$$

—the clause used in the definition $\sim$ —are in the equivalence relation generated by $\sim_1$. Take a covering chain

$$x \multimap\!\!\!\subset_S x_1 \multimap\!\!\!\subset_S \cdots x_m \multimap\!\!\!\subset_S x'$$

in $(\mathcal{C}(S), \sqsubseteq_S)$. Here $\multimap\!\!\!\subset_S$ is the covering relation w.r.t. the order $\sqsubseteq_s$, so $x \multimap\!\!\!\subset_S x_1$ means x, x_1 are distinct and $x \sqsubseteq_S x_1$ with nothing strictly in between. Via the map σ we obtain

$$\sigma_2 x \multimap\!\!\!\subset_B \sigma_2 x_1 \multimap\!\!\!\subset_B \cdots \sigma_2 x_m \multimap\!\!\!\subset_B \sigma_2 x'$$

in $\mathcal{C}(B)$ where $\sigma_2 x = \overline{\tau_1 y}$ and $\sigma_2 x' = \overline{\tau_1 y'}$. Via the discrete fibration $\tau``$ we obtain a covering chain in the reverse direction,

$$y \supset\!\!\!\multimap_T y_1 \supset\!\!\!\multimap_T \cdots y_m \supset\!\!\!\multimap_T y'$$

in $(\mathcal{C}(T), \sqsubseteq_T)$, where each each (x_i, y_i), for $1 \le i \le m$, is a matching pair. Moreover, $(x_i, y_i) \sim_1 (x_{i+1}, y_{i+1})$ at each i with $1 \le i \le m$. Hence (x,y) and (x',y') are in the equivalence relation generated by $\sim_1$. □

The profunctor composition $\tau``\circ\sigma``$ is given as the discrete fibration

$$\tau``\circ\sigma`` : M/\sim \;\to\; \mathcal{C}(A)^{\mathrm{op}}\times\mathcal{C}(C)$$

acting so

$$\{(x,y)\}_\sim \mapsto (\overline{\sigma_1 x}, \tau_2 y)\,.$$

It is *not* the case that $(\tau\odot\sigma)``$ and $\tau``\circ\sigma``$ coincide up to isomorphism. The profunctor composition $\tau``\circ\sigma``$ will generally contain extra equivalence classes $\{(x,y)\}_\sim$ for matching pairs (x,y) which are "unreachable." Although $\sigma_2 x = z = \overline{\tau_1 y}$ automatically for a matching pair (x,y), the configurations x and y may impose incompatible causal dependencies on their interface z so never be realized as a configuration in the synchronized composition $\mathcal{C}(T)\odot\mathcal{C}(S)$, used in building the composition of strategies $\tau\odot\sigma$.

Example 7.9. Let A and C both be the empty event structure $\varnothing$. Let B be the event structure consisting of the two concurrent events b_1, assumed −ve, and b_2, assumed +ve in B . Let the strategy $\sigma : \varnothing \rightarrow\!\!\!\!\!\!+\!\!\!\!\rightarrow B$ comprise the event structure $s_1 \rightarrowtriangle s_2$ with s_1 −ve and s_2 +ve, $\sigma(s_1) = b_1$ and $\sigma(s_2) = b_2$. In $B^\perp$ the polarities are reversed so there is a strategy $\tau : B \rightarrow\!\!\!\!\!\!+\!\!\!\!\rightarrow \varnothing$ comprising the event structure $t_2 \rightarrowtriangle t_1$ with t_2 −ve and t_1 +ve yet with $\tau(t_1) = \bar{b}_1$ and $\tau(t_2) = \bar{b}_2$. The equivalence class $\{(x,y)\}_\sim$, where $x = \{s_1, s_2\}$ and $y = \{t_1, t_2\}$, would be present in the profunctor composition $\tau``\circ\sigma``$ whereas $\tau\odot\sigma$ would be the empty strategy and accordingly the profunctor $(\tau\odot\sigma)``$ only has a single element, $\varnothing$.

Definition 7.10. For (x,y) a matching pair, define

$$\begin{aligned} x\cdot y =_{\mathrm{def}} &\{(s,*) \mid s\in x \;\&\; \sigma_1(s) \text{ is defined}\}\cup\\ &\{(*,t) \mid t\in y \;\&\; \tau_2(t) \text{ is defined}\}\cup\\ &\{(s,t) \mid s\in x \;\&\; t\in y \;\&\; \sigma_2(s) = \overline{\tau_1(t)}\} \end{aligned}$$

Say (x,y) is *reachable* if $x\cdot y \in \mathcal{C}(T)\odot\mathcal{C}(S)$, and *unreachable* otherwise.

For $z\in\mathcal{C}(T)\odot\mathcal{C}(S)$ say a *visible prime* of z is a prime of the form $[(s,*)]_z$, for $(s,*)\in z$, or $[(*,t)]_z$, for $(*,t)\in z$.

Lemma 7.11. *(i) If (x,y) is a reachable matching pair and $(x,y)\sim(x',y')$, then (x',y') is a reachable matching pair;*
(ii) For reachable matching pairs (x,y), (x',y'), $(x,y)\sim(x',y')$ iff $x\cdot y$ and $x'\cdot y'$ have the same visible primes.

Proof. We use the characterization of $\sim$ in terms of the single-step relation $\sim_1$ given in Lemma 7.8.
(i) Suppose $(x,y)\sim_1(x',y')$ or $(x',y')\sim_1(x,y)$. By inspection of the construction of the product of stable families in Section 3.3.1, if $x\cdot y\in\mathcal{C}(T)\odot\mathcal{C}(S)$ then $x'\cdot y'\in\mathcal{C}(T)\odot\mathcal{C}(S)$.

(ii) *"If":* Suppose $x \cdot y$ and $x' \cdot y'$ have the same visible primes, forming the set Q. Then $z =_{\text{def}} \bigcup Q \in \mathcal{C}(T) \odot \mathcal{C}(S)$, being the union of a compatible set of configurations in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Moreover, $z \subseteq x \cdot y, x' \cdot y'$. Take a covering chain

$$z \overset{e_1}{\longrightarrow\!\!\!\!\subset} \cdots z_i \overset{e_i}{\longrightarrow\!\!\!\!\subset} z_{i+1} \overset{e_n}{\longrightarrow\!\!\!\!\subset} x \cdot y$$

in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Each $(\pi_1 z_i, \pi_2 z_i)$ is a matching pair, from the definition of $\mathcal{C}(T) \odot \mathcal{C}(S)$. Necessarily, $e_i = (s_i, t_i)$ for some $s_i \in S$, $t_i \in T$, with $\sigma_2(s_i) = \overline{\tau_1(t_i)}$, again by the definition of $\mathcal{C}(T) \odot \mathcal{C}(S)$. Thus

$$(\pi_1 z_i, \pi_2 z_i) \sim_1 (\pi_1 z_{i+1}, \pi_2 z_{i+1}) \,.$$

Hence $(\pi_1 z, \pi_2 z) \sim (x, y)$, and similarly $(\pi_1 z, \pi_2 z) \sim (x', y')$, so $(x, y) \sim (x', y')$.

"Only if": It suffices to observe that if $(x, y) \sim_1 (x', y')$, then $x \cdot y$ and $x' \cdot y'$ have the same visible primes. But if $(x, y) \sim_1 (x', y')$ then $x \cdot y \overset{(s,t)}{\longrightarrow\!\!\!\!\subset} x' \cdot y'$, for some $s \in S, t \in T$, and no visible prime in $x' \cdot y'$ contains (s, t). □

Lemma 7.12. *Let $\sigma : A \mathrel{+\!\!\!\!\twoheadrightarrow} B$ and $\tau : B \mathrel{+\!\!\!\!\twoheadrightarrow} C$ be strategies. Defining*

$$\varphi_{\sigma,\tau} : \mathcal{C}(T \odot S) \to M/\sim \quad by \quad \varphi_{\sigma,\tau}(z) = \{(\Pi_1 z, \Pi_2 z)\}_\sim \,,$$

where $\Pi_1 z = \pi_1 \bigcup z$ and $\Pi_2 z = \pi_2 \bigcup z$, yields an injective, order-preserving function from $(\mathcal{C}(T \odot S), \sqsubseteq_{T \odot S})$ to $(M/\sim, \sqsubseteq)$—its range is precisely the equivalence classes $\{(x, y)\}_\sim$ for reachable matching pairs (x, y). The diagram

$$\begin{array}{ccc}
(\mathcal{C}(T \odot S), \sqsubseteq_{T \odot S}) & \xrightarrow{\varphi_{\sigma,\tau}} & (M/\sim, \sqsubseteq) \\
{\scriptstyle (\tau \odot \sigma)``} \downarrow & \swarrow {\scriptstyle \tau``\circ\sigma``} & \\
(\mathcal{C}(A), \sqsubseteq_A)^{\text{op}} \times (\mathcal{C}(C), \sqsubseteq_C) & &
\end{array}$$

commutes.

Proof. For $z \in \mathcal{C}(T \odot S)$, we obtain that $\varphi_{\sigma,\tau}(z) = (\Pi_1 z, \Pi_2 z) = (\pi_1 \bigcup z, \pi_2 \bigcup z)$ is a matching pair, from the definition of $\mathcal{C}(T) \odot \mathcal{C}(S)$; it is clearly reachable as $\pi_1 \bigcup z \cdot \pi_2 \bigcup z = \bigcup z \in \mathcal{C}(T) \odot \mathcal{C}(S)$. For any reachable matching pair (x, y) let z be the set of visible primes of $x \cdot y$. Then, $z \in \mathcal{C}(T \odot S)$ and, by Lemma 7.11(ii), $(\Pi_1 z, \Pi_2 z) \sim (x, y)$ so $\varphi_{\sigma,\tau}(z) = \{(x, y)\}_\sim$. Injectivity of $\varphi_{\sigma,\tau}$ follows directly from Lemma 7.11(ii).

To show that $\varphi_{\sigma,\tau}$ is order-preserving it suffices to show if $z {-\!\!\!\subset} z'$ in $(\mathcal{C}(T \odot S), \sqsubseteq)$ then $\varphi_{\sigma,\tau}(z) \sqsubseteq \varphi_{\sigma,\tau}(z')$ in $(M/\sim, \sqsubseteq)$. (The covering relation ${-\!\!\!\subset}$ is the same as that used in the proof of Lemma 7.8.) If $z {-\!\!\!\subset} z'$ then either $z \overset{p}{\longrightarrow\!\!\!\!\subset} z'$, with p +ve, or $z' \overset{p}{\longrightarrow\!\!\!\!\subset} z$, with p −ve, for p a visible prime of $\mathcal{C}(T) \odot \mathcal{C}(S)$, *i.e.* with $top(p)$ of the form $(s, *)$ or $(*, t)$. We concentrate on the case where p is +ve (the proof when p is −ve is similar). In the case where p is +ve,

$$\Pi_1 z \cdot \Pi_2 z = \bigcup z \subseteq \bigcup z' = \Pi_1 z' \cdot \Pi_2 z'$$

in $\mathcal{C}(T)\odot\mathcal{C}(S)$ and there is a covering chain

$$\bigcup z = w_0 \xrightarrow{(s_1,t_1)}\!\!\!\!\subset w_1 \cdots \xrightarrow{(s_n,t_n)}\!\!\!\!\subset w_n \xrightarrow{top(p)}\!\!\!\!\subset \bigcup z'$$

in $\mathcal{C}(T)\odot\mathcal{C}(S)$. Each w_i, for $0 \le i \le m$, is associated with a reachable matching pair $(\pi_1 w_i, \pi_2 w_i)$ where $\pi_1 w_i \cdot \pi_2 w_i = w_i$. Also $(\pi_1 w_i, \pi_2 w_i) \sim_1 (\pi_1 w_{i+1}, \pi_2 w_{i+1})$, for $0 \le i < m$. Hence $(\Pi_1 z, \Pi_2 z) \sim (\pi_1 w_n, \pi_2 w_n)$, by Lemma 7.8(ii). If $top(p) = (s, *)$ then $\pi_1 w_n \xrightarrow{s}\!\!\!\!\subset \Pi_1 z'$, with s +ve, and $\pi_2 w_n = \Pi_2 z'$. If $top(p) = (*, t)$ then $\pi_1 w_n = \Pi_1 z'$ and $\pi_2 w_n \xrightarrow{t}\!\!\!\!\subset \Pi_2 z'$, with t +ve. In either case $\pi_1 w_n \sqsubseteq_S \Pi_1 z'$ and $\pi_2 w_n \sqsubseteq_T \Pi_2 z'$ with $\sigma_2 \pi_1 w_n = \sigma_2 \Pi_1 z'$ and $\tau_1 \pi_2 w_n = \tau_1 \Pi_2 z'$. Hence, from the definition of $\sqsubseteq$ on $M/\sim$,

$$\varphi_{\sigma,\tau}(z) = \{(\Pi_1 z, \Pi_2 z)\}_\sim = \{(\pi_1 w_n, \pi_2 w_n)\}_\sim \sqsubseteq \{(\Pi_1 z', \Pi_2 z')\}_\sim = \varphi_{\sigma,\tau}(z')\,.$$

It remains to show commutativity of the diagram. Let $z \in \mathcal{C}(T\odot S)$. Then,

$$(\tau\text{``} \circ \sigma\text{``})(\varphi_{\sigma,\tau}(z)) = (\tau\text{``} \circ \sigma\text{``})(\{(\Pi_1 z, \Pi_2 z)\}_\sim) = (\overline{\sigma_1 \Pi_1 z},\ \tau_2 \Pi_2 z) = (\tau\odot\sigma)\text{``}(z)\,,$$

via the definition of $\tau\odot\sigma$—as required. □

Because $(-)$“ does not preserve composition up to isomorphism but only up to the transformation φ of Lemma 7.12, $(-)$“ forms a *lax* functor from the bicategory of strategies to that of profunctors.

7.5 Games as factorization systems

The results of Section 7.1 show an event structure with polarity determines a factorization system; the ‘left’ maps are given by $\supseteq^-$ and the ‘right’ maps by $\subseteq^+$. More specifically they form an instance of a *rooted* factorization system $(\mathbb{X}, \to_L, \to_R, 0)$ where maps $f : x \to_L x'$ are the ‘left’ maps and $g : x \to_R x'$ the ‘right’ maps of a factorization system on a small category $\mathbb{X}$, with distinguished object 0, such that any object x of $\mathbb{X}$ is reachable by a chain of maps:

$$0 \leftarrow_L \cdot \to_R \cdots \leftarrow_L \cdot \to_R x\,;$$

and two ‘confluence’ conditions hold:

$$x_1 \to_R x \;\&\; x_2 \to_R x \implies \exists x_0.\ x_0 \to_R x_1 \;\&\; x_0 \to_R x_2\,, \quad \text{and its dual}$$
$$x \to_L x_1 \;\&\; x \to_L x_2 \implies \exists x_0.\ x_1 \to_L x_0 \;\&\; x_2 \to_R x_0\,.$$

Think of objects of $\mathbb{X}$ as configurations, the R-maps as standing for (compound) Player moves and L-maps for the reverse, or undoing, of (compound) Opponent moves in a game.

The characterization of strategy, Proposition 4.20, exhibits a strategy as a discrete fibration w.r.t. $\sqsubseteq$ whose functor preserves $\supseteq^-$ and $\subseteq^+$. This generalizes. Define a strategy in a rooted factorization system to be a functor from another rooted factorization system preserving L-maps, R-maps, 0 and forming a discrete fibration. To obtain strategies *between* rooted factorization systems we again follow the methodology of Joyal [8], and take a strategy from $\mathbb{X}$ to $\mathbb{Y}$ to be a strategy in the dual of $\mathbb{X}$ in parallel composition with $\mathbb{Y}$. Now the dual operation becomes the opposite construction on a factorization system, reversing the roles and directions of the 'left' and 'right' maps. The parallel composition of factorization systems is given by their product. Composition of strategies is given essentially as that of profunctors, but restricting to reachable elements.

Chapter 8

Winning ways

What does it mean to win a nondeterministic concurrent game and what is a winning strategy? This chapter extends the work on games and strategies to games with winning conditions and winning strategies.

8.1 Winning strategies

A *game with winning conditions* comprises $G = (A, W)$ where A is an event structure with polarity and $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations* for Player. We define the *losing conditions* to be $L =_{\text{def}} \mathcal{C}^\infty(A) \setminus W$. Clearly a game with winning conditions is determined once we specify either its winning or losing conditions, and we can define such a game by specifying its losing conditions.

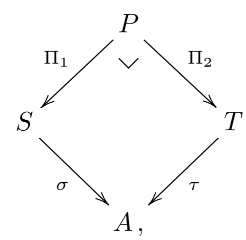
A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$—a configuration x is +-maximal if whenever $x \stackrel{s}{\longrightarrow\!\!\subset}$ then the event s has −ve polarity. Any achievable position $z \in \mathcal{C}^\infty(S)$ of the game can be extended to a +-maximal, so winning, configuration (via Zorn's Lemma). So a strategy prescribes Player moves to reach a winning configuration whatever state of play is achieved following the strategy. Note that for a game A, if winning conditions $W = \mathcal{C}^\infty(A)$, *i.e.* every configuration is winning, then any strategy in A is a winning strategy.

In the special case of a deterministic strategy $\sigma : S \to A$ in G it is winning iff $\sigma\varphi(x) \in W$ for all $x \in \mathcal{C}^\infty(S)$, where φ is the closure operator $\varphi : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S)$ determined by σ or, equivalently, the images under σ of fixed points of φ lie outside L. Recall from Section 6.2.3 that a deterministic strategy $\sigma : S \to A$ determines a closure operator φ on $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x\} .$$

Clearly, we can equivalently say a strategy $\sigma : S \to A$ in G is winning if it always prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent; a strategy $\sigma : S \to A$ in G is winning if $\sigma x \notin L$ for all $+$-maximal configurations $x \in \mathcal{C}^\infty(S)$

Informally, we can also understand a strategy as winning for Player if when played against any counter-strategy of Opponent, the final result is a win for Player. Suppose $\sigma : S \to A$ is a strategy in a game (A, W). A counter-strategy is strategy of Opponent, so a strategy $\tau : T \to A^\perp$ in the dual game. We can view σ as a strategy $\sigma : \varnothing \mathrel{\longrightarrow\!\!\!\!\!\!+\;\;} A$ and τ as a strategy $\tau : A \mathrel{\longrightarrow\!\!\!\!\!\!+\;\;} \varnothing$. Their composition $\tau \odot \sigma : \varnothing \mathrel{\longrightarrow\!\!\!\!\!\!+\;\;} \varnothing$ is not in itself so informative. Rather it is the status of the configurations in $\mathcal{C}^\infty(A)$ their full interaction induces which decides which of Player or Opponent wins. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,



to obtain the event structure P resulting from the interaction of σ and τ. (Note $P \cong \mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))$, in the terms of Chapter 4, by the remarks of Section 4.3.3.) Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^\infty(P)$. A maximal configuration z in $\mathcal{C}^\infty(P)$ images to a configuration $\sigma \Pi_1 z = \tau \Pi_2 z$ in $\mathcal{C}^\infty(A)$. Define the set of *results* of the interaction of σ and τ to be

$$\langle \sigma, \tau \rangle =_{\mathrm{def}} \{ \sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P) \} .$$

We shall show the strategy σ is a winning for Player iff all the results of the interaction $\langle \sigma, \tau \rangle$ lie within the winning configurations W, for any counter-strategy $\tau : T \to A^\perp$ of Opponent.

It will be convenient later to have proved facts about $+$-maximality in the broader context of the composition of arbitrary strategies.

Convention 8.1. Refer to the construction of the composition of pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : B^\perp \| C$ in Chapter 4 We shall say a configuration x of either $\mathcal{C}^\infty(S), \mathcal{C}^\infty(T)$ or $(\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ is $+$-maximal if whenever $x \stackrel{e}{-\!\!\!\subset}$ then the event e has $-$ve polarity. In the case of $(\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ an event of $-$ve polarity is deemed to be one of the form $(s, *)$, with s $-$ve in S, or $(*, t)$, with t $-$ve in T. We shall say a configuration z of $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)))$ is $+$-maximal if whenever $z \stackrel{p}{-\!\!\!\subset}$ then $top(p)$ has $-$ve polarity.

Lemma 8.2. *Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be receptive pre-strategies. Then,*

$$z \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty \text{ is } +\text{-maximal iff}$$
$$\pi_1 z \in \mathcal{C}^\infty(S) \text{ is } +\text{-maximal } \& \ \pi_2 z \in \mathcal{C}^\infty(T) \text{ is } +\text{-maximal.}$$

Proof. Let $z \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$. *"Only if":* Assume z is +-maximal. Suppose, for instance, $\pi_1 z$ is not +-maximal. Then, $\pi_1 z \overset{s}{\longrightarrow\!\subset}$ for some +ve event $s \in S$. Consider the two cases. *Case $\sigma_1(s)$ is defined:* Form the configuration $z \cup \{(s,*)\} \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$, to contradict the +-maximality of z. *Case $\sigma_2(s)$ is defined:* As s is +-ve by the receptivity of τ there is $t \in T$ such that $\pi_2 z \overset{t}{\longrightarrow\!\subset}$ and $\tau_1(t) = \overline{\sigma_2(s)}$. Form the configuration $z \cup \{(s,t)\} \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$, to contradict the +-maximality of z. The argument showing $\pi_2 z$ is +-maximal is similar.

"If": Assume both $\pi_1 z$ and $\pi_2 z$ are +-maximal. Suppose z were not +-maximal. Then, either

- $z \overset{(s,*)}{\longrightarrow\!\subset}$ or $z \overset{(s,t)}{\longrightarrow\!\subset}$ with s a +ve event of S, or
- $z \overset{(*,t)}{\longrightarrow\!\subset}$ or $z \overset{(s,t)}{\longrightarrow\!\subset}$ with t a +ve event of T.

But then either $\pi_1 z \overset{s}{\longrightarrow\!\subset}$, contradicting the +-maximality of $\pi_1 z$, or $\pi_2 z \overset{t}{\longrightarrow\!\subset}$, contradicting the +-maximality of $\pi_2 z$. □

Corollary 8.3. *Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be receptive pre-strategies. Then,*

$$x \in \mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))) \text{ is } +\text{-maximal iff}$$
$$\Pi_1 x \in \mathcal{C}^\infty(S) \text{ is } +\text{-maximal } \& \ \Pi_2 x \in \mathcal{C}^\infty(T) \text{ is } +\text{-maximal.}$$

Proof. From Lemma 8.2, noting the order isomorphism $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))) \cong (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ given by $x \mapsto \bigcup x$ and that $\Pi_1 x = \pi_1 \bigcup x$, $\Pi_2 x = \pi_2 \bigcup x$. □

Lemma 8.4. *Let $\sigma : S \to A$ be a strategy in a game (A, W). The strategy σ is winning for Player iff $\langle \sigma, \tau \rangle \subseteq W$ for all (deterministic) strategies $\tau : T \to A^\perp$.*

Proof. "Only if": Suppose σ is winning, *i.e.* $\sigma x \in W$ for all +-maximal $x \in \mathcal{C}^\infty(S)$. Let $\tau : T \to A^\perp$ be a strategy. By Corollary 8.3,

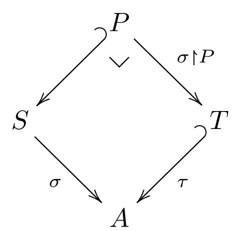
$$x \in \mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))) \text{ is } +\text{-maximal}$$
iff
$$\Pi_1 x \in \mathcal{C}^\infty(S) \text{ is } +\text{-maximal } \ \& \ \Pi_2 x \in \mathcal{C}^\infty(T) \text{ is } +\text{-maximal.}$$

Letting x be maximal in $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)))$ it is certainly +-maximal, whence $\Pi_1 x$ is +-maximal in $\mathcal{C}^\infty(S)$. It follows that $\sigma \Pi_1 x \in W$ as σ is winning. Hence $\langle \sigma, \tau \rangle \subseteq W$.

"If": Assume $\langle\sigma,\tau\rangle \subseteq W$ for all strategies $\tau : T \to A^\perp$. Suppose x is +-maximal in $\mathcal{C}^\infty(S)$. Define T to be the event structure given as the restriction

$$T =_{\text{def}} A^\perp \upharpoonright \sigma x \cup \{a \in A^\perp \mid pol_{A^\perp} = -\}\,.$$

Let $\tau : T \to A^\perp$ be the inclusion map $T \hookrightarrow A^\perp$. The pre-strategy τ can be checked to be receptive and innocent, so a strategy. (In fact, τ is a *deterministic* strategy as all its +ve events lie within the configuration σx.) One way to describe a pullback of τ along σ is as the "inverse image" $P =_{\text{def}} S \upharpoonright \{s \in S \mid \sigma(s) \in T\}$:



From the definition of T and P we see $x \in \mathcal{C}^\infty(P)$; and moreover that x is maximal in $\mathcal{C}^\infty(P)$ as x is +-maximal in $\mathcal{C}^\infty(S)$. Hence $\sigma x \in \langle\sigma,\tau\rangle$ ensuring $\sigma x \in W$, as required.

The proof is unaffected if we restrict to *deterministic* counter-strategies $\tau : T \to A^\perp$. □

Corollary 8.5. *There are the following four equivalent ways to say that a strategy $\sigma : S \to A$ is winning in (A, W)—we write L for the losing configurations $\mathcal{C}^\infty(A) \setminus W$:*

1. *$\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to reach a winning configuration, no matter what the activity or inactivity of Opponent;*

2. *$\sigma x \notin L$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent;*

3. *$\langle\sigma,\tau\rangle \subseteq W$ for all strategies $\tau : T \to A^\perp$, i.e. all plays against counter-strategies of the Opponent result in a win for Player;*

4. *$\langle\sigma,\tau\rangle \subseteq W$ for all deterministic strategies $\tau : T \to A^\perp$, i.e. all plays against deterministic counter-strategies of the Opponent result in a win for Player.*

Not all games with winning conditions have winning strategies. Consider the game A consisting of one player move $\oplus$ and one opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as its winning conditions. This game has no winning strategy; any strategy $\sigma : S \to A$, being receptive, will have an event $s \in S$ with $\sigma(s) = \ominus$, and so the losing $\{s\}$ as a +-maximal configuration.

8.2 Operations

8.2.1 Dual

There is an obvious dual of a game with winning conditions $G = (A, W_G)$:

$$G^\perp = (A^\perp, W_{G^\perp})$$

where, for $x \in \mathcal{C}^\infty(A)$,

$$x \in W_{G^\perp} \text{ iff } \overline{x} \notin W_G .$$

We are using the notation $a \leftrightarrow \overline{a}$, giving the correspondence between events of A and $A^\perp$, extended to their configurations: $\overline{x} =_{\text{def}} \{\overline{a} \mid a \in x\}$, for $x \in \mathcal{C}^\infty(A)$. As usual the dual reverses the roles of Player and Opponent and correspondingly the roles of winning and losing conditions.

8.2.2 Parallel composition

The parallel composition of two games with winning conditions $G = (A, W_G)$, $H = (B, W_H)$ is

$$G \| H =_{\text{def}} (A \| B,\ W_G \| \mathcal{C}^\infty(B) \cup \mathcal{C}^\infty(A) \| W_H)$$

where $X \| Y = \{\{1\} \times x \cup \{2\} \times y \mid x \in X \ \&\ y \in Y\}$ when X and Y are subsets of configurations. In other words, for $x \in \mathcal{C}^\infty(A \| B)$,

$$x \in W_{G\|H} \text{ iff } x_1 \in W_G \text{ or } x_2 \in W_H ,$$

where $x_1 = \{a \mid (1, a) \in x\}$ and $x_2 = \{b \mid (2, b) \in x\}$. To win in $G \| H$ is to win in either game. Its losing conditions are $L_A \| L_B$—to lose is to lose in both games G and H.[1] The unit of $\|$ is $(\varnothing, \varnothing)$. In order to disambiguate the various forms of parallel composition, we shall sometimes use the linear-logic notation $G \parr H$ for the parallel composition $G \| H$ of games with winning strategies.

8.2.3 Tensor

Defining $G \otimes H =_{\text{def}} (G^\perp \| H^\perp)^\perp$ we obtain a game where to win is to win in both games G and H—so to lose is to lose in either game. More explicitly,

$$(A, W_A) \otimes (B, W_B) =_{\text{def}} (A \| B,\ W_A \| W_B) .$$

The unit of $\otimes$ is $(\varnothing, \{\varnothing\})$.

[1] I'm grateful to Nathan Bowler, Pierre Clairambault and Julian Gutierrez for guidance in the definition of parallel composition of games with winning conditions.

8.2.4 Function space

With $G \multimap H =_{\rm def} G^\perp \| H$ a win in $G \multimap H$ is a win in H conditional on a win in G.

Proposition 8.6. *Let $G = (A, W_G)$ and $H = (B, W_H)$ be games with winning conditions. Write $W_{G\multimap H}$ for the winning conditions of $G \multimap H$, so $G \multimap H = (A^\perp \| B, W_{G\multimap H})$. For $x \in \mathcal{C}^\infty(A^\perp \| B)$,*

$$x \in W_{G\multimap H} \quad \textit{iff} \quad \overline{x_1} \in W_G \implies x_2 \in W_H \,.$$

Proof. Letting $x \in \mathcal{C}^\infty(A^\perp \| B)$,

$$\begin{aligned} x \in W_{G\multimap H} \ &\text{iff} \ \ x \in W_{G^\perp \| H} \\ &\text{iff} \ \ x_1 \in W_{G^\perp} \text{ or } x_2 \in W_H \\ &\text{iff} \ \ \overline{x_1} \notin W_G \text{ or } x_2 \in W_H \\ &\text{iff} \ \ \overline{x_1} \in W_G \implies x_2 \in W_H \,. \end{aligned}$$

□

8.3 The bicategory of winning strategies

We can again follow Joyal and define strategies between games now with winning conditions: a (winning) strategy from G, a game with winning conditions, to another H is a (winning) strategy in $G \multimap H = G^\perp \| H$. We compose strategies as before. We first show that the composition of winning strategies is winning.

Lemma 8.7. *Let σ be a winning strategy in $G^\perp \| H$ and τ be a winning strategy in $H^\perp \| K$. Their composition $\tau \odot \sigma$ is a winning strategy in $G^\perp \| K$.*

Proof. Let $G = (A, W_G)$, $H = (B, W_H)$ and $K = (C, W_K)$.

Suppose $x \in \mathcal{C}^\infty(T \odot S)$ is +-maximal. Then $\bigcup x \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$. By Zorn's Lemma we can extend $\bigcup x$ to a maximal configuration $z \supseteq \bigcup x$ in $(\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ with the property that all events of $z \setminus \bigcup x$ are synchronizations of the form (s, t) for $s \in S$ and $t \in T$. Then, z will be +-maximal in $(\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ with

$$\sigma_1 \pi_1 z = \sigma_1 \pi_1 \bigcup x \ \ \& \ \ \tau_2 \pi_2 z = \tau_2 \pi_2 \bigcup x \,. \tag{1}$$

By Lemma 8.2,

$$\pi_1 z \text{ is +-maximal in } S \ \ \& \ \ \pi_2 z \text{ is +-maximal in } T \,.$$

As σ and τ are winning,

$$\sigma \pi_1 z \in W_{G^\perp \| H} \ \ \& \ \ \tau \pi_2 z \in W_{H^\perp \| K} \,.$$

Now $\sigma \pi_1 z \in W_{G^\perp \| H}$ expreses that

$$\overline{\sigma_1 \pi_1 z} \in W_G \implies \sigma_2 \pi_1 z \in W_H \tag{2}$$

and $\tau\pi_2 z \in W_{H^\perp \| K}$ that

$$\overline{\tau_1\pi_2 z} \in W_H \implies \tau_2\pi_2 z \in W_K\,, \qquad (3)$$

by Proposition 8.6. But $\sigma_2\pi_1 z = \overline{\tau_1\pi_2 z}$, so (2) and (3) yield

$$\overline{\sigma_1\pi_1 z} \in W_G \implies \tau_2\pi_2 z \in W_K\,.$$

By (1)

$$\overline{\sigma_1\pi_1 \bigcup x} \in W_G \implies \tau_2\pi_2 \bigcup x \in W_K\,,$$

*i.e.*by Proposition 4.2,

$$\overline{\upsilon_1 x} \in W_G \implies \upsilon_2 x \in W_K$$

in the span of the composition $\tau\odot\sigma$. Hence $x \in W_{G^\perp\|K}$, as required. □

For a general game with winning conditions (A, W) the copy-cat strategy need not be winning, as shown in the following example.

Example 8.8. Let A consist of two events, one +ve event $\oplus$ and one −ve event $\ominus$, inconsistent with each other. Take as winning conditions the set $W = \{\{\oplus\}\}$. The event structure $\mathbb{C}\mathbb{C}_A$:

$$\begin{array}{cccc} A^\perp & \ominus \rightarrowtriangle \oplus & A \\ & \oplus \leftarrowtriangle \ominus & \end{array}$$

To see $\mathbb{C}\mathbb{C}_A$ is not winning consider the configuration x consisting of the two −ve events in $\mathbb{C}\mathbb{C}_A$. Then x is +-maximal as any +ve event is inconsistent with x. However, $\overline{x}_1 \in W$ while $x_2 \notin W$, failing the winning condition of $(A, W) \multimap (A, W)$.

Recall from Chapter 7, that each event structure with polarity$A$ possesses a Scott order on its configurations $\mathcal{C}^\infty(A)$:

$$x' \sqsubseteq x \;\text{ iff }\; x' \supseteq^- x \cap x' \subseteq^+ x\,.$$

A necessary and sufficient for copy-cat to be winning w.r.t. a game (A, W):

$$\begin{array}{ll} \forall x, x' \in \mathcal{C}^\infty(A). & \text{if } x' \sqsubseteq x \;\&\; x' \text{ is +-maximal} \;\&\; x \text{ is −-maximal,} \\ & \text{then } \; x \in W \implies x' \in W\,. \end{array} \qquad \textbf{(Cwins)}$$

Lemma 8.9. *Let (A, W) be a game with winning conditions. The copy-cat strategy $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp \| A$ is winning iff (A, W) satisfies* **(Cwins)**.

Proof. By Lemma 7.5,

$$z \in \mathcal{C}^\infty(\mathbb{C}\mathbb{C}_A) \;\text{ iff }\; z = \{1\} \times \overline{x} \cup \{2\} \times x' \;\text{ with }\; x' \sqsubseteq_A x\,,$$

for $x, x' \in \mathcal{C}^\infty(A)$. In this situation z is +-maximal iff both x is −-maximal and x' is +-maximal. Thus **(Cwins)** expresses precisely that copy-cat is winning. □

A robust sufficient condition on an event structure with polarity A which ensures that copy-cat is a winning strategy for all choices of winning conditions is the property

$$\forall x \in \mathcal{C}(A).\ x \overset{a}{\longrightarrow\!\!\subset}\ \&\ x \overset{a'}{\longrightarrow\!\!\subset}\ \&\ pol(a) = +\ \&\ pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(race-free)}$$

This property, which says immediate conflict respects polarity, is seen earlier in Lemma 5.3 (characteriziing those A for which copy-cat is deterministic).

Proposition 8.10. *Let A be an event structure with polarity. Copy-cat is a winning strategy for all games (A, W) with winning conditions W iff A satisfies* (**race-free**)*.*

Proof. *"If":* Assume (**race-free**). Let $W \subseteq \mathcal{C}^\infty(A)$. We show **(Cwins)** holds for the game with winning conditions (A, W). For $x, x' \in \mathcal{C}^\infty(A)$, assume

$$x' \sqsubseteq x\ \&\ x' \text{ is } +\text{-maximal}\ \&\ x \text{ is } -\text{-maximal}.$$

Then, as $x' \supseteq^- x \cap x' \subseteq^+ x$, there are covering chains associated with purely +ve and −ve events from $x \cap x'$ to x and x', respectively:

$$x \cap x' \overset{+}{\longrightarrow\!\!\subset} \cdots \overset{+}{\longrightarrow\!\!\subset} x\,,$$
$$x \cap x' \overset{-}{\longrightarrow\!\!\subset} \cdots \overset{-}{\longrightarrow\!\!\subset} x'\,.$$

If one of the covering chains is of zero length then so must the other be—otherwise we contradict one or other of the maximality assumptions. On the other hand, if both are nonempty, by repeated use of (**race-free**) we again contradict a maximality assumption, *e.g.*

$$\begin{array}{ccccccc}
y_1 & \overset{+}{\longrightarrow\!\!\subset} & x_1 \cup x_1' & \overset{+}{\longrightarrow\!\!\subset} & \cdots & \overset{+}{\longrightarrow\!\!\subset} & x \cup x_1' \\
{\scriptstyle -}\big\uparrow & & {\scriptstyle -}\big\uparrow & & & & {\scriptstyle -}\big\uparrow \\
x \cap x' & \overset{+}{\longrightarrow\!\!\subset} & x_1 & \overset{+}{\longrightarrow\!\!\subset} & \cdots & \overset{+}{\longrightarrow\!\!\subset} & x
\end{array}$$

shows how a repeated use of (**race-free**) contradicts the −-maximality of x. We conclude $x = x \cap x' = x'$ so certainly $x \in W \implies x' \in W$, as required to fulfil **(Cwins)**.

"Only if": Suppose A failed (**race-free**), *i.e.* $x \overset{a}{\longrightarrow\!\!\subset} x_1\ \&\ x \overset{a'}{\longrightarrow\!\!\subset} x_2$ with $x_1 \not\uparrow x_2$ and $pol_A(a) = +$ and $pol(a') = -$ within the finite configurations of A. The set $\{1\} \times \overline{x}_1 \cup \{2\} \times x_2$ is certainly a finite configuration of $A^\perp \| A$ and is easily checked to also be a configuration of $\mathbb{C}_A$. Define winning conditions by

$$W = \{x \in \mathcal{C}^\infty(A) \mid a \in x\}\,.$$

Let $z \in \mathcal{C}^\infty(\mathbb{C}_A)$ be a +-maximal extension of $\{1\} \times \overline{x}_1 \cup \{2\} \times x_2$ (the maximal extension exists by Zorn's Lemma). Take $z_1 = \{a \mid (1, a) \in z\}$ and $z_2 = \{a \mid (2, a) \in z\}$. Then $\overline{z}_1 \supseteq x_1$ and $z_2 \supseteq x_2$. As $a \in \overline{z}_1$ we obtain $\overline{z}_1 \in W$, whereas $z_2 \notin W$ because z_2 extends y which is inconsistent with a. Hence copy-cat is not winning in $(A, W)^\perp \| (A, W)$. $\square$

We can now refine the bicategory of strategies **Games** to the bicategory **WGames** with objects games with winning conditions $G, H, \cdots$ satisfying (**Cwins**) and arrows winning strategies $G \twoheadrightarrow H$; 2-cells, their vertical and horizontal composition is as before. Its restriction to deterministic strategies yields a bicategory **WDGames** equivalent to a simpler order-enriched category.

8.4 Total strategies

As an application of winning conditions we apply them to pick out a subcategory of "total strategies," informally strategies in which Player can always answer a move of Opponent.[2]

We restrict attention to 'simple games' (games and strategies are alternating and begin with opponent moves—see Section 6.2.4). Here a strategy is *total* if all its finite maximal sequences are even, so ending in a +ve move, *i.e.* a move of Player. In general, the composition of total strategies need not be total—see the Exercise below. However, as we will see, we can pick out a subcategory of 'simple games' with suitable winning conditions. Within this full subcategory of games with winning conditions winning strategies will be total and moreover compose.

Exercise 8.11. *Exhibit two total strategies whose composition is not total.* □

As objects of the subcategory we choose simple games with winning strategies,

$$(A, W_A)$$

where A is a simple game and W_A is a subset of possibly infinite sequences $s_1 s_2 \cdots$ satisfying

$$W_A \cap \text{Finite}(A) = \text{Even}(A) \qquad \textbf{(Tot)}$$

i.e. the finite sequences in W_A are precisely those of even length. Note that winning strategies in such a game will be total. (Below we use 'sequence' to mean allowable finite or infinite sequences of the appropriate simple game.)

The function space $(A, W_A) \multimap (B, W_B)$, given as $(A, W_A)^\perp \| (B, W_B)$, has winning conditions W such that

$$s \in W \text{ iff } s \upharpoonright A \in W_A \implies s \upharpoonright B \in W_B\,.$$

Lemma 8.12. *For s a sequence of $A^\perp \| B$, s is even iff $s \upharpoonright A$ is odd or $s \upharpoonright B$ is even.*

Proof. By parity, considering the final move of the sequence.
"Only if": Assume s is even, *i.e.* its final event is +ve. If s ends in B, $s \upharpoonright B$ ends in + so is even. If s ends in A, $s \upharpoonright A$ ends in − so is odd.
"If": Assume $s \upharpoonright A$ is odd or $s \upharpoonright B$ is even. Suppose, to obtain a contradiction, that s is not even, *i.e.* s is odd so ends in −. If s ends in B, $s \upharpoonright B$ ends in − so

[2]This section is inspired by [26], though differs in several respects.

is odd and consequently $s \upharpoonright A$ even (as the length of s is the sum of the lengths of $s \upharpoonright A$ and $s \upharpoonright B$). Similarly, if s ends in A, $s \upharpoonright A$ ends in $+$ so $s \upharpoonright A$ is even and $s \upharpoonright B$ is odd. Either case contradicts the initial assumption. Hence s is even. □

It follows that W, the winning conditions of the function space, satisfies (**Tot**): Let s be a finite sequence of a strategy in $A^\perp \| B$. Then,

$$\begin{aligned} s \in W \text{ iff } & s \upharpoonright A \in W_A \implies s \upharpoonright B \in W_B \\ \text{iff } & s \upharpoonright A \notin W_A \text{ or } s \upharpoonright B \in W_B \\ \text{iff } & s \upharpoonright A \text{ is odd or } s \upharpoonright B \text{ is even} \\ \text{iff } & s \text{ is even.} \end{aligned}$$

All maps in the subcategory (which are winning strategies in its function spaces $(A, W_A) \multimap (B, W_B)$) compose (because winning strategies do) and are total (because winning conditions of its function spaces satisfy (**Tot**)).

8.5 On determined games

A game with winning conditions G is said to be *determined* when either Player or Opponent has a winning strategy, *i.e.* either there is a winning strategy in G or in $G^\perp$.[3] Not all games are determined. Neither the game G consisting of one player move $\oplus$ and one opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as winning conditions, nor the game $G^\perp$ have a winning strategy.

Notation 8.13. *Let $\sigma : S \to A$ be a strategy. We say $y \in \mathcal{C}^\infty(A)$ is* σ-reachable *iff $y = \sigma x$ for some $x \in \mathcal{C}^\infty(S)$. Let $y' \subseteq y$ in $\mathcal{C}^\infty(A)$. Say y' is* $-$-maximal in y *iff $y \xrightarrow{-}\!\!\subset y''$ implies $y'' \notin y$. Similarly, say y' is* $+$-maximal in y *iff $y \xrightarrow{+}\!\!\subset y''$ implies $y'' \notin y$.*

Lemma 8.14. *Let (A, W) be a game with winning conditions. Let $y \in \mathcal{C}^\infty(A)$. Suppose*

$$\begin{aligned} & \forall y' \in \mathcal{C}^\infty(A). \\ & y' \subseteq y \;\&\; y' \text{ is } -\text{-maximal in } y \;\&\; \text{not } +\text{-maximal in } y \\ & \implies \\ & \{y'' \in \mathcal{C}(A) \mid y' \subseteq^+ y'' \;\&\; (y'' \setminus y') \cap y = \varnothing\} \cap W = \varnothing\,. \end{aligned}$$

Then y is σ-reachable in all winning strategies σ.

Proof. Assume the property above of $y \in \mathcal{C}^\infty(A)$. Suppose, to obtain a contradiction, that y is not σ-reachable in a winning strategy $\sigma : S \to A$.

Let $x' \in \mathcal{C}^\infty(A)$ be $\subseteq$-maximal such that $\sigma x' \subseteq y$ (this uses Zorn's lemma).

By the receptivity of σ, the configuration $\sigma x'$ is $-$-maximal in y. By supposition, $\sigma x' \subsetneq y$, so we must therefore have $\sigma x' \xrightarrow{+}\!\!\subset y_0 \subseteq y$ in $\mathcal{C}^\infty(A)$, *i.e.* $\sigma x'$ is not $+$-maximal in y. From the property assumed of y we deduce both

$$\sigma x' \notin W \;\;\&\;\; (\forall y'' \in W.\ \sigma x' \subseteq^+ y'' \implies (y'' \setminus \sigma x') \cap y \neq \varnothing)\,.$$

[3]This section is based on work with Julian Gutierrez.

As σ is winning, there is +-maximal extension $x' \subseteq^+ x''$ in $\mathcal{C}^\infty(S)$ such that $\sigma x'' \in W$. Hence

$$(\sigma x'' \setminus \sigma x') \cap y \neq \varnothing\,.$$

Taking a $\leq_A$-minimal event a_1, necessarily +ve, in the above set we obtain

$$\sigma x' \overset{a_1}{\longrightarrow\!\subset} y_1 \subseteq^+ \sigma x''\,.$$

By Corollary 4.22, $y_1 = \sigma x_1$ for some $x_1 \in \mathcal{C}^\infty(S)$ with $x' \overset{+}{\longrightarrow\!\subset} x_1 \subseteq x''$. But this contradicts the choice of x' as $\subseteq$-maximal such that $\sigma x' \subseteq y$. Hence the original assumption that y is not σ-reachable must be false. □

Recall the property (**race-free**) of an event structure with polarity A, first seen in Lemma 5.3, though here rephrased a little:

$$\forall y, y_1, y_2 \in \mathcal{C}(A).\ y \overset{-}{\longrightarrow\!\subset} y_1 \ \&\ y \overset{+}{\longrightarrow\!\subset} y_2 \implies y_1 \uparrow y_2\,. \qquad \textbf{(race-free)}$$

Corollary 8.15. *If A, an event structure with polarity, fails to satisfy* (**race-free**)*, then there are winning conditions W, for which the game (A, W) is not determined.*

Proof. Suppose (**race-free**) failed, that $y \overset{-}{\longrightarrow\!\subset} y_1$ and $y \overset{+}{\longrightarrow\!\subset} y_2$ and $y_1 \not\uparrow y_2$ in $\mathcal{C}(A)$. Assign configurations $\mathcal{C}^\infty(A)$ to winning conditions W or its complement as follows:

(i) for y'' with $y_1 \subseteq^+ y''$, assign $y'' \notin W$;

(ii) for y'' with $y_2 \subseteq^- y''$, assign $y'' \in W$;

(iii) for y'' with $y' \subseteq^+ y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' −-maximal and not +-maximal in y, assign $y'' \notin W$;

(iv) for y'' with $y' \subseteq^- y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' +-maximal and not −-maximal in y, assign $y'' \in W$;

(v) assign arbitrarily in all other cases.

We should check the assignment is well-defined, that we do not assign a configuration both to W and its complement.

Clearly the first two cases (i) and (ii) are disjoint as $y_1 \not\uparrow y_2$.

The two cases (iii) and (iv) are also disjoint. Suppose otherwise, that both (iii) and (iv) hold for y'', *viz.*

$$\begin{aligned} &y_1' \subseteq^+ y'' \ \&\ (y'' \setminus y_1') \cap y = \varnothing\ \& \\ &\qquad\qquad y_1' \text{ is −-maximal } \ \&\ \text{ not +-maximal in } y\,, \text{ and} \\ &y_2' \subseteq^- y'' \ \&\ (y'' \setminus y_2') \cap y = \varnothing\ \& \\ &\qquad\qquad y_2' \text{ is +-maximal } \ \&\ \text{ not −-maximal in } y\,. \end{aligned}$$

As

$$y_1' \subseteq^+ y'' \supseteq^- y_2'$$

we deduce $y_2'^{-} \subseteq y_1'$, *i.e.* all the −ve events of y_2' are in y_1'. Now let $a \in y_2'^{+}$. Then $a \in y$ as $y_2' \subseteq y$. Therefore $a \notin y'' \setminus y_1'$, by assumption. But $a \in y''$ as $y_2' \subseteq^{-} y''$, so $a \in y_1'$. We conclude $y_2' \subseteq y_1'$. A similar dual argument shows $y_1' \subseteq y_2'$. Thus $y_1' = y_2'$. But this implies that y_1' is both −-maximal and not −maximal in y —a contradiction.

Suppose both the conditions (i) and (iv) are met by y''. From (vi), as y' is +-maximal & not −-maximal in y,

$$y' \overset{a}{\longrightarrow\!\subset} y_0 \subseteq y\,,$$

for some event a with $pol_A(a) = -$ and $y_0 \in \mathcal{C}^\infty(A)$. From (i), $y \subseteq y''$, so

$$y' \overset{a}{\longrightarrow\!\subset} y_0 \subseteq y''\,.$$

Therefore

$$a \in y'' \setminus y' \ \&\ a \in y\,,$$

which contradicts (iv). Similarly the cases (ii) and (iii) are disjoint.

We conclude that the assignment of winning conditions is well-defined.

Then y is reachable for both winning strategies in (A, W) and winning strategies in $(A, W)^\perp$. Suppose σ is a winning strategy σ in (A, W). By (iii) and Lemma 8.14, y is σ-reachable. From receptivity y_1 is σ-reachable, say $y_1 = \sigma x_1$ for some $x_1 \in \mathcal{C}(S)$. There is a +-maximal extension x_1' of x_1 in $\mathcal{C}^\infty(S)$. By (i), $\sigma x_1'$ cannot be a winning configuration. Hence there can be no winning strategy in (A, W). In a dual fashion, there can be no winning strategy in $(A, W)^\perp$. □

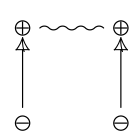
It is tempting to believe that a nondeterministic winning strategy always has a winning (weakly-)deterministic sub-strategy. However, this is not so, as the following examples show.

Example 8.16. A winning strategy need not have a winning deterministic sub-strategy. Consider the game (A, W) where A consists of two inconsistent events $\ominus$ and $\oplus$, of the indicated polarity, and $W = \{\{\ominus\}, \{\oplus\}\}$. Consider the strategy σ in A given by the identity map $\mathrm{id}_A : a \to A$. Then σ is a nondeterministic winning strategy—all +-maximal configurations in A are winning. However any sub-strategy must include $\ominus$ by receptivity and cannot include $\oplus$ if it is to be deterministic, wherepon it has $\varnothing$ as a +-maximal configuration which is not winning.

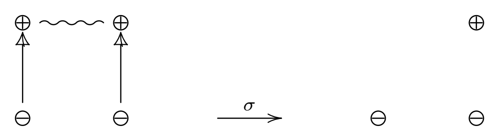
Example 8.17. Observe that the strategy σ of Example 8.16 is already weakly-deterministic—*cf.* Corollary 5.6. A winning strategy need not have a winning *weakly*-deterministic sub-strategy. Consider the game (A, W) where A consists of two −ve events $1, 2$ and one +ve event 3 all consistent with each other and

$$W = \{\varnothing, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Let S be the event structure



and $\sigma : S \to A$ the only possible total map of event structures with polarity:



Then σ is a winning strategy for which there is no weakly-deterministic sub-strategy.

8.6 Determinacy for well-founded games

Definition 8.18. *A game A is* well-founded *if every configuration in $\mathcal{C}^\infty(A)$ is finite.*

It is shown that any well-founded concurrent game satisfying (**race-free**) is determined.

8.6.1 Preliminaries

Proposition 8.19. *Let $\mathcal{Q}$ be a non-empty family of finite partial orders closed under rigid inclusions, i.e. if $q \in \mathcal{Q}$ and $q' \hookrightarrow q$ is a rigid inclusion (regarded as a map of event structures) then $q' \in \mathcal{Q}$. The family $\mathcal{Q}$ determines an event structure $(P, \leq, \mathrm{Con})$ as follows:*

- *the events P are the prime partial orders in $\mathcal{Q}$, i.e. those finite partial orders in $\mathcal{Q}$ with a top element;*
- *the causal dependency relation $p' \leq p$ holds precisely when there is a rigid inclusion from $p' \hookrightarrow p$;*
- *a finite subset $X \subseteq P$ is consistent, $X \in \mathrm{Con}$, iff there is $q \in \mathcal{Q}$ and rigid inclusions $p \hookrightarrow q$ for all $p \in X$.*

If $x \in \mathcal{C}(P)$ then $\bigcup x$, the union of the partial orders in x, is in $\mathcal{Q}$. The function $x \mapsto \bigcup x$ is an order-isomorphism from $\mathcal{C}(P)$, ordered by inclusion, to $\mathcal{Q}$, ordered by rigid inclusions.

Call a non-empty family of finite partial orders closed under rigid inclusions a *rigid family*. Observe:

Proposition 8.20. *Any stable family $\mathcal{F}$ determines a rigid family: its configurations x possess a partial order $\leq_x$ such that whenever $x \subseteq y$ in $\mathcal{F}$ there is a rigid inclusion $(x, \leq_x) \hookrightarrow (y, \leq_y)$ between the corresponding partial orders.*

Notation 8.21. *We shall use $\mathrm{Pr}(\mathcal{Q})$ for the construction described in Proposition 8.19. The construction extends that on stable families with the same name.*

Lemma 8.22. *Let $\sigma : S \to A$ be a strategy. Letting $x, y \in \mathcal{C}(S)$,*

$$x^+ \subseteq y^+ \ \&\ \sigma x \subseteq \sigma y \implies x \subseteq y\,.$$

Proof. The proof relies on Proposition 4.20, characterising strategies. We first prove two special cases of the lemma.
Special case $\sigma x \subseteq^- \sigma y$. By assumption $x^+ \subseteq y^+$. Supposing $s \in y^+ \setminus x^+$, via the injectivity of σ on y, we obtain $\sigma y \setminus \sigma x$ contains $\sigma(s)$ a +ve event—a contradiction. Hence $x^+ = y^+$.

From Proposition 4.20(ii), as $\sigma x \subseteq^- \sigma y$, we obtain (a unique) $x' \in \mathcal{C}(S)$ such that $x \subseteq x'$ and $\sigma x' = \sigma y$:

$$\begin{array}{ccc} x & \subseteq & x' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \sigma x & \subseteq^- & \sigma y\,. \end{array}$$

Now $[x^+] \subseteq^- x$, from which

$$\begin{array}{ccc} [x^+] & \subseteq & x \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \sigma[x^+] & \subseteq^- & \sigma x\,. \end{array}$$

Combining the two diagrams:

$$\begin{array}{ccc} [x^+] & \subseteq & x' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \sigma[x^+] & \subseteq^- & \sigma y\,. \end{array}$$

As $[y^+] \subseteq^- y$,

$$\begin{array}{ccc} [y^+] & \subseteq & y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \sigma[y^+] & \subseteq^- & \sigma y\,. \end{array}$$

where, by Proposition 4.20(ii), y is the unique such configuration of S. But $y^+ = x^+$ so this same property is shared by x'. Hence $x' = y$ and $x \subseteq y$.

Thus

$$x^+ \subseteq y^+ \ \&\ \sigma x \subseteq^- \sigma y \implies x \subseteq y\,. \tag{1}$$

Note that, in particular,

$$x^+ = y^+ \ \&\ \sigma x = \sigma y \implies x = y\,. \tag{2}$$

Special case $\sigma x \subseteq^+ \sigma y$. By Proposition 4.20(i), there is (a unique) $y_1 \in \mathcal{C}(S)$ with $y_1 \subseteq y$ such that $\sigma y_1 = \sigma x$:

$$\begin{array}{ccc} y_1 & \overset{\subseteq}{\cdots\cdots} & y \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \sigma x & \subseteq^+ & \sigma y\,, \end{array}$$

Now $x^+, y_1^+ \subseteq y$ and $\sigma\, x^+ = (\sigma x)^+ = \sigma\, y_1^+$. So by the local injectivity of σ we obtain $x^+ = y_1^+$. By (2) above, $x = y_1$, whence $x \subseteq y$. Thus

$$x^+ \subseteq y^+ \ \&\ \sigma x \subseteq^+ \sigma y \implies x \subseteq y\,. \tag{3}$$

Any inclusion $\sigma x \subseteq \sigma y$ can be built as a composition of inclusions $\subseteq^-$ and $\subseteq^+$, so the lemma follows from the special cases (1) and (3). □

Lemma 8.23. *Let $\sigma : S \to A$ be a strategy for which no +ve event of S appears as a −ve event in A. Defining*

$$\mathcal{F}_\sigma \ =_{\text{def}}\ \{x^+ \cup (\sigma x)^- \mid x \in \mathcal{C}(S)\}$$

yields a stable family for which

$$\alpha_\sigma(s) = \begin{cases} s & \text{if } s \text{ is +ve,} \\ \sigma(s) & \text{if } s \text{ is −ve.} \end{cases}$$

is a map of stable families $\alpha_\sigma : \mathcal{C}(S) \to \mathcal{F}_\sigma$ which induces an order-isomorphism

$$(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$$

taking $x \in \mathcal{C}(S)$ to $\alpha_\sigma\, x = x^+ \cup (\sigma x)^-$. Defining

$$f_\sigma(e) = \begin{cases} \sigma(e) & \text{if } e \text{ is +ve,} \\ e & \text{if } e \text{ is −ve} \end{cases}$$

on events e of $\mathcal{F}_\sigma$ yields a map of stable families $f_\sigma : \mathcal{F}_\sigma \to \mathcal{C}(A)$ such that

$$\begin{array}{ccc} \mathcal{C}(S) & \xrightarrow{\alpha_\sigma} & \mathcal{F}_\sigma \\ & {\scriptstyle\sigma}\searrow & \downarrow{\scriptstyle f_\sigma} \\ & & \mathcal{C}(A) \end{array}$$

commutes.

Proof. A configuration $x \in \mathcal{C}(S)$ has direct image

$$\alpha_\sigma x = x^+ \cup (\sigma x)^-$$

under the function α_σ. Direct image under α_σ is clearly surjective and preserves inclusions, and by Lemma 8.22 yields an order-isomorphism $(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$: if $\alpha_\sigma x \subseteq \alpha_\sigma y$, for $x, y \in \mathcal{C}(S)$, then $x^+ \subseteq y^+$ and $(\sigma x)^- \subseteq (\sigma y)^-$ by the disjointness of S^+ and A, whence $\sigma x \subseteq \sigma y$ so $x \subseteq y$.

It is now routine to check that $\mathcal{F}_\sigma$ is a stable family and α_σ is a map of stable families. For instance to show the stability property required of $\mathcal{F}_\sigma$, assume $\alpha_\sigma x, \alpha_\sigma y \subseteq \alpha_\sigma z$. Then $x, y \subseteq z$ so $\sigma\, x \cap y = (\sigma x) \cap (\sigma y)$ as σ is a map of event structures, and consequently $(\sigma\, x \cap y)^- = (\sigma x)^- \cap (\sigma y)^-$. Now reason

$$\begin{aligned}
(\alpha_\sigma x) \cap (\alpha_\sigma y) &= (x^+ \cup (\sigma x)^-) \cap (y^+ \cup (\sigma y)^-)\\
&= (x^+ \cap y^+) \cup ((\sigma x)^- \cap (\sigma y)^-)\\
&\quad \text{—by distributivity with the disjointness of } S^+ \text{ and } A\,,\\
&= (x \cap y)^+ \cup (\sigma\, x \cap y)^-\\
&= (\alpha_\sigma\, x \cap y) \in \mathcal{F}_\sigma\,.
\end{aligned}$$

From the definitions of α_σ and f_σ it is clear that $f_\sigma \alpha_\sigma(s) = \sigma(s)$ for all events of S. Any configuration of $\mathcal{F}_\sigma$ is sent under f_σ to a configuration in $\mathcal{C}(A)$ in a locally injective fashion, making f_σ a map of stable families; this follows from the matching properties of σ. □

When we "glue" strategies together it can be helpful to assume that all the initial −ve moves of the strategies are exactly the same:

Lemma 8.24. *Let $\sigma : S \to A$ be a strategy. Then $\sigma \cong \sigma'$, a strategy $\sigma' : S' \to A$ for which*

$$\forall s' \in S'.\ pol_{S'}[s']_{S'} = \{-\} \implies s' = [\sigma(s')]_A\,.$$

Proof. Without loss of generality we may assume no +ve event of S appears as a −ve event in A. Take $f_\sigma : \mathcal{F}_\sigma \to \mathcal{C}(A)$ given by Lemma 8.24 and construct σ' as the composite map

$$\mathrm{Pr}(\mathcal{F}_\sigma) \xrightarrow{\mathrm{Pr}(\sigma)} \mathrm{Pr}(\mathcal{C}(A)) \overset{top}{\cong} A$$

—recall *top* takes a prime $[a]_A$ to a, where $a \in A$. □

8.6.2 Determinacy proof

Definition 8.25. *Let A be an event structure with polarity. Let $W \subseteq \mathcal{C}^\infty(A)$. Let $y \in \mathcal{C}^\infty(A)$. Define A/y to be the event structure with polarity comprising events*

$$\{a \in A \setminus y \mid y \cup [a]_A \in \mathcal{C}^\infty(A)\}\,,$$

also called A/y, with consistency relation

$$X \in \mathrm{Con}_{A/y} \text{ iff } X \subseteq_{\mathrm{fin}} A/y \ \&\ y \cup [X]_A \in \mathcal{C}^\infty(A)\,,$$

and causal dependency the restriction of that on A. Define $W/y \subseteq \mathcal{C}^\infty(A/y)$ by

$$z \in W/y \text{ iff } z \in \mathcal{C}^\infty(A/y) \ \&\ y \cup z \in W\,.$$

Finally, define $(A, W)/y =_{\mathrm{def}} (A/y, W/y)$.

Proposition 8.26. *Let A be an event structure with polarity and $y \in \mathcal{C}^\infty(A)$. Then,*

$$z \in \mathcal{C}^\infty(A/y) \text{ iff } z \subseteq A/y \ \& \ y \cup z \in \mathcal{C}^\infty(A)\,.$$

Assume A is a *well-founded* event structure with polarity with winning conditions $W \subseteq \mathcal{C}(A)$. Assume the property (**race-free**) of A:

$$\forall y, y_1, y_2 \in \mathcal{C}(A).\ y \stackrel{-}{\longrightarrow\!\!\!\subset} y_1 \ \& \ y \stackrel{+}{\longrightarrow\!\!\!\subset} y_2 \implies y_1 \uparrow y_2\,. \qquad \textbf{(race-free)}$$

Observe that by repeated use of (**race-free**), if $x, y \in \mathcal{C}(A)$ with $x \cap y \subseteq^+ x$ and $x \cap y \subseteq^- y$, then $x \cup y \in \mathcal{C}(A)$.

We show that the game (A, W) is determined. Assuming Player has no winning strategy we build a winning (counter) strategy for Opponent based on the following lemma.

Lemma 8.27. *Assume game A is well-founded and satisfies* (**race-free**). *Let $W \subseteq \mathcal{C}(A)$. Assume (A, W) has no winning strategy (for Player). Then,*

$$\forall x \in \mathcal{C}(A).\ \varnothing \subseteq^+ x \ \& \ x \in W$$
$$\Longrightarrow$$
$$\exists y \in \mathcal{C}(A).\ x \subseteq^- y \ \& \ y \notin W \ \& \ (A, W)/y \textit{ has no winning strategy.}$$

Proof. Suppose otherwise, that under the assumption that (A, W) has no winning strategy, there is some $x \in \mathcal{C}(A)$ such that

$$\varnothing \subseteq^+ x \ \& \ x \in W$$
$$\&$$
$$\forall y \in \mathcal{C}(A).\ x \subseteq^- y \ \& \ y \notin W \implies (A, W)/y \text{ has a winning strategy.}$$

We shall establish a contradiction by constructing a winning strategy for Player.

For each $y \in \mathcal{C}(A)$ with $x \subseteq^- y$ and $y \notin W$, choose a winning strategy

$$\sigma_y : S_y \to A/y\,.$$

By Lemma 8.24, we can replace σ_y by a stable family $\mathcal{F}_y$ with all −ve events in A and a map of stable families $f_y : \mathcal{F}_y \to \mathcal{C}(A)$. It is easy to arrange that, within the collection of all such stable families, $\mathcal{F}_{y_1}$ and $\mathcal{F}_{y_2}$ are disjoint on +ve events whenever y_1 and y_2 are distinct. We build a putative stable family as

$$\begin{aligned}\mathcal{F} =_{\text{def}} &\ \{y \in \mathcal{C}(A) \mid pol_A(y \setminus x) \subseteq \{-\}\} \ \cup \\ &\ \{y \cup v \mid y \in \mathcal{C}(A) \ \& \ pol_A(y \setminus x) \subseteq \{-\} \ \& \ x \cup y \notin W \ \& \\ &\qquad v \in \mathcal{F}_{x \cup y} \ \& \ + \in pol\, v \ \& \ y \cup f_{x \cup y} v \in \mathcal{C}(A)\}\,.\end{aligned}$$

[Note, in the second set-component, that $x \cup y$ is a configuration by (**race-free**).] We assign events of $\mathcal{F}$ the same polarities they have in A and the families $\mathcal{F}_y$.

We check that $\mathcal{F}$ is indeed a stable family.

Clearly $\varnothing \in \mathcal{F}$. Assuming $z_1, z_2 \subseteq z$ in $\mathcal{F}$, we require $z_1 \cup z_2$, $z_1 \cap z_2 \in \mathcal{F}$.

It is easily seen that if both z_1 and z_2 belong to the first set-component, so do their union and intersection. Suppose otherwise, without loss of generality, that z_2 belongs to the second set-component. Then, necessarily, z is in the second set-component of $\mathcal{F}$ and has the form $z = y \cup v$ described there.

Consider the case where $z_1 = y_1 \cup v_1$ and $z_2 = y_2 \cup v_2$, both belonging to the second set-component of $\mathcal{F}$. Then

$$x \cup y_1 = x \cup y_2 = x \cup y\,,$$

from the assumption that families $\mathcal{F}_y$ are disjoint on +ve events for distinct y, and

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x\cup y}\,.$$

It follows that $x \cup (y_1 \cup y_2) = x \cup y \notin W$ and $v_1 \cup v_2 \in \mathcal{F}_{x\cup y} = \mathcal{F}_{x\cup(y_1\cup y_2)}$. As $z_1, z_z \subseteq z$,

$$(y_1 \cup f_{x\cup y} v_1),\ (y_2 \cup f_{x\cup y} v_2) \subseteq (y \cup f_{x\cup y} v)$$

so

$$(y_1 \cup y_2) \cup f_{x\cup y}(v_1 \cup v_2) = (y_1 \cup f_{x\cup y} v_1) \cup (y_2 \cup f_{x\cup y} v_2) \in \mathcal{C}(A)\,.$$

This ensures $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$. Similarly, $x \cup (y_1 \cap y_2) = (x \cup y_1) \cap (x \cup y_2) = x \cup y \notin W$ and $v_1 \cap v_2 \in \mathcal{F}_{x\cup y} = \mathcal{F}_{x\cup(y_1\cap y_2)}$. Checking

$$(y_1 \cap y_2) \cup f_{x\cup y}(v_1 \cap v_2) = (y_1 \cup f_{x\cup y} v_1) \cap (y_2 \cup f_{x\cup y} v_2) \in \mathcal{C}(A)$$

ensures $z_1 \cap z_2 = (y_1 \cap y_2) \cup (v_1 \cap v_2) \in \mathcal{F}$.

Consider the case where $z_1 \in \mathcal{C}(A)$ belongs to the first and $z_2 = y_2 \cup v_2$ to the second set-component of $\mathcal{F}$. As $z_1 \subseteq y \cup v$ it has the form $z_1 = y_1 \cup v_1$ where $y_1 \in \mathcal{C}(A)$ with $y_1 \subseteq y$ and $v_1 \in \mathcal{F}_{x\cup y}$ with $v_1 \subseteq v$; all the events of $v_1 = z_1 \setminus (x \cup y)$ have −ve polarity which ensures $v_1 \in \mathcal{F}_{x\cup y}$ by the receptivity of σ_y. Because v_2 and v have +ve events in common,

$$x \cup y_2 = x \cup y\,,$$

while clearly

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x\cup y}\,.$$

We deduce $x \cup (y_1 \cup y_2) = x \cup y \notin W$ and $v_1 \cup v_2 \in \mathcal{F}_{x\cup y} = \mathcal{F}_{x\cup(y_1\cup y_2)}$ whence $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$ after an easy check that $(y_1 \cup y_2) \cup f_{x\cup y}(v_1 \cup v_2) \in \mathcal{C}(A)$. We have $y_2 \cup f_{x\cup y} v_2 \in \mathcal{C}(A)$. But $f_{x\cup y}$ is constant on −ve events so

$$z_1 \cap z_2 = z_1 \cap (y_2 \cup v_2) = z_1 \cap (y_2 \cup f_{x\cup y} v_2) \in \mathcal{C}(A)\,,$$

and $z_1 \cap z_2$ belongs to the first set-component of $\mathcal{F}$.

A routine check establishes that $\mathcal{F}$ is coincidence-free, and uses that each family $\mathcal{F}_y$ is coincidence-free when considering configurations of the second set-component.

Having established that $\mathcal{F}$ is a stable family, we define a total map of stable families

$$f : \mathcal{F} \to \mathcal{C}(A)$$

by taking

$$f(e) = \begin{cases} e & \text{if } e \in x \text{ or } e \text{ is } -\text{ve}, \\ f_y(e) & \text{if } e \text{ is a } +\text{ve event of } \mathcal{F}_y. \end{cases}$$

Defining σ to be the composite map of stable families

$$\mathcal{C}(\Pr(\mathcal{F})) \xrightarrow{top} \mathcal{F} \xrightarrow{f} \mathcal{C}(A)$$

we also obtain a map of event structures

$$\sigma : \Pr(\mathcal{F}) \to A$$

as the embedding of event structures in stable families is full and faithful. Ascribe to events p of $\Pr(\mathcal{F})$ the same polarities as events $top(p)$ of $\mathcal{F}$. Clearly σ preserves polarities as f does, so σ is a total map of event structures with polarity. In fact, σ is a winning strategy for (A, W).

To show receptivity of σ it suffices to show for all $z \in \mathcal{F}$ that $fz \overset{-}{\longrightarrow\!\!\subset} y'$ in $\mathcal{C}(A)$ implies $z \overset{z}{\longrightarrow\!\!\subset}{}'$ with $\sigma z' = z$ for some unique $z' \in \mathcal{F}$. If z belongs to the first set-component of $\mathcal{F}$ this is obvious—take $z' = y'$. Otherwise z belongs to the second set-component, and takes the form $y \cup v$, when receptivity follows from the receptivity of $\sigma_{x \cup y}$. No extra causal dependencies, over those of A, are introduced into y in the first set-component of $\mathcal{F}$. Considering $y \cup v$ in the second set-component of $\mathcal{F}$, the only extra causal dependencies introduced in $y \cup v$, above those inherited from its image $y \cup f_{x \cup y} v$ in A, are from v in $\mathcal{F}_{x \cup y}$ and those making a +ve event of v in $y \cup v$ depend on −ve events $y \setminus x$. For these reasons σ is also innocent, and a strategy in A.

To show σ is a winning strategy for (A, W) it suffices to show that $fz \in W$ for every +-maximal configuration $z \in \mathcal{F}$. Let z be a +-maximal configuration of $\mathcal{F}$.

Suppose that z belongs to the first set-component of $\mathcal{F}$ and, to obtain a contradiction, that $fz \notin W$. Then $z = fz \in \mathcal{C}(A)$ and $pol\, z \setminus x \subseteq \{-\}$. By axiom **(race-free)**, $x \uparrow y$, so $x \subseteq z$ from the +-maximality of z. As $x \subseteq^- z$ and $z \notin W$ the strategy σ_z is winning in $(A, W)/z$. Because z is +-maximal in $\mathcal{F}$ we must have $\varnothing$ is +-maximal in $\mathcal{F}_z$. It follows that $\varnothing \in W/z$, *i.e.* $z \in W$—a contradiction.

Suppose that z belongs to the second set-component of $\mathcal{F}$, so that z has the form $y \cup v$ with $y \in \mathcal{C}(A)$ and $v \in \mathcal{F}_{x \cup y}$. By **(race-free)**, $x \subseteq y$, as z is +-maximal in $\mathcal{F}$. Hence $v \in \mathcal{F}_y$ and is necessarily +-maximal in $\mathcal{F}_y$, again from the +-maximality of z. As σ_y is winning, $f_y v \in W/y$. Therefore $fz = y \cup f_y v \in W$.

Finally, we have constructed a winning strategy σ in (A, W)—the contradiction required to establish the lemma. □

Remark. In the proof above we could instead build the strategy for Player, on which the proof by contradiction depends, out of a rigid family of finite partial orders. Recall that stable families, including configurations of event structures, are rigid families w.r.t. the order induced on configurations; finite configurations

x determine finite partial orders $(x, \leq_x)$, which we call $q(x)$ in the construction below. Define

$$\begin{aligned}\mathcal{Q} =_{\mathrm{def}} & \{q(y) \mid y \in \mathcal{C}(A) \ \&\ pol_A(y \smallsetminus x) \subseteq \{-\}\} \cup \\ & \{q(y); q(v) \mid y \in \mathcal{C}(A) \ \&\ pol_A(y \smallsetminus x) \subseteq \{-\} \ \&\ x \cup y \notin W \ \& \\ & \qquad v \in \mathcal{F}_{x \cup y} \ \&\ + \in pol\, v \ \&\ y \cup f_{x \cup y} v \in \mathcal{C}(A)\}\end{aligned}$$

where above $q(y); q(v)$ is the least partial order on $y \cup v$ in which events inherit causal dependencies from $q(v)$, from their images in $q(y \cup f_{x \cup y} v)$ and in addition have the causal dependencies $y^- \times v^+$. The family $\mathcal{Q}$ can be shown to be closed under rigid inclusions, and so a rigid family. □

Theorem 8.28. *Assume game A is well-founded, satisfies* (**race-free**) *and has winning conditions $W \subseteq \mathcal{C}(A)$. If (A, W) has no winning strategy for Player, then there is a winning (counter) strategy for Opponent.*

Proof. Assume (A, W) has no winning strategy for Player.

We build a winning counter-strategy for Opponent out of a rigid family of partial orders, themselves constructed from 'alternating sequences' of configurations of A.

Define an *alternating sequence* to be a sequence

$$x_1, y_1, x_2, y_2, \cdots, x_i, y_i, \cdots, x_k, y_k, x_{k+1}$$

of length $k + 1 \geq 1$ of configurations of A such that

$$\varnothing \subseteq^+ x_1 \subseteq^- y_1 \subseteq^+ x_2 \subseteq^- y_2 \subseteq^- \cdots \subseteq^+ x_i \subseteq^- y_i \subseteq^+ \cdots \subseteq^+ x_k \subseteq^- y_k \subseteq^+ x_{k+1}$$

with

$$x_i \in W \ \&\ y_i \notin W \ \&\ (A, W)/y_i \text{ has no winning strategy,}$$

when $1 \leq i \leq k$. It is important that x_{k+1}, which may be $\varnothing$, need not be in W. In particular, we allow the alternating singleton sequence x_1 comprising a single configuration of A with $\varnothing \subseteq^+ x_1$ without necessarily having $x_1 \in W$.

For each alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$ define the partial order $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ to comprise the partial order on x_{k+1} inherited from A together with additional causal dependencies given by the pairs in

$$x_i^+ \times (y_i \smallsetminus x_i)\,, \text{ where } 1 \leq i \leq k.$$

We define $\mathcal{Q}$ to be the rigid family comprising the set of all partial orders got from alternating sequences, closed under rigid inclusions.

Form the event structure $\Pr(\mathcal{Q})$ as described in Proposition 8.19. Assign the same polarity to an event in $\Pr(\mathcal{Q})$ as its top event in A. Recall from Proposition 8.19 the order-isomorphism $\mathcal{C}(\Pr(\mathcal{Q})) \cong \mathcal{Q}$ given by $x \mapsto \bigcup x$ for $x \in \mathcal{C}(\Pr(\mathcal{Q}))$. The map

$$\tau : \Pr(\mathcal{Q}) \to A$$

taking $p \in \mathrm{Pr}(\mathcal{Q})$ to its top event is a total map of event structures with polarity. Writing $T : \mathcal{Q} \to \mathcal{C}(A)$ for the function taking $q \in \mathcal{Q}$ to its set of underlying events, $\tau x = T(\bigcup x)$ for all $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{Q}))$, *i.e.* the diagram

$$\begin{array}{ccc} \mathcal{C}(\mathrm{Pr}(\mathcal{Q})) & \cong & \mathcal{Q} \\ & \underset{\tau}{\searrow} & \downarrow T \\ & & \mathcal{C}(A) \end{array}$$

commutes. We shall reason about order-properties of τ via the function T.

We claim that τ is a winning counter-strategy, in other words a winning strategy for Opponent, in which the roles of + and − are reversed.

Because the construction of the partial orders in $\mathcal{Q}$ only introduces extra causal dependencies of −ve events on +ve events, τ is innocent (remember the reversal of polarities). To check receptivity of τ it suffices to show that for $q \in \mathcal{Q}$ assuming $T(q) \stackrel{a}{-\!\!\!\subset} z'$ in $\mathcal{C}(A)$, where $pol_A(a) = +$, there is a unique $q' \in \mathcal{Q}$ such that $q -\!\!\!\subset q'$ and $T(q') = z'$. Any such extension q' must comprise the partial order q extended by the event a. As a is +ve the events on which it immediately depends in q' will coincide with those on which a immediately depends in z', guaranteeing the uniqueness of q'. It remains to show the existence of q'.

By assumption, q rigidly embeds in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ for some alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$. In the case where q consists of purely +ve events, take $q' =_{\mathrm{def}} Q(z')$. Otherwise, consider the largest i for which $T(q) \cap (y_i \smallsetminus x_i) \neq \varnothing$. Then,

$$pol_A\, T(q) \smallsetminus y_i \subseteq \{+\}\,. \tag{1}$$

From the construction of $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ and the rigidity of the inclusion of q in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ we obtain

$$x_i^+ \subseteq T(q)\,. \tag{2}$$

From (2), $T(q) \subseteq^- T(q) \cup y_i$ and, by assumption, $T(q) \stackrel{a}{-\!\!\!\subset} z'$ with $pol_A(a) = +$. Using (**race-free**), their union remains in $\mathcal{C}(A)$, and we can define

$$x' =_{\mathrm{def}} T(q) \cup y_i \cup \{a\} \in \mathcal{C}(A)\,.$$

Note that

$$x_1, y_1, \cdots, x_i, y_i, x'$$

is an alternating sequence because $y_i \subseteq^+ x'$ by (1) and it is built from an alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$. Restricting $Q(x_1, y_1, \cdots, x_i, y_i, x')$ to events z we obtain a partial order q' for which $q -\!\!\!\subset q'$ in $\mathcal{Q}$ and $T(q') = z$.

We now show that τ is winning for Opponent. For this it suffices to show that if $q \in \mathcal{Q}$ is −-maximal then $T(q) \notin W$. Assume $q \in \mathcal{Q}$ is −-maximal in $\mathcal{Q}$. Necessarily q embeds rigidly in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ for some alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$.

In the case where q consists of purely +ve events

$$\varnothing \subseteq^{+} T(q) \text{ in } \mathcal{C}(A)\,.$$

Suppose $T(q) \in W$. By Lemma 8.27, for some $y \in \mathcal{C}(A)$,

$$T(q) \subseteq^{-} y \ \&\ y \notin W\,.$$

But then there is a strict extension $q \hookrightarrow Q(T(q), y, \varnothing)$ of q by −ve events in $\mathcal{Q}$, and q is not −-maximal—a contradiction.

In the case where q has −ve events, we may take the largest i for which $T(q) \cap (y_i \setminus x_i) \neq \varnothing$. As earlier,

$$(1)\ pol_A\, T(q) \setminus y_i \subseteq \{+\} \quad \& \quad (2)\ x_i^{+} \subseteq T(q)\,.$$

As q is −-maximal, $y_i \subseteq T(q)$, whence by (1),

$$y_i \subseteq^{+} T(q)\,.$$

Suppose, to obtain a contradiction, that $T(q) \in W$. The game $(A, W)/y_i$ has no winning strategy. By Lemma 8.27, given

$$\varnothing \subseteq^{+} x =_{\text{def}} T(q) \setminus y_i$$

in $\mathcal{C}((A, W)/y_i)$ there is $y \in \mathcal{C}((A, W)/y_i)$ with

$$x \subseteq^{-} y \ \ \&\ \ y \notin W/y_i\,.$$

Let $x'_{i+1} =_{\text{def}} T(q)$ and $y'_{i+1} =_{\text{def}} y_i \cup y \notin W$. Then,

$$x_1, y_1, \cdots, x_i, y_i, x'_{i+1}, y'_{i+1}, \varnothing$$

is an alternating sequence which strictly extends q by −ve events, contradicting its −-maximality.

We conclude that τ is a winning strategy for Opponent. □

Corollary 8.29. *If a well-founded game A satisfies* (**race-free**) *then* (A, W) *is determined for any winning conditions W.*

8.7 Satisfaction in the predicate calculus

The syntax for predicate calculus: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\ \phi \mid \forall x.\ \phi$$

where R ranges over basic relation symbols of a fixed arity and $x, x_1, x_2, \cdots, x_k$ over variables.

A model M for the predicate calculus comprises a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M. Following Tarski we can then define by structural induction the truth of a formula of predicate logic w.r.t. an assignment of values in V_M to the variables of the formula. We write

$$\rho \vDash_M \phi$$

iff formula ϕ is true in M w.r.t. environment ρ; we take an environment to be a function from variables to values.

W.r.t. a model M and an environment ρ, we can denote a formula ϕ by $[\![\phi]\!]_M\rho$, a concurrent game with winning conditions, so that $\rho \vDash_M \phi$ iff the game $[\![\phi]\!]_M\rho$ has a winning strategy.

The denotation as a game is defined by structural induction:

$$\begin{aligned}
[\![R(x_1,\cdots,x_k)]\!]_M\rho &= \begin{cases} (\varnothing,\{\varnothing\}) & \text{if } \rho \vDash_M R(x_1,\cdots,x_k)\,, \\ (\varnothing,\varnothing) & \text{otherwise.}\end{cases}\\
[\![\phi \wedge \psi]\!]_M\rho &= [\![\phi]\!]_M\rho \otimes [\![\psi]\!]_M\rho\\
[\![\phi \vee \psi]\!]_M\rho &= [\![\phi]\!]_M\rho \parr [\![\psi]\!]_M\rho\\
[\![\neg\phi]\!]_M\rho &= ([\![\phi]\!]_M\rho)^\perp\\
[\![\exists x.\ \phi]\!]_M\rho &= \bigoplus_{v\in V_M} [\![\phi]\!]_M\rho[v/x]\\
[\![\forall x.\ \phi]\!]_M\rho &= \bigominus_{v\in V_M} [\![\phi]\!]_M\rho[v/x]\,.
\end{aligned}$$

We use $\rho[v/x]$ to mean the environment ρ updated to assign value v to variable x. The game $(\varnothing,\{\varnothing\})$ the unit w.r.t. $\otimes$ is the game used to denote true and the game $(\varnothing,\{\varnothing\})$ the unit w.r.t. $\parr$ to denote false. Denotations of conjunctions and disjunctions are denoted by the operations of $\otimes$ and $\parr$ on games, while negations denote dual games. Universal and existential quantifiers denote *prefixed sums* of games, operations which we now describe.

The prefixed game $\oplus.(A,W)$ comprises the event structure with polarity $\oplus.A$ in which all the events of A are made to causally depend on a fresh +ve event $\oplus$. Its winning conditions are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\}\cup y$ for some $y \in W$. The game $\bigoplus_{v\in V}(A_v,W_v)$ has underlying event structure with polarity the sum (=coproduct) $\sum_{v\in V}\oplus.A_v$ with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. Note in particular that the empty configuration of $\bigoplus_{v\in V} G_v$ is not winning—Player must make a move in order to win. The game $\bigominus_{v\in V} G_v$ is defined dually, as $(\bigoplus_{v\in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move. More explicitly, the prefixed game $\ominus.(A,W)$ comprises the event structure with polarity $\ominus.A$ in which all the events of A are made to causally depend on the previous occurrence of an opponent event $\ominus$, with winning configurations either the empty configuration or of the form $\{\ominus\}\cup y$ where $y \in W$. Writing $G_v = (A_v,W_v)$, the underlying event structure of $\bigominus_{v\in V} G_v$ is the sum $\sum_{v\in V}\ominus.A_v$ with a configuration winning iff it is empty or the image under injection of a winning configuration in a prefixed component.

It is easy to check by structural induction that:

Proposition 8.30. *For any formula ϕ the game $[\![\phi]\!]_M\rho$ is well-founded and race-free (i.e. satisfies Axiom* (**race-free**)*), so a determined game by the result of the last section.*

The following facts are useful for building strategies.

Proposition 8.31.

(i) If $\sigma : S \to A$ is a strategy in A and $\tau : T \to B$ is a strategy in B, then $\sigma \| \tau : S \| T \to A \| B$ is a strategy in $A \| B$.

(ii) If $\sigma : S \to T$ is a strategy in T and $\tau : T \to B$ is a strategy in B, then their composition as maps of event structures with polarity $\tau\sigma : S \to B$ is a strategy in B.

Proof. It is easy to check that the properties of receptivity and innocence are preserved by parallel composition and composition of maps. □

There are 'projection' strategies from a tensor product of games to its components:

Proposition 8.32. *Let $G = (A, W_G)$ and $H = (B, W_H)$ be race-free games with winning conditions. The map of event structures with polarity*

$$\mathrm{id}_{A^\perp} \| \gamma_B : A^\perp \| \mathbb{C}_B \to A^\perp \| B^\perp \| B$$

is a winning strategy $p_H : G \otimes H \twoheadrightarrow H$. The map of event structures with polarity

$$\mathrm{id}_{B^\perp} \| \gamma_A : B^\perp \| \mathbb{C}_A \to B^\perp \| A^\perp \| A \cong A^\perp \| B^\perp \| A$$

is a winning strategy $p_G : G \otimes H \twoheadrightarrow G$.

Proof. By Proposition 8.31, as $\mathrm{id}_{A^\perp}$ is a strategy in $A^\perp$ and γ_B is a strategy in $B^\perp \| B$ the map $p_H = \mathrm{id}_{A^\perp} \| \gamma_B$ is certainly a strategy in $A^\perp \| B^\perp \| B$.

We need to check that p_H is a winning strategy in $G \otimes H \multimap H$. Consider x, a $+$-maximal configuration of $A^\perp \| \mathbb{C}_B$. As B is race-free, the copy-cat strategy γ_B is winning in $H \multimap H$. Consequently if x images to a winning configuration in $G \otimes H$ on the left of $G \otimes H \multimap H$ it will image to a winning configuration in H on the right of $G \otimes H \multimap H$. (Recall a winning configuration of $G \otimes H$ is essentially the union of a winning configuration in G together with a winning configuration in H.) Consequently, x images to a winning configuration in $G \otimes H \multimap H$, as is required for p_H to be a winning strategy.

The strategy p_G is defined analogously but for the isomorphism $B^\perp \| A^\perp \| A \cong A^\perp \| B^\perp \| A$ which does not disturb its winning nature. □

The following lemma is used to build and deconstruct strategies in prefixed sums of games. The lemma concerns the more basic prefixed sums of event structures. These are built as coproducts $\sum_{i \in I} \bullet.B_i$ of event structures $\bullet.B_i$ in which an event $\bullet$ is prefixed to B_i, making all the events in B_i causally depend on $\bullet$.

Lemma 8.33. *Suppose $f : A \to \sum_{i \in I} \bullet.B_i$ is a total map of event structures, with codomain a prefixed sum. Then, A is isomorphic to an prefixed sum, $A \cong \sum_{j \in J} \bullet.A_j$, and there is a function $r : J \to I$ and total maps of event structures $f_j : A_j \to B_{r(j)}$ for which*

$$\begin{array}{ccc} \sum_{j \in J} \bullet.A_j & \cong & A \\ {\scriptstyle [\bullet.f_j]_{j \in J}} \big\downarrow & \swarrow {\scriptstyle f} & \\ \sum_{i \in I} \bullet.B_i & & \end{array}$$

commutes.

Proof. Let J be the subset of events of A whose images are prefix events $\bullet$ in $\sum_{i \in I} \bullet.B_i$. As f is a map of event structures any distinct pairs of events in J are inconsistent. Moreover, every event of A is $\leq_A$-above a necessarily unique event in J. It follows that the events of J are $\leq_A$-minimal with $A \cong \sum_{j \in J} \bullet.A_j$; the event structure A_j is $A/\{j\}$, that part of the event structure strictly above the event j. Each event $j \in J$ is sent to a unique prefix event $f(j)$ in $\sum_{i \in I} \bullet.B_i$. Thus f determines a function $r : J \to I$ and maps $f_j : A_j \to B_{r(i)}$ for all $j \in J$. By construction the map f is reassembled, up to isomorphism, as the unique mediating map $[\bullet.f_j]_{j \in J}$ for which

$$\begin{array}{ccccc} \bullet.A_j & \xrightarrow{in_j^A} & \sum_{j \in J} \bullet.A_j & \cong & A \\ {\scriptstyle \bullet.f_j} \big\downarrow & & {\scriptstyle [\bullet.f_j]_{j \in J}} \big\downarrow & \swarrow {\scriptstyle f} & \\ \bullet.B_{r(j)} & \xrightarrow[in_{r(j)}^B]{} & \sum_{i \in I} \bullet.B_i & & \end{array}$$

commutes for all $j \in J$. □

Lemma 8.34. *Let G, H, G_v, where $v \in V$, be race-free games with winning conditions. Then,*

(i) *$G \otimes H$ has a winning strategy iff G has a winning strategy and H has a winning strategy.*

(ii) *$\bigoplus_{v \in V} G_v$ has a winning strategy iff G_v has a winning strategy for some $v \in V$.*

(iii) *$\ominus_{v \in V} G_v$ has a winning strategy iff G_v has a winning strategy for all $v \in V$.*

If in addition G and H are determined,

(iv) *$G \parr H$ has a winning strategy iff G has a winning strategy or H has a winning strategy.*

Proof. Throughout write $G_v = (A_v, W_v)$, where $v \in V$.

(i) *'Only if':* If $G \otimes H$ has a winning strategy $\sigma : (\varnothing, \{\varnothing\}) \dashrightarrow G \otimes H$, then the compositions $p_G \odot \sigma$ and $p_H \odot \sigma$ provide winning strategies in G and H, respectively. *'If':* If $G = (A, W_G)$ and $H = (B, W_H)$ have winning strategies given as maps of event structures with polarity $\sigma : S \to A$ and $\tau : T \to B$ then the map $\sigma \| \tau : S \| T \to A \| B$ is a winning strategy in $G \otimes H$.

(ii) *'Only if':* Suppose $\sigma : S \to \sum_{v \in V} \oplus.A_v$ is a winning strategy in $\bigoplus_{v \in V} G_v$. As $\varnothing$ is not winning in the game, S must be nonempty. By Lemma 8.33, S decomposes into a prefixed sum necessarily nonempty and of the form $\sum_{j \in J} \oplus.S_j$ with maps, now necessarily total maps of event structures with polarity, $\sigma_j : S_j \to A_{v(j)}$. Because σ is winning any such map will be a winning strategy in $G_{v(j)}$. *'If':* Suppose $\sigma_v : S_v \to A_v$ is a winning strategy in G_v. Prefixing we obtain $\oplus.\sigma_v : \oplus.S_v \to \oplus.A_v$, a winning strategy in $\oplus.G_v$. Composing with the winning 'injection' strategy $In_v : \oplus.G_v \dashrightarrow \sum_{v \in V} \oplus.G_v$ defined below we obtain a winning strategy in $\bigoplus_{v \in V} G_v$. The injection strategy is built from the injection map of event structures with polarity

$$in_v : \oplus.A_v \to \sum_{v \in V} \oplus.A_v .$$

as the composite map

$$In_v : \mathbb{C}_{\oplus.A_v} \xrightarrow{\gamma_{\oplus.A_v}} (\oplus.A_v)^\perp \| \oplus .A_v \xrightarrow{\mathrm{id}_{(\oplus.A_v)^\perp} \| in_v} (\oplus.A_v)^\perp \| \textstyle\sum_{v \in V} \oplus.A_v .$$

Proposition 8.31 is used to show In_v is a strategy. It can be seen that in_v is both receptive and innocent so a strategy in $\sum_{v \in V} \oplus.A_v$. The map $\mathrm{id}_{(\oplus.A_v)^\perp}$ is a strategy. Hence $\mathrm{id}_{(\oplus.A_v)^\perp} \| in_v$ is a strategy. As the composition of two strategy maps, In_v is a strategy in $(\oplus.A_v)^\perp \| \sum_{v \in V} \oplus.A_v$. It is a winning strategy because, as is easily seen from the explicit composite form of In_v, the image under In_v of a +-maximal configuration in $\mathbb{C}_{\oplus.A_v}$ is winning.

(iii) *'Only if':* Defining $P_v =_{\mathrm{def}} In_v^\perp$, where $In_v : \oplus.G_v^\perp \dashrightarrow \bigoplus_{v \in V} G_v^\perp$ is an instance of an injection strategy defined above, we obtain by duality a winning strategy

$$P_v : \bigominus_{v \in V} G_v \dashrightarrow \ominus .G_v ,$$

for any $v \in V$. Let $v \in V$. By composition with P_v a winning strategy in $\bigominus_{v \in V} G_v$ yields a winning strategy in the component $\ominus.G_v$. By Lemma 8.33 in a strategy $\sigma : S \to \ominus.A_v$ the event structure S decomposes into a prefixed sum, where the prefixing events are necessarily all −ve. As σ is receptive the sum must be a unary prefixed sum of the form $\ominus.S'$. Lemma 8.33 provides a map $\sigma' : S' \to A_v$. From σ being winning the map σ' will be a winning strategy in G_v. *'If':* Suppose $\sigma_v : S_v \to A_v$ is a winning strategy in G_v, for all $v \in V$. Prefixing we obtain winning strategies $\ominus.\sigma_v : \ominus.S_v \to \ominus.A_v$ in $\ominus.G_v$. Forming the sum $\sum_{v\in V} \ominus.\sigma_v : \sum_{v\in V} \ominus.S_v \to \ominus.\sigma_v : \sum_{v\in V} \ominus.A_v$ we obtain a strategy winning in $\bigominus_{v\in V} G_v$.

(iv) Now suppose G and H are determined. *'If':* The dual winning strategies $p^{\perp}_{G^{\perp}} : G \twoheadrightarrow G \parr H$ and $p^{\perp}_{H^{\perp}} : H \twoheadrightarrow G \parr H$ compose with a winning strategy $(\varnothing, \{\varnothing\}) \twoheadrightarrow G$, or respectively a winning strategy $(\varnothing, \{\varnothing\}) \twoheadrightarrow H$, to yield a winning strategy $(\varnothing, \{\varnothing\}) \twoheadrightarrow G \parr H$. *'Only if':* Suppose $G \parr H$ has a winning strategy. Then $G^{\perp} \otimes H^{\perp} = (G \parr H)^{\perp}$ has no winning strategy. Hence by (i), $G^{\perp}$ has no winning strategy or $H^{\perp}$ has no winning strategy. From determinacy, G has a winning strategy or H has a winning strategy. □

Theorem 8.35. *For all predicate-calculus formulae ϕ and environments ρ, $\rho \vDash_M \phi$ iff the game $[\![\phi]\!]_M \rho$ has a winning strategy.*

Proof. By Proposition 8.30 the games $[\![\phi]\!]_M \rho$ obtained from formulae ϕ are race-free and determined. The proof is by structural induction on ϕ.

The base case where ϕ is $R(x_1, \cdots, x_k)$ is obvious; the game $(\varnothing, \{\varnothing\})$ has as (unique) winning strategy the map $\varnothing \to \varnothing$, while $(\varnothing, \varnothing)$ has no winning strategy.

For the case $\phi \wedge \psi$, reason

$$\begin{aligned}
\rho \vDash_M \phi \wedge \psi &\iff \rho \vDash_M \phi \ \&\ \rho \vDash_M \psi \\
&\iff [\![\phi]\!]_M \rho \text{ has a winning strategy } \& \ [\![\psi]\!]_M \rho \text{ has a winning strategy, by induction,} \\
&\iff [\![\phi]\!]_M \rho \otimes [\![\psi]\!]_M \rho \text{ has a winning strategy, by Lemma 8.34(i),} \\
&\iff [\![\phi \wedge \psi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\phi \vee \psi$,

$$\begin{aligned}
\rho \vDash_M \phi \vee \psi &\iff \rho \vDash_M \phi \text{ or } \rho \vDash_M \psi \\
&\iff [\![\phi]\!]_M \rho \text{ has a winning strategy or } [\![\psi]\!]_M \rho \text{ has a winning strategy, by induction,} \\
&\iff [\![\phi]\!]_M \rho \parr [\![\psi]\!]_M \rho \text{ has a winning strategy, by Lemma 8.34(iv),} \\
&\iff [\![\phi \wedge \psi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\neg\phi$,

$$\begin{aligned}
\rho \vDash_M \neg\phi &\iff \rho \nvDash_M \phi \\
&\iff [\![\phi]\!]_M \rho \text{ has no winning strategy, by induction,} \\
&\iff ([\![\phi]\!]_M \rho)^{\perp} \text{ has a winning strategy, by determinacy.}
\end{aligned}$$

In the case $\exists x.\ \phi$,

$$\begin{aligned}
\rho \vDash_M \exists x.\phi &\iff \rho[v/x] \vDash_M \phi \text{ for some } v \in V \\
&\iff [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, for some } v \in V, \text{ by induction,} \\
&\iff \bigoplus_{v\in V} [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, by Lemma 8.34(ii),} \\
&\iff [\![\exists x.\phi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\forall x.\ \phi$,

$$\begin{aligned}
\rho \vDash_M \forall x.\phi &\iff \rho[v/x] \vDash_M \phi \text{ for all } v \in V \\
&\iff [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, for all } v \in V\text{, by induction,} \\
&\iff \bigominus_{v \in V} [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, by Lemma 8.34(iii),} \\
&\iff [\![\forall x.\phi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

□

Chapter 9

Borel determinacy

9.1 Introduction

We show the determinacy of concurrent games with Borel sets as winning conditions, provided they are race-free and bounded-concurrent. Both restrictions are necessary. The proof of determinacy of concurrent games proceeds via a reduction to the determinacy of tree games, and the determinacy of these in turn reduces to the determinacy of traditional Gale-Stewart games.

9.2 Tree games and Gale-Stewart games

We introduce tree games as a special case of concurrent games, traditional Gale-Stewart games as a variant, and show how to reduce the determinacy of tree games to that of Gale-Stewart games. Via Martin's theorem for the determinacy of Gale-Stewart games with Borel winning conditions we show that tree games with Borel winning conditions are determined.

9.2.1 Tree games

Definition 9.1. Say E, an event structure with polarity, is *tree-like* iff it is race-free, has empty concurrency relation (so $\leq_E$ forms a forest) and is such that polarities alternate along branches, *i.e.* if $e \rightarrowtriangle e'$ then $pol_E(e) \neq pol_E(e')$.

A *tree game* is (E, W), a concurrent game with winning conditions, in which E is tree-like.

Proposition 9.2. *Let E be a tree-like event structure with polarity. Then, its configurations $\mathcal{C}(E)$ form a tree w.r.t. $\subseteq$. Its root is the empty configuration $\varnothing$. Its (maximal) branches may be finite or infinite; finite sub-branches correspond to finite configurations of E; infinite branches correspond to infinite configurations of E. Its arcs, associated with $x \overset{e}{\longrightarrow\!\!\subset} x'$, are in 1-1 correspondence with events $e \in E$. The events e associated with initial arcs $\varnothing \overset{e}{\longrightarrow\!\!\subset} x$ all share the same* *polarity. Along a branch*

$$\varnothing \stackrel{e_1}{\longrightarrow\!\!\!\!\subset} x_1 \stackrel{e_2}{\longrightarrow\!\!\!\!\subset} x_2 \stackrel{e_3}{\longrightarrow\!\!\!\!\subset} \cdots \stackrel{e_i}{\longrightarrow\!\!\!\!\subset} x_i \stackrel{e_{i+1}}{\longrightarrow\!\!\!\!\subset} \cdots$$

the polarities of the events $e_1, e_2, \ldots, e_i, \ldots$ *alternate.*

Proposition 9.2 gives the precise sense in which 'arc,' 'sub-branch' and 'branch' are synonyms for 'events,' 'configurations' and 'maximal configurations' when an event structure is tree-like. Notice that for a non-empty tree-like event structure with polarity, all the events that can occur initially share the same polarity.

Definition 9.3. We say a a non-empty tree game (E, W) has polarity + or − according as its initial events are +ve or −ve. It is convenient to adopt the convention that the empty game $(\varnothing, \varnothing)$ has polarity +, and the empty game $(\varnothing, \{\varnothing\})$ has polarity −.

Observe that:

Proposition 9.4. *Let* $f : S \to A$ *be a total map of event structures with polarity, where* A *is tree-like. Then,* S *is also tree-like and the map* f *is innocent. The map* f *is a strategy iff it is receptive.*

Proof. As f preserves the concurrency relation, being a map of event structures, S must be tree-like. Innocence of f now follows so that only its receptivity is required for it to be a strategy. □

9.2.2 Gale-Stewart games

For the sake of uniformity we shall present Gale-Stewart games as a slight variant of tree games, a variant in which all maximal configurations of the tree game are infinite, and where Player and Opponent must play to a maximal, infinite configuration.

Definition 9.5. A *Gale-Stewart* game (G, V) comprises

- a tree-like event structure G for which all maximal configurations are infinite, and
- a subset V of infinite configurations—the *winning* configurations.

A *winning strategy* in a Gale-Stewart game (G, V) is a deterministic strategy $\sigma : S \to G$ such that $\sigma x \in V$ for all maximal configurations x of S.

This is not how a Gale-Stewart game and, particularly, a winning strategy in a Gale-Stewart game are traditionally defined. However, because the strategy σ is deterministic it is injective as a map on configurations, so corresponds to the subfamily of configurations $T = \{\sigma x \mid x \in \mathcal{C}^\infty(S)\}$ of $\mathcal{C}^\infty(G)$. The family T forms a subtree of the tree of configurations of G. Its properties, detailed below, reconcile our definition with the traditional one.

Proposition 9.6. *A winning strategy in a Gale-Stewart game* (G, V) *corresponds to a non-empty subset* $T \subseteq \mathcal{C}^\infty(G)$ *such that*

(i) $\forall x, y \in \mathcal{C}^\infty(G).\ \ y \subseteq x \in T \implies y \in T\,,$

(ii) $\forall x, y \in \mathcal{C}(G).\ \ x \in T \ \&\ x \overset{-}{\longrightarrow\!\!\subset} y \implies y \in T\,,$

(iii) $\forall x, y_1, y_2 \in T.\ x \overset{+}{\longrightarrow\!\!\subset} y_1 \ \&\ x \overset{+}{\longrightarrow\!\!\subset} y_2 \implies y_1 = y_2\,,$ *and*

(iv) all $\subseteq$*-maximal members of* T *are infinite and in* V*.*

Proof. Given σ, a winning strategy in the Gale-Stewart game we define T as above. Then, (i) follows because σ is a map of event structures and G is tree-like; (ii) and (iii) follow from σ being receptive and deterministic; (iv) is a consequence of all winning configurations being infinite. Conversely, given T a subfamily of $\mathcal{C}^\infty(G)$ satisfying (i)-(iv) it is a relatively routine matter to construct a tree-like event structure S and map $\sigma : S \to G$ which is a winning strategy in (G, V). □

A Gale-Stewart game (G, V) has a *dual* game $(G, V)^* =_{\text{def}} (G^\perp, V^*)$, where V^* is the set of all maximal configurations in $\mathcal{C}^\infty(G)$ not in V. A winning strategy for Opponent in (G, V) is a winning strategy (for Player) in the dual game $(G, V)^*$.

For any event structure A there is a topology on $\mathcal{C}^\infty(A)$ given by the Scott open subsets. The $\subseteq$-maximal configurations in $\mathcal{C}^\infty(A)$ inherit a sub-topology from that on $\mathcal{C}^\infty(A)$. The Borel subsets of a topological space are those subsets of configurations in the sigma-algebra generated by the Scott open subsets. Donald Martin proved in his celebrated theorem [27] that Gale-Stewart games (G, V) are determined, *i.e.* there is a either a winning strategy for Player or a winning strategy for Opponent, when V is a Borel subset of the maximal configurations of $\mathcal{C}^\infty(A)$.

9.2.3 Determinacy of tree games

We show the determinacy of tree games with Borel winning conditions through a reduction of the determinacy of tree games to the determinacy of Gale-Stewart games.

Let (E, W) be a tree game. We construct a Gale-Stewart game $\mathrm{GS}(E, W) = (G, V)$ and a partial map $\mathit{proj} : G \to E$. The events of G are built as sequences of events in E together with two new symbols δ^- and δ^+ decreed to have polarity $-$ and $+$, respectively; the symbols δ^- and δ^+ represent delay moves by Opponent and Player, respectively.

Precisely, an event of G is a non-empty finite sequence

$$[e_1, \cdots, e_k]$$

of symbols from $E \cup \{\delta^-, \delta^+\}$ where: e_1 has the same polarity as (E, W); polarities alternate along the sequence; and for all subsequences $[e_1, \cdots, e_i]$, with

$i \le k$,

$$\{e_1, \cdots, e_i\} \cap E \in \mathcal{C}(E)\,.$$

The immediate causal dependency relation of G is given by

$$[e_1, \cdots, e_k] \le_G [e_1, \cdots, e_k, e_{k+1}]$$

and consistency by compatibility w.r.t. $\le_G$. Events $[e_1, \cdots, e_k]$ of G have the same polarity as their last entry e_k. It is easy to see that G is tree-like, and that the only maximal configurations are infinite (because of the possibility of delay moves).

The map $\mathit{proj} : G \to E$ takes an event $[e_1, \cdots, e_k]$ of G to e_k if $e_k \in E$, and is undefined otherwise. The winning set V consists of all those infinite configurations x of G for which $\mathit{proj}\, x \in W$.

We have constructed a Gale-Stewart game $\mathrm{GS}(E, W) = (G, V)$. The construction respects the duality on games.

Lemma 9.7. *Letting (E, W) be a tree game,*

$$\mathrm{GS}((E, W)^{\perp}) = (\mathrm{GS}(E, W))^{*}\,.$$

Proof. Directly from the definition of the operation GS. □

Suppose $\sigma : S \to G$ is a winning strategy for (G, V). The composite

$$S \xrightarrow{\ \sigma\ } G \xrightarrow{\ \mathit{proj}\ } E \qquad (F1)$$

is a partial map of event structures with polarity. Letting $D \subseteq S$ be the subset of events on which $\mathit{proj} \circ \sigma$ is defined, the map $\mathit{proj} \circ \sigma$ factors as

$$S \longrightarrow S{\downarrow}D \xrightarrow{\ \sigma_0\ } E \qquad (F2)$$

where: the first partial map acts like the identity on events in D and is undefined otherwise—it sends a configuration $x \in \mathcal{C}^{\infty}(S)$ to $x \cap D \in \mathcal{C}^{\infty}(S{\downarrow}D)$; and σ_0 is the total map that acts like σ on D. We shall show that σ_0 is a (possibly nondeterministic) winning strategy for (E, W).

Lemma 9.8. *The map σ_0 is a winning strategy for (E, W).*

Proof. Write $S_0 =_{\mathrm{def}} S{\downarrow}D$. By Proposition 9.4, for $\sigma_0 : S_0 \to E$ to be a strategy we only require its receptivity. From the construction of G and proj,

$$\mathit{proj}\, x \longrightarrow\!\!\!\!\subset y \text{ in } \mathcal{C}(E) \implies \exists! x' \in \mathcal{C}(G).\ x \longrightarrow\!\!\!\!\subset x' \ \&\ \mathit{proj}\, x' = y\,.$$

This together with the receptivity of σ entails the receptivity of σ_0.

To show σ_0 is winning, suppose z is a +-maximal configuration of S_0; we require $\sigma_0 z \in W$. We will show this by exhibiting an infinite configuration $x \in \mathcal{C}^{\infty}(S)$ such that $x \cap D = z$. Then, according to the factorisation $(F2)$, $x \mapsto z \mapsto \sigma_0 z$, so we will have $\sigma_0 z = \mathit{proj}\, \sigma x$. The configuration x being infinite will ensure $\sigma x \in V$ because σ is winning in the Gale-Stewart game (G, V). By definition, $\sigma x \in V$ implies $proj\, \sigma x \in W$, so $\sigma_0 z \in W$.

It remains to exhibit an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$. When z is infinite this is readily achieved by defining $x =_{\rm def} [z]_S \in \mathcal{C}^\infty(S)$. Suppose z is finite. Define $x_0 =_{\rm def} [z]_S \in \mathcal{C}(S)$, ensuring $x_0 \cap D = z$. We inductively build an infinite chain

$$x_0 \xrightarrow{s_1}\!\!\subset x_1 \xrightarrow{s_2}\!\!\subset \cdots \xrightarrow{s_n}\!\!\subset x_n \xrightarrow{s_{n+1}}\!\!\subset \cdots$$

in $\mathcal{C}(S)$ where all the events s_n are 'delay' moves not in D. Then $x_n \cap D = z$ for all $n \in \omega$. By the definition of a winning strategies in Gale-Stewart games, no x_n can be $\subseteq$-maximal in $\mathcal{C}(S)$. For each Opponent move s_n choose to delay—as we may do by the receptivity of σ. For each Player move s_n we have no choice as only a delay move is possible—otherwise we would contradict the $+$-maximality assumed of z. Taking $x =_{\rm def} \bigcup_n x_n$ produces an infinite configuration $x \in \mathcal{C}^\infty(S)$ such that $x \cap D = z$, as required. □

Corollary 9.9. *Let H be a tree game. If the Gale-Stewart game* GS(H) *has a winning strategy, then H has a winning strategy.*

Theorem 9.10. *Tree games with Borel winning conditions are determined.*

Proof. Assume (E, W) is a tree game where W is a Borel set. Construct GS$(E, W) = (G, V)$ as above. The function *proj*, acting as $x \mapsto proj\, x$ on configurations, is easily seen to be a Scott-continuous function from $\mathcal{C}^\infty(G) \to \mathcal{C}^\infty(E)$. It restricts to a continuous function from the subspace of maximal configurations in $\mathcal{C}^\infty(G)$. Hence V, as the inverse image of W under this restricted function, is a Borel subset. By Martin's Borel-determinacy theorem [27], the game (G, V) is determined, so has either a winning strategy for Player or a winning strategy for Opponent.

Suppose first that GS(E, W) has a winning strategy (for Player). By Corollary 9.9 we obtain a winning strategy for (E, W). Suppose, on the other hand, that GS(E, W) has a winning strategy for Opponent, *i.e.* there is a winning strategy in the dual game GS$(E, W)^*$. By Lemma 9.7, GS$((E, W)^\perp) =$ GS$(E, W)^*$ has a winning strategy. By Corollary 9.9, $(E, W)^\perp$ has a winning strategy, *i.e.* there is a winning strategy for Opponent in (E, W). □

9.3 Race-freeness and bounded-concurrency

Not all games are determined; We have seen the necessity of race-freeness for the determinacy of well-founded games. However, a determinacy theorem holds for well-founded games (games where all configurations are finite) which are **(race – free)**

$$x \xrightarrow{a}\!\!\subset \;\&\; x \xrightarrow{a'}\!\!\subset \;\&\; pol(a) \neq pol(a') \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(Race – free)}$$

However race-freeness is not sufficient to ensure determinacy when the game is not well-founded, as is illustrated in the following example.

Example 9.11. Let A be the event structure with polarity consisting of one positive event $\oplus$ which is concurrent with an infinite chain of alternating negative and positive events, *i.e.* for each i we have both $\oplus\ co\ \oplus_i$ and $\oplus\ co\ \ominus_i$, $i \in \mathbb{N}$,

$$A = \quad \oplus \quad \ominus_1 \rightarrowtriangle \oplus_1 \rightarrowtriangle \ominus_2 \rightarrowtriangle \oplus_2 \rightarrowtriangle \cdots$$

and Borel winning conditions (for Player) given by

$$W = \{\varnothing, \{\ominus_1, \oplus_1\}, ..., \{\ominus_1, \oplus_1, ..., \ominus_i, \oplus_i\}, ..., A\}.$$

So, Player wins if (i) no event is played, or (ii) the event $\oplus$ is not played and the play is finite and finishes in some $\oplus_i$, or (iii) all of the events in A are played. Otherwise, Opponent wins.

Player does not have a winning strategy because Opponent has an infinite family of *spoiler* strategies, not all be dominated by a single strategy of Player. The inclusion maps $\tau_\infty : T_\infty \to A^\perp$ and $\tau_i : T_i \to A^\perp$, $i \in \mathbb{N}$, are strategies for Opponent where $T_\infty^\perp =_{\text{def}} A$ and $T_i^\perp =_{\text{def}} A \smallsetminus \{e' \in A \mid \ominus_i \le e'\}$, for $i \in \mathbb{N}$.

Any strategy for Player that plays $\oplus$ is dominated by some strategy τ_i for Opponent; likewise, any strategy for Player that does not play $\oplus$ and plays only finitely many positive events $\oplus_i$ is also dominated by some strategy τ_i for Opponent. Moreover, a strategy for Player that does not play $\oplus$ and plays all of the events $\oplus_i$ in A is dominated by τ_∞. So, Player does not have a winning strategy in this game. Similarly, Opponent does not have a winning strategy in A because Player has two strategies that cannot be both dominated by any strategy for Opponent. Let $\sigma_{\overline{\oplus}} : S_{\overline{\oplus}} \to A$ and $\sigma_\oplus : S_\oplus \to A$ be strategies for Player such that $S_{\overline{\oplus}} =_{\text{def}} A \smallsetminus \{\oplus\}$ and $S_\oplus =_{\text{def}} A$.

On the one hand, any strategy for Opponent that plays only finitely many (possibly zero) negative events $\ominus_i$ is dominated by $\sigma_{\overline{\oplus}}$; on the other, any strategy for Opponent that plays all of the negative events $\ominus_i$ in A is dominated by $\sigma_\oplus$. Thus neither player has a winning strategy in this game! □

In the above example, to win Player should only make the move $\oplus$ when Opponent has played an infinite number of moves. We can banish such difficulties by insisting that in a game no event is concurrent with infinitely many events of the opposite polarity. This property is called *bounded-concurrency*:

$$\forall y \in \mathcal{C}^\infty(A).\ \forall e \in y.\ \{e' \in y \mid e\ co\ e'\ \&\ pol(e) \ne pol(e')\} \text{ is finite.}$$

$$\textbf{(Bounded – concurrent)}$$

Bounded concurrency is in fact a *necessary* structural condition for determinacy with respect to Borel winning conditions.

Notation 9.12. *For a concurrent game A with configurations y, y', write $max_+(y', y)$ iff y' is $\oplus$-maximal in y, i.e. $y' \xrightarrow{e}\!\subset\ \&\ pol(e) = + \implies e \notin y$; in a dual way, we write $\overline{max}_+(y', y)$ iff y' is not $\oplus$-maximal in y. We use max_- analogously when $pol(e) = -$.*

We show that if a countable, race-free A is not bounded-concurrent, then there is Borel W so that the game (A, W) is not determined. Since A is not bounded-concurrent, there is $y \in \mathcal{C}^\infty(A)$ and $e \in y$ such that e is concurrent with infinitely many events of opposite polarity in y. W.l.o.g. assume that $pol(e) = +$, that $y \setminus \{e\}$ is a configuration and that $y = [e] \cup [\{a \in y \mid pol_A(a) = -\}]$. The following rules determine whether $y' \in \mathcal{C}^\infty(A)$ is in W or L:

1. $y' \supseteq y \Longrightarrow y' \in W$;
2. $y' \subset y$ & $e \in y' \Longrightarrow y' \in L$;
3. $y' \subset y$ & $e \notin y'$ & $max_+(y', y \setminus \{e\})$ & $\overline{max}_-(y', y \setminus \{e\}) \Longrightarrow y' \in W$;
4. $y' \subset y$ & $e \notin y'$ & $\overline{max}_+(y', y \setminus \{e\})$ or $max_-(y', y \setminus \{e\}) \Longrightarrow y' \in L$;
5. $y' \not\supseteq y$ & $(y' \cap y) \subset^- y' \Longrightarrow y' \in W$;
6. $y' \not\supseteq y$ & $(y' \cap y) \subset^+ y' \Longrightarrow y' \in L$;
7. otherwise assign y' (arbitrarily) to W.

No y' is assigned as winning for both Player and Opponent: the implications' antecedents are all pair-wise mutually exclusive.[1] The countability of A is important in showing that W is Borel.

Lemma 9.13. *Let A be a countable race-free game. If A is not bounded-concurrent, then there is Borel $W \subseteq \mathcal{C}^\infty(A)$ such that the game (A, W) is not determined.*

Proof. The set W is Borel because it is defined by clauses such as $y' \subset y$ which have extensions, in this case $\{y' \in \mathcal{C}^\infty(A) \mid y' \subset y\}$, which are Borel sets by virtue of the countability of A. For instance, a clause such as $e \in y'$ has extension

$$\{y' \in \mathcal{C}^\infty(A) \mid e \in y'\} = \widehat{[e]},$$

a basic open set. In general, for $x \in \mathcal{C}(A)$, we use $\widehat{x}$ to denote the basic open set $\{x' \in \mathcal{C}^\infty(A) \mid x \subseteq x'\}$. The clause $y' \supseteq y$, equivalent to $\forall a \in y.\, a \in y'$, has extension

$$\{y' \in \mathcal{C}^\infty(A) \mid y' \supseteq y\} = \bigcap_{a \in y} \widehat{[a]};$$

because A is assumed countable so is y and the intersection is an intersection of countably many open sets. To see that $\{y' \in \mathcal{C}^\infty(A) \mid y' \subset y\}$ is Borel is a bit more complicated. Observe that

$$\{y' \in \mathcal{C}^\infty(A) \mid y' \subset y\} = \bigcap_{a \notin y} (\mathcal{C}^\infty(A) \setminus \widehat{[a]}) \cap \bigcup_{a \in y} (\mathcal{C}^\infty(A) \setminus \widehat{[a]});$$

the big intersection is the extension of $y' \subseteq y$ and the big union that of $\exists a \in y.\, a \notin y'$—because A is assumed countable the intersection and union are countable.

We first show:

[1] The winning conditions W in Example 9.11 are instance of this scheme.

(i) If σ is a winning strategy for Player then y is σ-reachable, *i.e.* $\sigma : S \to A$, there is $x \in \mathcal{C}^\infty(S)$ s.t. $\sigma x = y$.
(ii) If τ is a winning strategy for Opponent then y is τ-reachable.
Write $y_e =_{\text{def}} y \setminus \{e\}$.

(i) This part uses rules (2), (4) and (6). Suppose $\sigma : S \to A$ is a winning strategy for Player. There is a $\subseteq$-maximal configuration of S s.t. $\sigma x_0 \subseteq y$ (via Zorn's lemma). By receptivity, σx_0 is $-$-maximal in y. As σ is winning, there is a $+$-maximal $x \in \mathcal{C}^\infty(S)$ with $x_0 \subseteq^+ x$ and $\sigma x \in W$ (Zorn).

If $\sigma x \supseteq y$ then necessarily $\sigma x \supseteq^+ y$ and by a general property of strategies we obtain y is σ-reachable. For completeness we include the argument. Take $x' =_{\text{def}} \{s \in x \mid \sigma(s) \notin (\sigma x) \setminus y\}$. Suppose $s' \twoheadrightarrow s$ in x. Then

$$\sigma(s') \in (\sigma x) \setminus y \implies \sigma(s) \in (\sigma x) \setminus y$$

by $+$-innocence. Hence its contrapositive, *viz.*

$$\sigma(s) \notin (\sigma x) \setminus y \implies \sigma(s') \notin (\sigma x) \setminus y\,,$$

so that $s \in x'$ implies $s' \in x'$. Thus, being down-closed and consistent, $x' \in \mathcal{C}^\infty(S)$, with $\sigma x' = y$ from the definition of x'.

The remaining case $\sigma x \not\supseteq y$ is impossible. Suppose $x_0 \neq x$, so $x_0 \subset x$. Then we also have $(\sigma x) \cap y \subset^+ \sigma x$, using the $\subseteq$-maximality of x_0. By (6), $\sigma x \in L$—a contradiction. Suppose, on the other hand, that $x_0 = x$. If $e \in \sigma x$, by (2) we obtain the contradiction $\sigma x \in L$. If $e \notin \sigma x$, by (4) we obtain the contradiction $\sigma x \in L$; recall $\sigma x = \sigma x_0$ is $-$-maximal in y so in y_e when $e \notin \sigma x$.

(ii) This part uses rules (1), (3) and (5). Suppose $\tau : T \to A^\perp$ is a winning strategy for Opponent. It is sufficient to show y_e is τ-reachable as then y will also be τ-reachable by receptivity. Assume to obtain a contradiction that y_e is not τ-reachable. Then there is a $\subseteq$-maximal $x_0 \in \mathcal{C}^\infty(T)$ s.t. $\tau x_0 \subseteq y$ (via Zorn's lemma). By assumption, $\tau x_0 \subset y$. By receptivity, τx_0 is $+$-maximal in y_e and necessarily τx_0 is not $-$-maximal in y_e. By (3), $\tau x_0 \in W$. As τ is winning, there is a $-$-maximal $x \in \mathcal{C}^\infty(T)$ with $x_0 \subseteq^- x$ and $\tau x \in L$ (Zorn); from the latter $x_0 \subset x$. We claim that by (1)&(5), $\tau x \subseteq y_e$, contradicting the $\subseteq$-maximality of x_0. To show the claim, suppose to obtain a contradiction that $\tau x \nsubseteq y_e$. Then $\tau x \nsubseteq y$, as e is +ve , so $(\tau x) \cap y \subset^- \tau x$. By (1), $\tau x \not\supseteq y$. Now by (5), $\tau x \in W$, the required contradiction.

To conclude the proof we show there is no winning strategy for either player.

If σ is a winning strategy for Player then by (i) there is $x \in \mathcal{C}^\infty(S)$ s.t. $\sigma x = y$; in particular there is $s_e \in x$ s.t. $\sigma(s_e) = e$. Define the inclusion map $\tau_0 : A^\perp \upharpoonright (\sigma[s_e]_S \cup \{a \in A^\perp \mid pol_A(a) = +\} \hookrightarrow A^\perp$. Then τ_0 s a strategy for Opponent for which there is $y' \in \langle \sigma, \tau_0 \rangle$ with $e \in y'$ and where y' only contains finitely many $-$-events. Either $y' \subset y$ whence $y' \in L$ by (2), or $y' \nsubseteq y$ whereupon $(y' \cap y) \subset^+ y'$ so $y' \in L$ by (6). Hence as τ_0 is a strategy for Opponent not dominated by σ the latter cannot be a winning strategy for Player.

If τ is a winning strategy for Opponent then y is τ-reachable. Define the inclusion map $\sigma_0 : A \upharpoonright (y \cup \{a \in A \mid pol_A(a) = -\} \hookrightarrow A$. Then σ_0 is a strategy for Player for which there is $y' \in \langle \sigma_0, \tau \rangle$ with $y' \supseteq y$. By (1) $y' \in W$, so σ_0 is not dominated by τ, which cannot be a winning strategy for Opponent. □

9.4 Determinacy of concurrent games

We now construct a tree game $\mathrm{TG}(A, W)$ from a concurrent game (A, W). We can think of the events of $\mathrm{TG}(A, W)$ as corresponding to (non-empty) *rounds* of −ve or +ve events in the original concurrent game (A, W). When (A, W) is race-free and bounded-concurrent, a winning strategy for $\mathrm{TG}(A, W)$ will induce a winning strategy for (A, W). In this way we reduce determinacy of concurrent games to determinacy of tree games.

9.4.1 The tree game of a concurrent game

From a concurrent game (A, W) we construct a tree game

$$\mathrm{TG}(A, W) = (\,TA, TW\,)\,.$$

The construction of TA depends on whether $\varnothing \in W$.

In the case where $\varnothing \in W$, define an alternating sequence of (A, W) to be a sequence

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2i} \subset^- x_{2i+1} \subset^+ x_{2i+2} \subset^- \cdots$$

of configurations in $\mathcal{C}^\infty(A)$—the sequence need not be maximal. Define the −ve events of $\mathrm{TG}(W, A)$ to be

$$[\varnothing, x_1, x_2, \ldots, x_{2k-2}, x_{2k-1}\,]\,,$$

finite alternating sequences of the form

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^+ x_{2k-2} \subset^- x_{2k-1}\,,$$

and the +ve events to be

$$[\varnothing, x_1, x_2, \ldots, x_{2k-1}, x_{2k}\,]\,,$$

finite alternating sequences

$$\varnothing \subset^- x_1 \subset^+ x_2 \subset^- \cdots \subset^- x_{2k-1} \subset^+ x_{2k}\,,$$

where $k \geq 1$. The causal dependency relation on TA is given by the relation of initial sub-sequence, with a finite subset of events being consistent iff the events are all initial sub-sequences of a common alternating sequence.

It is easy to see that a configuration of TA corresponds to an alternating sequence, the −ve events of TA matching arcs $x_{2k-2} \subset^- x_{2k-1}$ and the +ve events arcs $x_{2k-1} \subset^+ x_{2k}$. As such, we say a configuration $y \in \mathcal{C}^\infty(TA)$ is winning, and in TW, iff y corresponds to an alternating sequence

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots$$

for which $\bigcup_i x_i \in W$.

In the case where $\varnothing \notin W$, we define an alternating sequence of (A, W) as a sequence

$$\varnothing \subset^+ x_1 \subset^- x_2 \subset^+ \cdots \subset^- x_{2i} \subset^+ x_{2i+1} \subset^- x_{2i+2} \subset^+ \cdots$$

of configurations in $\mathcal{C}^\infty(A)$. In this case, the −ve events of $\mathrm{TG}(W, A)$ are finite alternating sequences ending in x_{2k}, while the +ve events end in x_{2k-1}, for $k \geq 1$. The remaining parts of the definition proceed analogously.

We have constructed a tree game $\mathrm{TG}(A, W)$ from a concurrent game (A, W). The construction respects the duality on games.

Lemma 9.14. *Let (A, W) be a concurrent game.*

$$\mathrm{TG}((A, W)^\perp) = (\mathrm{TG}(A, W))^\perp .$$

Proof. From the construction TG, because alternating sequences

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots$$

in $\mathcal{C}^\infty(A)$ correspond to alternating sequences

$$\varnothing \cdots \subset^- x_i \subset^+ x_{i+1} \subset^- \cdots$$

in $\mathcal{C}^\infty(A^\perp)$. □

Proposition 9.15. *Suppose (A, W) is a bounded-concurrent game. Maximal alternating sequences have one of two forms,*

(i) finite:

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots x_k ,$$

where x_i is finite for all $0 < i < k$ (where possibly x_k is infinite), or

(iii) infinite:

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots ,$$

where each x_i is finite.

Proof. Otherwise, taking the first infinite x_i, within configuration x_{i+1} there would be an event of $x_{i+1} \setminus x_i$ concurrent with infinitely many events of x_i of opposite polarity—contradicting the bounded-concurrency of A. □

9.4.2 Borel determinacy of concurrent games

Now assume that the concurrent game (A,W) is race-free and bounded-concurrent. Suppose that $str : T \to TA$ is a (winning) strategy in the tree game $TG(A,W)$. Note that T is necessarily tree-like. We construct $\sigma_0 : S \to A$, a (winning) strategy in the original concurrent game (A,W). We construct S indirectly, from a prime-algebraic domain $\mathcal{Q}$, built as follows. For technical reasons, in the construction of $\mathcal{Q}$ it is convenient to assume—as can easily be arranged—that

$$A \cap (A \times T) = \varnothing\,.$$

Via *str* a sub-branch

$$\vec{t} = (t_1, \cdots, t_i, \cdots)$$

of T determines a *tagged alternating sequence*

$$\varnothing \;\cdots\; \overset{t_{i-1}}{\subset^-}\; x_{i-1} \;\overset{t_i}{\subset^+}\; x_i \;\overset{t_{i+1}}{\subset^-}\; \cdots$$

where $str(t_i) = [\varnothing, \dots, x_{i-1}, x_i]$. (Informally, the arc t_i is associated with a round extending x_{i-1} to x_i in the original concurrent game.)

Define $q(\vec{t})$ to be the partial order comprising events

$$\bigcup \{(x_i \setminus x_{i-1}) \mid t_i \text{ is a } -\text{ve arc of } \vec{t}\} \,\cup$$
$$\bigcup \{(x_i \setminus x_{i-1}) \times \{t_i\} \mid t_i \text{ is a } +\text{ve arc of } \vec{t}\}$$

—so a copy of the events $\bigcup_i x_i$ but with +ve events tagged by the +ve arc of T at which they occur[2]—with order a copy of that $\bigcup_i x_i$ inherits from A with additional causal dependencies pairs from

$$x_{i-1}^- \;\times\; ((x_i \setminus x_{i-1}) \times \{t_i\})$$

—making the +ve events occur after the −ve events which precede them in the alternating sequence.

Define the partial order $\mathcal{Q}$ as follows. Its elements are partial orders q, not necessarily finite, for which there is a rigid inclusion

$$q \hookrightarrow q(t_1, t_2, \cdots, t_i, \cdots)\,,$$

for some sub-branch $(t_1, t_2, \cdots, t_i, \cdots)$ of T. The order on $\mathcal{Q}$ is that of rigid inclusion. Define the function $\sigma : \mathcal{Q} \to \mathcal{C}^\infty(A)$ by taking

$$\sigma q = \{a \in A \mid a \text{ is } -\text{ve } \& \ a \in q\} \cup \{a \in A \mid \exists t \in T.\ a \text{ is } +\text{ve } \& \ (a,t) \in q\}$$

for $q \in \mathcal{Q}$. We should check that σq is indeed a configuration of A. Clearly, $\sigma q(\vec{t}) = \bigcup_{i \in I} x_i$ where

$$\varnothing \;\cdots\; \overset{t_{i-1}}{\subset^-}\; x_{i-1} \;\overset{t_i}{\subset^+}\; x_i \;\overset{t_{i+1}}{\subset^-}\; \cdots$$

is the tagged alternating sequence determined by $\vec{t} =_{\text{def}} (t_1, \cdots, t_i, \cdots)$. Any q for which there is a rigid inclusion $q \hookrightarrow q(\vec{t})$ will be sent to a sub-configuration of $\bigcup_i x_i$.

[2]It is so that the two components remain disjoint under tagging that we make the technical assumption above.

Proposition 9.16. *Let* $(t_1, \cdots, t_i, \cdots)$ *be a sub-branch of* T*, so corresponding to a configuration* $\{t_1, \cdots, t_i, \cdots\} \in \mathcal{C}^\infty(T)$*. Then,*

$$str\{t_1, \cdots, t_i, \cdots\} \in TW \iff \sigma q(t_1, \cdots, t_i, \cdots) \in W\,.$$

Proof. Let $\vec{t} =_{\text{def}} (t_1, \cdots, t_i, \cdots)$. We have $str(t_i) = [\varnothing, \ldots, x_{i-1}, x_i]$ for some

$$\varnothing \;\cdots\; \subset^- x_{i-1} \subset^+ x_i \subset^- \cdots,$$

an alternating sequence of (A, W). Directly from the definitions of TW, $q(\vec{t})$ and σ,

$$\begin{aligned} str\{\vec{t}\} \in TW &\iff \bigcup_i x_i \in W \\ &\iff \sigma q(\vec{t}) \in W\,. \end{aligned}$$

□

We shall make use of the following proposition.

Proposition 9.17. *For all* $q, q' \in \mathcal{Q}$*, whenever there is an inclusion of the events of* q *in the events of* q' *there is a rigid inclusion* $q \hookrightarrow q'$*.*

Proof. To see this, suppose the events of q are included in the events of q'. To establish the rigid inclusion $q \hookrightarrow q'$ we require that, for all $a \in q, b \in q'$,

$$b \rightarrowtriangle_q a \iff b \rightarrowtriangle_{q'} a\,. \tag{\dagger}$$

However, in the construction of $q(t_1, t_2, \cdots, t_i, \cdots)$ the only immediate dependencies introduced beyond those of A are those of the form $b \rightarrowtriangle (a', t)$, of tagged +ve events on −ve rounds specified earlier in the branch on which the +ve arc t occurs. This property is inherited by q and q' in $\mathcal{Q}$. Thus in checking (†) we can restrict attention to the case where b is −ve and a is +ve and of the form (a', t) for some $a' \in A$ and arc t of T. The arc t determines a sub-branch $t_1, \cdots, t_k = t$ of T and a corresponding tagged alternating sequence

$$\varnothing \;\cdots\; \overset{t_{k-1}}{\subset^-} x_{k-1} \overset{t_k}{\subset^+} x_k\,.$$

So in this case,

$$\begin{aligned} b \rightarrowtriangle_q a &\iff b \text{ is } \le_A\text{-maximal in } x_{k-1}^- \;\&\; a' \text{ is } \le_A\text{-maximal in } x_k \setminus x_{k-1} \\ &\iff b \rightarrowtriangle_{q'} a\,, \end{aligned}$$

which ensures (†), and the proposition. □

Notation 9.18. Proposition 9.17, justifies us in writing $\subseteq$ for the order of $\mathcal{Q}$. We shall also write $q \subseteq^- q'$ when all the events in q' above those of q are −ve, and similarly $q \subseteq^+ q'$ when all the events in q' above those of q are +ve. □

The following lemma is crucial and depends critically on (A, W) being race-free and bounded-concurrent.

Lemma 9.19. *The order* $(\mathcal{Q}, \subseteq)$ *is a prime algebraic domain in which the primes are precisely those (necessarily finite) partial orders with a maximum.*

Proof. Any compatible finite subset X of $\mathcal{Q}$ has a least upper bound: if all the members of X include rigidly in a common q then taking the union of their images in q, with order inherited from q, provides their least upper bound. Provided $\mathcal{Q}$ has least upper bounds of directed subsets it will then be consistently complete with the additional property that every $q \in \mathcal{Q}$ is the least upper bound of the primes below it—this will make $\mathcal{Q}$ a prime algebraic domain.

To establish prime algebraicity it remains to show that $\mathcal{Q}$ has least upper bounds of directed sets.

Let S be a directed subset of $\mathcal{Q}$. The +ve events of orders $q \in S$ are tagged by +ve arcs of T. Because S is directed the +ve tags which appear throughout all $q \in S$ must determine a common sub-branch of T, *viz.*

$$\vec{t} =_{\rm def} (t_1, t_2, \cdots, t_i, \cdots) .$$

Every +ve arc of the sub-branch appears in some $q \in S$ and all −ve arcs are present only by virtue of preceding a +ve arc. The sub-branch $\vec{t}$ may be

(1) infinite and necessarily a full branch of T, if the elements of S together mention infinitely many tags;

(2) finite with $q(\vec{t})$ infinite, and necessarily finishing with a +ve arc;

(3) finite and non-empty with $q(\vec{t})$ finite, and necessarily finishing with a +ve arc; or

(4) empty with $\vec{t} = ()$.

(1) Consider the case where $\vec{t}$ forms an infinite branch of T. We shall argue that for all $q \in S$, there is a rigid inclusion

$$q \hookrightarrow q(\vec{t}) .$$

Then, forming the partial order $\bigcup S$ comprising the union of the events of all $q \in S$ with order the restriction of that on $q(\vec{t})$ we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(\vec{t}) ,$$

so a least upper bound of S in $\mathcal{Q}$.

Let $q \in S$. By Proposition 9.17, to establish the rigid inclusion $q \hookrightarrow q(\vec{t})$ it suffices to show the events of q are included in those of $q(\vec{t})$. From the nature of the sub-branch determined by S, we must have that all the +ve events of q are included in those of $q(\vec{t})$—all +ve events of q are tagged by a +ve arc of $\vec{t}$. Suppose, to obtain a contradiction, that there is some −ve event a of q not in $q(\vec{t})$. For every +ve arc t_i in $\vec{t}$ there is $q_i \in S$ with a +ve tagged event (a_i, t_i). Let

$$I \subseteq_{\rm fin} \{i \mid t_i \text{ is a +ve arc of } \vec{t}\} .$$

As S is directed, there is an upper bound in S of

$$\{q\} \cup \{q_i \mid i \in I\}\,.$$

It follows that

$$\{a\} \cup \{a_i \mid i \in I\} \in \mathrm{Con}_A\,,$$

Hence, forming the down-closure in A of $\{a\} \cup \{a_i \mid t_i \text{ is a +ve arc in } \vec{t}\}$, we obtain

$$[\{a\} \cup \{a_i \mid t_i \text{ is a +ve arc in } \vec{t}\}] \in \mathcal{C}^\infty(A)\,.$$

Moreover it is a configuration which violates the assumption of bounded-concurrency—the −ve event a is concurrent with infinitely many of the +ve events a_i. From this contradiction we deduce that the events of q are included in the events of $q(\vec{t})$.

(2) Consider the case where $\vec{t}$ is a finite branch $(t_1, \cdots, t_k)$, where necessarily t_k is a +ve arc, and where $q(\vec{t})$ is infinite. By bounded-concurrency, all $q(t_1, \cdots, t_i)$, for $0 \le i < k$, are finite with only $q(\vec{t}) = q(t_1, \cdots, t_k)$ infinite.

Let $q \in S$. By Proposition 9.17, we can show there is a rigid inclusion

$$q \hookrightarrow q(\vec{t})$$

by showing all the events of q are in $q(\vec{t})$. Again, all the +ve events of q are in $q(\vec{t})$. Suppose, to obtain a contradiction, that $b \in q$ with $b \notin q(\vec{t})$, so b has to be −ve. There is a member of S with an event tagged by t_k. Thus, using the directedness of S, there has to be $q_1 \in S$ with $q \subseteq q_1$ and where q_1 has an event tagged by t_k. Because of the extra dependencies introduced in the construction of $q(\vec{t})$, all the −ve events of $q(\vec{t})$ are included in q_1. Note in addition that

$$[q_1^+] \subseteq q(\vec{t})$$

because all the +ve events of q_1 are in $q(\vec{t})$. We deduce

$$[q_1^+] \subseteq^+ q(\vec{t})\,. \qquad (i)$$

Also,

$$[q_1^+] \subset^- q_1\,, \qquad (ii)$$

where the inclusion has to be strict because $b \in q_1 \setminus q(\vec{t})$. Consider the images of (i) and (ii) in $\mathcal{C}^\infty(A)$:

$$\sigma[q_1^+] \subseteq^+ \sigma q(\vec{t}) \quad \text{and} \quad \sigma[q_1^+] \subset^- \sigma q_1\,.$$

As A is race-free, we obtain the configuration $x =_{\mathrm{def}} \sigma q(\vec{t}) \cup \sigma q_1 \in \mathcal{C}^\infty(A)$ and the strict inclusion

$$\sigma q(\vec{t}) \subset^- x\,,$$

making x a configuration which contains the −ve event b concurrent with infinitely many +ve events—the images of those tagged by t_k. But this contradicts the bounded-concurrency of A. Hence all the events of q are in $q(\vec{t})$.

As in case (1) we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(\vec{t}) \,,$$

and a least upper bound of S in $\mathcal{Q}$.

(3) The case where $\vec{t}$ is a non-empty finite branch $(t_1, \cdots, t_k)$ and $q(\vec{t})$ is finite. Again, t_k is necessarily a +ve arc. As S is directed, the set of events $\bigcup_{q \in S} \sigma q$ is a configuration in $\mathcal{C}^\infty(A)$. Again, all the +ve events of any $q \in S$ are in $q(\vec{t})$, from which it follows that as sets,

$$(\bigcup_{q \in S} \sigma q)^+ \subseteq \sigma q(\vec{t}) \,.$$

Hence, the down-closure

$$[(\bigcup_{q \in S} \sigma q)^+]_A \subseteq \sigma q(\vec{t}) \text{ in } \mathcal{C}^\infty(A) \,. \qquad (iii)$$

There is $q_1 \in S$ with an event tagged by t_k. Because of the extra dependencies introduced in the construction of $q(\vec{t})$, all the −ve events of $q(\vec{t})$ are included in q_1. Consequently, all the −ve events of $\sigma q(\vec{t})$ are included in $\bigcup_{q \in S} \sigma q$. From this and (iii) we deduce

$$[(\bigcup_{q \in S} \sigma q)^+] \subseteq^+ \sigma q(\vec{t}) \text{ in } \mathcal{C}^\infty(A) \,. \qquad (iv)$$

Also, straightforwardly,

$$[(\bigcup_{q \in S} \sigma q)^+] \subseteq^- \bigcup_{q \in S} \sigma q \text{ in } \mathcal{C}^\infty(A) \,. \qquad (v)$$

From (iv) and (v), because A is race-free, we obtain the configuration

$$y =_{\text{def}} (\sigma q(\vec{t}) \cup \bigcup_{q \in S} \sigma q) \in \mathcal{C}^\infty(A)$$

for which

$$\sigma q(\vec{t}) \subseteq^- y \in \mathcal{C}^\infty(A) \,.$$

But by receptivity of the original strategy $str : T \to TA$, there is a unique extension of the branch $\vec{t} = (t_1, \cdots, t_k)$ to $(t_1, \cdots, t_k, t_{k+1})$ in T such that

$$\sigma q(t_1, \cdots, t_k, t_{k+1}) = y \,.$$

W.r.t. this extension, forming the partial order $\bigcup S$ comprising the union of the events of all $q \in S$ with order the restriction of that on $q(t_1, \cdots, t_k, t_{k+1})$, we obtain a rigid inclusion

$$\bigcup S \hookrightarrow q(t_1, \cdots, t_k, t_{k+1}) \,,$$

so a least upper bound of S in $\mathcal{Q}$.

(4) Finally, consider the case where $\vec{t} = ()$. Then all $q \in S$ consist purely of −ve events. As S is directed, $\bigcup_{q\in S} \sigma q \in \mathcal{C}^\infty(A)$. If $\bigcup_{q\in S} \sigma q = \varnothing$ we have $\bigcup S = q()$. Assume $\bigcup_{q\in S} \sigma q$ is non-empty.

Suppose first that $\varnothing \in W$. We can form the alternating sequence

$$\varnothing \subset^- \bigcup_{q\in S} \sigma q\,.$$

By the receptivity of $str : T \to TA$ there is a unique 1-arc branch (t_1) of T with $\bigcup_{q\in S} \sigma q = \sigma q(t_1)$. Then $\bigcup S = q(t_1)$.

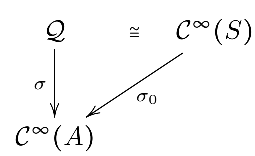
Now suppose $\varnothing \notin W$. In this case all alternating sequences must begin $\varnothing \subset^+ x_1 \cdots$ and consequently all initial arcs of T must be +ve. We are assuming $\bigcup_{q\in S} \sigma q$ is non-empty so contains some non-empty q. There must therefore be a rigid inclusion $q \hookrightarrow q(\vec{u})$ for some non-empty sub-branch $\vec{u} = (u_1, \cdots)$. Via str the sub-branch $\vec{u}$ determines the alternating sequence $\varnothing \subset^+ x_1 \subset^- \cdots$. Noting $\varnothing \subset^- \bigcup_{q\in S} \sigma q$, because A is race-free there is $x_1 \cup \bigcup_{q\in S} \sigma q \in \mathcal{C}^\infty(A)$. Form the alternating sequence

$$\varnothing \subset^+ x_1 \subset^- x_1 \cup \bigcup_{q\in S} \sigma q\,.$$

From the receptivity of str there is a sub-branch (u_1, u_2') such that $x_1 \cup \bigcup_{q\in S} \sigma q = \sigma q(u_1, u_2')$. We obtain $\bigcup S \hookrightarrow q(u_1, u_2')$. □

Definition 9.20. Define S to be the event structure with polarity, with events the primes of $\mathcal{Q}$; causal dependency the restriction of the order on $\mathcal{Q}$; with a finite subset of events consistent if they include rigidly in a common element of $\mathcal{Q}$. The polarity of event of S is the polarity in A of its top element (recall the event is a prime in $\mathcal{Q}$). Define $\sigma_0 : S \to A$ to be the function which takes a prime with top element an untagged event $a \in A$ to a and top element a tagged event (a, t) to a.

Lemma 9.21. *The function which takes $q \in \mathcal{Q}$ to the set of primes below q in $\mathcal{Q}$ gives an order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$. The function $\sigma_0 : S \to A$ is a strategy for which*



commutes.

Proof. The isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ is established in [2]. The diagram is easily seen to commute. Via the order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ we can carry out the argument that σ_0 is a strategy in terms of $\mathcal{Q}$ and σ. Innocence follows because the only additional causal dependencies introduced in $q(\vec{t})$ are of +ve events on −ve events. To show receptivity, suppose $q \in \mathcal{Q}$ is finite and $\sigma q \subset^- y$ in $\mathcal{C}(A)$.

There is a rigid inclusion $q \hookrightarrow q(\vec{t})$ for some $\vec{t} = (t_1, \cdots, t_i, \cdots)$, a sub-branch of T. Let

$$\varnothing \ \cdots \ \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

be the tagged sequence determined by $\vec{t}$.

First consider when $(\sigma q)^+ \neq \varnothing$. Suppose x_k is the earliest configuration at which $(\sigma q)^+ \subseteq x_k$. Then, t_k has to be +ve and

$$q^+ \cap ((x_k \smallsetminus x_{k-1}) \times \{t_k\}) \neq \varnothing .$$

The latter entails

$$x_k^- \subseteq \sigma q$$

because of the extra causal dependencies introduced in the definition of $q(\vec{t})$. It follows that

$$(\sigma q) \cap x_k \subseteq^+ x_k .$$

Moreover, as $(\sigma q)^+ \subseteq x_k$, we deduce

$$(\sigma q) \cap x_k \subseteq^- \sigma q \subseteq^- y .$$

By race-freeness, $x_k \cup y \in \mathcal{C}(A)$ with

$$x_k \subseteq^- x_k \cup y \text{ in } \mathcal{C}(A) .$$

In fact $x_k \subset^- x_k \cup y$ as $x_k^- \subseteq \sigma q \subset^- y$. Now

$$\varnothing \cdots \ \subset^+ x_k \subset^- x_k \cup y$$

is seen to form an alternating sequence, so a sub-branch of *TA*. From the receptivity of *str* there is a unique sub-branch $t_1, \ldots, t_k, t'_{k+1}$ of T which has this alternating sequence as image. Take q' to be the down-closure of y in $q(t_1, \ldots, t_k, t'_{k+1})$. This gives the unique q' such that $q \subseteq q'$ and $\sigma q' = y$.

Now consider when $(\sigma q)^+ = \varnothing$. Then $\varnothing \subseteq^- \sigma q \subset^- y$.

In the case where $\varnothing \in W$ we may form the alternating sequence

$$\varnothing \subset^- y .$$

The receptivity of *str* ensures there is a unique 1-arc branch (u_1) of T such that $\sigma q(u_1) = y$.

In the case where $\varnothing \notin W$ we also have $\varnothing \notin TW$. In this case all alternating sequences must begin $\varnothing \subset^+ x_1 \cdots$ and consequently all initial arcs of T must be +ve. Also, the empty configuration (or branch) of T cannot be +-maximal because its image under *str* is the empty configuration (or branch) of TW—impossible because *str* is a winning strategy. Thus there must be v_1, an initial, necessarily +ve arc of T. Via *str* the sub-branch (v_1) yields the alternating sequence $\varnothing \subset^+ x_1$, say. As A is race-free we obtain $x_1 \cup y \in \mathcal{C}^\infty(A)$ and the alternating sequence

$$\varnothing \subset^+ x_1 \subset^- x_1 \cup y .$$

From the receptivity of *str* there is a unique sub-branch (v_1, v_2) of T for which $\sigma q(v_1, v_2) = x_1 \cup y$. Take q' to be the down-closure of y in $q(v_1, v_2)$. This furnishes the unique q' such that $q \subseteq q'$ and $\sigma q' = y$.

We have shown the receptivity of σ, as required. □

Theorem 9.22. *Suppose that $str : T \to TA$ is a winning strategy in the tree game* $\mathrm{TG}(A, W)$. *Then $\sigma_0 : S \to A$ is a winning strategy in (A, W).*

Proof. For σ_0 to be winning we require that $\sigma_0 x \in W$ for any +-maximal $x \in \mathcal{C}^\infty(S)$. Via the order isomorphism $\mathcal{Q} \cong \mathcal{C}^\infty(S)$ we can carry out the proof in $\mathcal{Q}$ rather than $\mathcal{C}^\infty(S)$. For any q which is +-maximal in $\mathcal{Q}$ (*i.e.* whenever $q \subseteq^+ q'$ in $\mathcal{Q}$ then $q = q'$) we require that $\sigma q \in W$.

Let q be +-maximal in $\mathcal{Q}$. We will show that $q = q(\vec{u})$ for some +-maximal branch $\vec{u}$ of T. Certainly there is a rigid inclusion $q \hookrightarrow q(\vec{t})$ for some sub-branch $\vec{t} = (t_1, \cdots, t_i, \cdots)$ of T. Let

$$\varnothing \ \cdots \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots$$

be the tagged sequence determined by $\vec{t}$.

Consider the case in which the set q^+ is infinite. There are two possibilities. Suppose first that

$$q^+ \cap ((x_i \setminus x_{i-1}) \times \{t_i\}) \neq \varnothing .$$

for infinitely many +ve t_i. Because of the extra causal dependencies introduced in the definition of $q(\vec{t})$, the set of −ve events $q(\vec{t})^-$ is included in q. Hence $q \subseteq^+ q(\vec{t})$. But q is +-maximal, so $q = q(\vec{t})$. The second possibility is that $(\sigma q)^+ \subseteq x_k$ for some necessarily terminal configuration in the tagged alternating sequence, which now has to be of the form

$$\varnothing \ \cdots \overset{t_{i-1}}{\subset^-} x_{i-1} \overset{t_i}{\subset^+} x_i \overset{t_{i+1}}{\subset^-} \cdots \subset^+ x_k .$$

Because of the causal dependencies in $q(\vec{t})$, the set $q(\vec{t})^-$ is included in q. Hence $q \subseteq^+ q(\vec{t})$, so $q = q(\vec{t})$ because q is +-maximal.

Now consider the case where the set q^+ is finite. Then the set $(\sigma q)^+$, also finite, must be included in some x_k of the tagged alternating sequence, which we may assume is the earliest. Then t_k must be +ve. If $\sigma q \subseteq q(t_1, \cdots, t_k)$, then the set $q(t_1, \cdots, t_k)^-$ is included in q—again because of the causal dependencies there; and again $q \subseteq^+ q(t_1, \cdots, t_k)$ so $q = q(t_1, \cdots, t_k)$ because q is +-maximal. Otherwise, $x_k \subset^- x_k \cup (\sigma q)$ and we can extend the alternating sequence to

$$\varnothing \cdots \ \subset^+ x_k \subset^- x_k \cup (\sigma q) .$$

From the receptivity of *str* there is a sub-branch $t_1, \ldots, t_k, t'_{k+1}$ of T which has this alternating sequence as image. Now $q \subseteq^+ q(t_1, \ldots, t_k, t'_{k+1})$ so $q = q(t_1, \ldots, t_k, t'_{k+1})$ from the +-maximality of q.

Thus any $q \in \mathcal{Q}$ which is +-maximal has the form $q = q(\vec{u})$ for some sub-branch $\vec{u}$ of T. Any extension of $\vec{u}$ by a +-ve arc would yield a +-ve extension of $q(\vec{u})$, contradicting the +-maximality of q. Therefore $\vec{u}$ is +-maximal, so its image $str\{\vec{u}\}$ is in TW, as str is a winning strategy in $(TG(A,W), TW)$. But, by Proposition 9.16,

$$str\{\vec{u}\} \in TW \iff \sigma q(\vec{u}) \in W\,.$$

Hence, $\sigma q \in W$, as required. □

Corollary 9.23. *Let (A,W) be a race-free, bounded-concurrent game. If the tree game* $\mathrm{TG}(A,W)$ *has a winning strategy, then (A,W) has a winning strategy.*

Theorem 9.24. *Any race-free, concurrent-bounded game (A,W), in which W is a Borel subset of $\mathcal{C}^\infty(A)$, is determined.*

Proof. Assuming (A,W) is race-free, concurrent-bounded and W is Borel, we obtain a tree game $\mathrm{TG}(A,W) = (TA, TW)$ in which TW is also Borel. To see that TW is Borel, recall that a configuration y of TA corresponds to an alternating sequence

$$\varnothing \cdots \subset^+ x_i \subset^- x_{i+1} \subset^+ \cdots,$$

so determines $f(y) =_{\mathrm{def}} \bigcup_i x_i \in \mathcal{C}^\infty(A)$. This yields a Scott-continuous function $f : \mathcal{C}^\infty(TA) \to \mathcal{C}^\infty(A)$. The set TW is the inverse image $f^{-1}W$, so Borel. As the tree game $\mathrm{TG}(A,W)$ is determined—Theorem 9.10—we obtain a winning strategy for Player or a winning strategy for Opponent in the tree game.

Suppose first that $\mathrm{TG}(A,W)$ has a winning strategy (for Player). By Corollary 9.23 we obtain a winning strategy for (A,W). Suppose, on the other hand, that $\mathrm{TG}(A,W)$ has a winning strategy for Opponent, *i.e.* there is a winning strategy in the dual game $(\mathrm{TG}(A,W))^\perp$. By Lemma 9.14, $\mathrm{TG}((A,W)^\perp) = \mathrm{TG}(A,W)^\perp$ has a winning strategy. By Corollary 9.23, $(A,W)^\perp$ has a winning strategy, *i.e.* there is a winning strategy for Opponent in (A,W). □

Chapter 10

Games with imperfect information

10.1 Motivation

Consider the game "rock, scissors, paper" in which the two participants Player and Opponent independently sign one of r ("rock"), s ("scissors") or p ("paper"). The participant with the dominant sign w.r.t. the relation

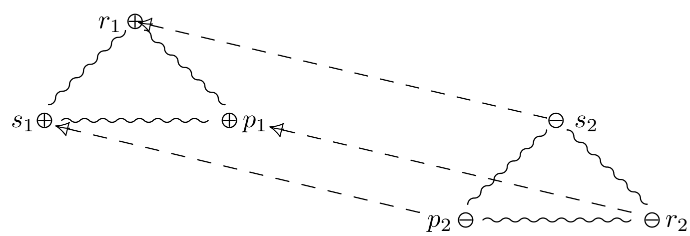
$$r \textit{ beats } s, \; s \textit{ beats } p \text{ and } p \textit{ beats } r$$

wins. It seems sensible to represent this game by RSP, the event structure with polarity

comprising the three mutually inconsistent possible signings of Player in parallel with the three mutually inconsistent signings of Opponent. In the absence of neutral configurations, a reasonable choice is to take the *losing* configurations (for Player) to be

$$\{s_1, r_2\}, \; \{p_1, s_2\}, \; \{r_1, p_2\}$$

and all other configurations as winning for Player. In this case there is a winning strategy for Player, *viz.* await the move of Opponent and then beat it with a dominant move. Explicitly, the winning strategy $\sigma : S \to RSP$ is given as the obvious map from S, the following event structure with polarity:



But this strategy cheats. In "rock, scissors, paper" participants are intended to make their moves *independently*. The problem with the game RSP as it stands is that it is a game of *perfect information* in the sense that all moves are visible to both participants. This permits the winning strategy above with its unwanted dependencies on moves which should be unseen by Player. To adequately model "rock, scissors, paper" requires a game of *imperfect information* where some moves are masked, or inaccessible, and strategies with dependencies on unseen moves are ruled out.

10.2 Games with imperfect information

We extend concurrent games to games with imperfect information. To do so in way that respects the operations of the bicategory of games we suppose a fixed preorder of *levels* $(\Lambda, \leq)$. The levels are to be thought of as levels of access, or permission. Moves in games and strategies are to respect levels: moves will be assigned levels in such a way that a move is only permitted to causally depend on moves at equal or lower levels; it is as if from a level only moves of equal or lower level can be seen.

An Λ-game (G, l) comprises a game $G = (A, W, L)$ with winning/losing conditions together with a *level function* $l : A \to \Lambda$ such that

$$a \leq_A a' \implies l(a) \leq l(a')$$

for all $a, a' \in A$. A Λ-strategy in the Λ-game (G, l) is a strategy $\sigma : S \to A$ for which

$$s \leq_S s' \implies l\sigma(s) \leq l\sigma(s')$$

for all $s, s' \in S$.

For example, for "rock, scissors, paper" we can take Λ to be the discrete preorder consisting of levels 1 and 2 unrelated to each other under $\leq$. To make RSP into a suitable Λ-game the level function l takes +ve events in RSP to level 1 and −ve events to level 2. The strategy above, where Player awaits the move of Opponent then beats it with a dominant move, is now disallowed because it is not a Λ-strategy—it introduces causal dependencies which do not respect levels. If instead we took Λ to be the unique preorder on a single level the Λ-strategies would coincide with all the strategies.

10.2.1 The bicategory of Λ-games

The introduction of levels meshes smoothly with the bicategorical structure on games.

For a Λ-game (G, l_G), define its dual $(G, l_G)^\perp$ to be $(G^\perp, l_{G^\perp})$ where $l_{G^\perp}(\overline{a}) = l_G(a)$, for a an event of G.

For Λ-games (G, l_G) and (H, l_H), define their parallel composition $(G, l_G) \| (H, l_H)$ to be $(G \| H, l_{G\|H})$ where $l_{G\|H}((1,a)) = l_G(a)$, for a an event of G, and $l_{G\|H}((2,b)) = l_H(b)$, for b an event of H.

A strategy between Λ-games from (G, l_G) to (H, l_H) is a strategy in $(G, l_G)^\perp \| (H, l_H)$.

Proposition 10.1.
(i) Let (G, l_G) be a Λ-game where G satisfies **(Cwins)**. *The copy-cat strategy on G is a Λ-strategy.*
(ii) The composition of Λ-strategies is a Λ-strategy.

Proof. (i) The additional causal links introduced in the construction of the copy-cat strategy are between complementary events in $G^\perp$ and G, at the same level in Λ, and so respect $\leq$.

(ii) Let (G, l_G), (H, l_H) and (K, l_K) be Λ-games. Let $\sigma : G \longrightarrow\!\!\!\!\!\!+\,\, H$ and $\tau : H \longrightarrow\!\!\!\!\!\!+\,\, K$ be Λ-strategies. We show their composition $\tau \odot \sigma$ is a Λ-strategy.

It suffices to show $p \rightarrowtriangle p'$ in $T \odot S$ implies $l_{G^\perp\|K} \tau\odot\sigma(p) \leq l_{G^\perp\|K} \tau\odot\sigma(p')$. Suppose $p \rightarrowtriangle p'$ in $T \odot S$ with $top(p) = e$ and $top(p') = e'$. Take $x \in \mathcal{C}(T \odot S)$ containing p' so p too. Then,

$$e \rightarrowtriangle_{\bigcup x} e_1 \rightarrowtriangle_{\bigcup x} \cdots \rightarrowtriangle_{\bigcup x} e_{n-1} \rightarrowtriangle_{\bigcup x} e'$$

where $e, e' \in V_0$ and $e_i \notin V_0$ for $1 \leq i \leq n-1$. (V_0 consists of 'visible' events of the stable family, those of the form $(s, *)$ with $\sigma_1(s)$ defined, or $(*, t)$, with $\tau_2(t)$ defined.) The events e_i have the form (s_i, t_i) where $\sigma_2(s_i) = \tau_1(t_i)$, for $1 \leq i \leq n-1$.

Any individual link in the chain above has one of the forms:

$$(s,t) \rightarrowtriangle_{\bigcup x} (s',t')\,,\ (s,*) \rightarrowtriangle_{\bigcup x} (s',t')\,,$$
$$(*,t) \rightarrowtriangle_{\bigcup x} (s',t')\,,\ (s,t) \rightarrowtriangle_{\bigcup x} (s',*)\,,\ \text{ or }\ (s,t) \rightarrowtriangle_{\bigcup x} (*,t')\,.$$

By Lemma 3.21, for any link either $s \rightarrowtriangle_S s'$ or $t \rightarrowtriangle_T t'$. As σ and τ are Λ-strategies, this entails

$$l_{G^\perp\|H} \sigma(s) \leq l_{G^\perp\|H} \sigma(s') \ \text{ or }\ l_{H^\perp\|K} \tau(t) \leq l_{H^\perp\|K} \tau(t')$$

for any link. Consequently $\leq$ is respected across the chain and $l_{G^\perp\|K} \tau\odot\sigma(p) \leq l_{G^\perp\|K} \tau\odot\sigma(p')$, as required. □

W.r.t. a particular choice of access levels $(\Lambda, \leq)$ we obtain a bicategory $\mathbf{WGames}_\Lambda$. Its objects are Λ-games (G, l) where G satisfies **(Cwins)** with arrows the Λ-strategies and 2-cells maps of spans. It restricts to a sub-bicategory of deterministic Λ-strategies, which as before is equivalent to an order-enriched category.

10.3 Hintikka's IF logic

We present a variant of Hintikka's Independence-Friendly (IF) logic and propose a semantics in terms of concurrent games with imperfect information. Assume a preorder $(\Lambda, \leq)$. The syntax for IF logic is essentially that of the predicate calculus, but with levels in Λ associated with quantifiers: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists^\lambda x.\ \phi \mid \forall^\lambda x.\ \phi$$

where $\lambda \in \Lambda$, R ranges over basic relation symbols of a fixed arity and $x, x_1, x_2, \cdots$ over variables.

Assume M, a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M; so M is a model for the predicate calculus in which the quantifier levels are stripped away. Again, an environment ρ is a function from variables to values; again, $\rho[v/x]$ means the environment ρ updated to value v at variable x. W.r.t. a model M and an environment ρ, we denote each closed formula ϕ of IF logic by a Λ-game, following very closely the definitions in Section ??. The differences are the assignment of levels to events and that the order on Λ has to be respected by the (modified) prefixed sums which quantified formulae denote.

The prefixed game $\oplus^\lambda.(A, W, l)$ comprises the event structure with polarity $\oplus.A$ in which all the events of $a \in A$ where $\lambda \leq l(a)$ are made to causally depend on a fresh +ve event $\oplus$, itself assigned level λ. Its winning conditions are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. The game $\bigoplus^\lambda_{v \in V}(A_v, W_v, l_v)$ has underlying event structure with polarity the sum $\sum_{v \in V} \oplus^\lambda.A_v$, maintains the same levels as its components, with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. The game $\bigominus^\lambda_{v \in V} G_v$ is defined dually, as $(\bigoplus^\lambda_{v \in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move.

True denotes the Λ-game the unit w.r.t. $\otimes$ and false denotes he unit w.r.t. $\invamp$. Denotations of conjunctions and disjunctions are given by the operations of $\otimes$ and $\invamp$ on Λ-games, while negations denote dual games. W.r.t. an environment ρ, universal and existential quantifiers denote the *prefixed sums* of games:

$$[\![\exists^\lambda x.\ \phi]\!]^\Lambda_M \rho = \bigoplus^\lambda_{v \in V_M} [\![\phi]\!]^\Lambda_M \rho[v/x]$$

$$[\![\forall^\lambda x.\ \phi]\!]^\Lambda_M \rho = \bigominus^\lambda_{v \in V_M} [\![\phi]\!]^\Lambda_M \rho[v/x]\,.$$

As a definition, an IF formula ϕ is satisfied w.r.t. an environment ρ, written

$$\rho \vDash^\Lambda_M \phi\,,$$

iff the Λ-game $[\![\phi]\!]^\Lambda_M \rho$ has a winning strategy.

Chapter 11

Probabilistic strategies

The chapter provides a new definition of probabilistic event structures, extending existing definitions, and characterised as event structures together with a continuous valuation on their domain of configurations. Probabilistic event structures possess a probabilistic measure on their domain of configurations. This prepares the ground for a very general definition of a probabilistic strategies, which are shown to compose, with probabilistic copy-cat strategies as identities. The result of the play-off of a probabilistic strategy and counter-strategy in a game is a probabilistic event structure so that a measurable pay-off function from the configurations of a game is a random variable, for which the expectation (the expected pay-off) is obtained as the standard Legesgue integral.

11.1 Probabilistic event structures

A probabilistic event structure comprises an event structure $(E, \leq, \mathrm{Con})$ together with a continuous valuation on its open sets of configurations, *i.e.* a function w from the open subsets of configurations $\mathcal{C}^\infty(E)$ to $[0,1]$ which is:

(normalized) $w(\mathcal{C}^\infty(E)) = 1$ *(strict)* $w(\varnothing) = 0$;

(monotone) $U \subseteq V \implies w(U) \leq w(V)$;

(modular) $w(U \cup V) + w(U \cap V) = w(U) + w(V)$;

(continuous) $w(\bigcup_{i\in I} U_i) = \sup_{i\in I} w(U_i)$ for *directed* unions $\bigcup_{i\in I} U_i$.

Continuous valuations play a central role in probabilistic powerdomains [28]. Continuous valuations are determined by their restrictions to basic open sets $\widehat{x} =_{\mathrm{def}} \{y \in \mathcal{C}^\infty(E) \mid x \subseteq y\}$, for x a finite configuration. The intuition: $w(U)$ is the probability of the resulting configuration being in the open set U. Indeed, continuous valuations extend to unique probabilistic measures on the Borel sets.

This description of a probabilistic event structure extends the definitions in [23]. It turns out to be equivalent to a more workable definition, which relates more directly to the configurations of E, that we develop now.

11.1.1 Preliminaries

Notation 11.1. *Let $\mathcal{F}$ be a stable family. Extend $\mathcal{F}$ to a lattice $\mathcal{F}^\top$ by adjoining an extra top element $\top$. Write its order as $x \sqsubseteq y$ and its join and meet operations as $x \vee y$ and $x \wedge y$ respectively.*

Definition 11.2. Let $\mathcal{F}$ be a stable family. Assume a function $v : \mathcal{F} \to \mathbb{R}$. Extend v to $v^\top : \mathcal{F}^\top \to \mathbb{R}$ by taking $v^\top(T) = 0$.

W.r.t. $v : \mathcal{F} \to \mathbb{R}$, for $n \in \omega$, define the *drop functions* $d_v^{(n)}[y; x_1, \cdots, x_n] \in \mathbb{R}$ for $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$ in $\mathcal{F}^\top$ as follows:

$$d_v^{(0)}[y;] =_{\text{def}} v^\top(y)$$
$$d_v^{(n)}[y; x_1, \cdots, x_n] =_{\text{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n]\,.$$

Throughout this section assume $\mathcal{F}$ is a stable family and $v : \mathcal{F} \to \mathbb{R}$.

Proposition 11.3. *Let $n \in \omega$. For $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i)\,.$$

For $y, x_1, \cdots, x_n \in \mathcal{F}$ with $y \subseteq x_1, \cdots, x_n$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} x_i)\,,$$

where the index I ranges over sets satisfying $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$.

Proof. We prove the first statement by induction on n. For the basis, when $n = 0$, $d_v^{(n)}[y;] = v(y)$, as required. For the induction step, with $n > 0$, we reason

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &=_{\text{def}} d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= v(y) - \sum_{\varnothing \neq I \subseteq \{1, \cdots, n-1\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \\ &\quad - v(x_n) + \sum_{\varnothing \neq J \subseteq \{1, \cdots, n-1\}} (-1)^{|I|+1} v(\bigvee_{j \in J} x_i \vee x_n)\,, \end{aligned}$$

making use of the induction hypothesis. Consider subsets K for which $\varnothing \neq K \subseteq \{1, \cdots, n\}$. Either $n \notin K$, in which case $\varnothing \neq K \subseteq \{1, \cdots, n-1\}$, or $n \in K$, in which case $K = \{n\}$ or $J =_{\text{def}} K \setminus \{n\}$ satisfies $\varnothing \neq J \subseteq \{1, \cdots, n-1\}$. From this observation, the sum above amounts to

$$v(y) - \sum_{\varnothing \neq K \subseteq \{1, \cdots, n\}} (-1)^{|K|+1} v(\bigvee_{k \in K} x_k)\,,$$

as required to maintain the induction hypothesis.

The second expression of the proposition is got by discarding all terms $v(\bigvee_{i \in I} x_i)$ for which $\bigvee_{i \in I} x_i = \top$ which leaves the sum unaffected as they contribute 0. □

Corollary 11.4. *Let $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. For ρ an n-permutation,*

$$d_v^{(n)}[y; x_{\rho(1)}, \cdots, x_{\rho(n)}] = d_v^{(n)}[y; x_1, \cdots, x_n].$$

Proof. As by Proposition 11.3, the value of $d_v^{(n)}[y; x_1, \cdots, x_n]$ is insensitive to permutations of its arguments. □

Proposition 11.5. *Assume $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If $y = x_i$ for some i with $1 \leq i \leq n$ then $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$.*

Proof. By Corollary 11.4, it suffices to show $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$ when $y = x_n$. In this case,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] \\ &= 0. \end{aligned}$$

□

Corollary 11.6. *Assume $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If $x_i \sqsubseteq x_j$ for distinct i, j with $1 \leq i, j \leq n$ then*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n].$$

Proof. By Corollary 11.4, it suffices to show

$$d_v^{(n)}[y; x_1, \cdots, x_{n-1}, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}]$$

when $x_{n-1} \sqsubseteq x_n$. Then,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-2}, x_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - 0, \end{aligned}$$

by Proposition 11.5. □

Proposition 11.7. *Assume $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. Then, $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$ if $y = \top$ and $d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$ if $x_i = \top$ with $1 \leq i \leq n$.*

Proof. When $n = 0$, $d_v^{(0)}[\top;] = v^\top(\top) = 0$. When $n \geq 1$, $d_v^{(n)}[\top; x_1, \cdots, x_n] = 0$ by Proposition 11.5 as *e.g.* $x_n = \top$. For the remaining statement, w.l.og. we may assume $i = n$ and that $x_n = \top$, yielding

$$d_v^{(n)}[y; x_1, \cdots, \top] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[\top; x_1 \vee \top, \cdots, x_{n-1} \vee \top] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}].$$

□

Lemma 11.8. *Let $n \geq 1$. Let $y, x_1, \cdots, x_n, x'_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. Assume $x_n \sqsubseteq x'_n$. Then,*

$$d_v^{(n)}[y; x_1, \cdots, x'_n] = d_v^{(n)}[y; x_1, \cdots, x_n] + d_v^{(n)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n, x'_n].$$

Proof. By definition,

$$\begin{aligned} \text{the r.h.s.} &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] \\ &\quad + d_v^{(n-1)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n] - d_v^{(n-1)}[x'_n; x_1 \vee x'_n, \cdots, x_{n-1} \vee x'_n] \\ &= d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x'_n; x_1 \vee x'_n, \cdots, x_{n-1} \vee x'_n] \\ &= d_v^{(n)}[y; x_1, \cdots, x_{n-1}, x'_n] \\ &= \text{the l.h.s..} \end{aligned}$$

□

11.1.2 The definition

Definition 11.9. Let $\mathcal{F}$ be a stable family. A *configuration-valuation* is function $v : \mathcal{F} \to [0,1]$ such that $v(\varnothing) = 1$ and which satisfies the "drop condition:"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$$

for all $n \geq 1$ and $y, x_1, \cdots, x_n \in \mathcal{F}$ with $y \subseteq x_1, \cdots, x_n$.

A *probabilistic stable family* comprises a stable family $\mathcal{F}$ together with a configuration-valuation $v : \mathcal{F} \to [0,1]$.

A *probabilistic event structure* comprises an event structure E together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$.

Proposition 11.10. *Let $v : \mathcal{F} \to [0,1]$. Then, v is a configuration-valuation iff $v^\top(\varnothing) = 1$ and $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}^\top$ with $y \sqsubseteq x_1, \cdots, x_n$. If v is a configuration-valuation, then*

$$y \sqsubseteq x \implies v^\top(y) \geq v^\top(x),$$

for all $x, y \in \mathcal{F}^\top$.

Proof. By Proposition 11.7 and as $d_v^{(1)}[y; x] = v^\top(y) - v^\top(x)$. □

In showing we have a probabilistic event structure or stable family it suffices to verify the "drop condition" only for covering intervals.

Lemma 11.11. *Let $\mathcal{F}$ be a stable family and $v : \mathcal{F} \to [0,1]$.*

(i) Let $y \subseteq x_1, \cdots, x_n$ in $\mathcal{F}$. Then, $d_v^{(n)}[y; x_1, \cdots, x_n]$ is expressible as a sum of terms

$$d_v^{(k)}[u; w_1, \cdots, w_k]$$

where $y \subseteq u \multimap w_i$ *in* $\mathcal{F}$ *and* $w_i \subseteq x_1 \cup \cdots \cup x_n$*, for all* i *with* $1 \le i \le k$*. [The set* $x_1 \cup \cdots \cup x_n$ *need not be in* $\mathcal{F}$*.]*

(ii) A fortiori, v *is a configuration-valuation iff* $v(\emptyset) = 1$ *and*

$$d_v^{(n)}[y; x_1, \cdots, x_n] \ge 0$$

for all $n \ge 1$ *and* $y \multimap x_1, \cdots, x_n$ *in* $\mathcal{F}$*.*

Proof. Define the *weight* of a term $d_v^{(n)}[y; x_1, \cdots, x_n]$, where $y \subseteq x_1, \cdots, x_n$ in $\mathcal{F}$, to be the product $|x_1 \setminus y| \times \cdots \times |x_n \setminus y|$.

Assume $y \subseteq x_1, \cdots, x'_n$ in $\mathcal{F}$. By Proposition 11.5, if y equals x'_n or some x_i, then $d_v^{(n)}[y; x_1, \cdots, x'_n] = 0$, so may be deleted as a contribution to a sum. Otherwise, if $y \subsetneq x_n \subsetneq x'_n$, by Lemma 11.8 we can rewrite $d_v^{(n)}[y; x_1, \cdots, x'_n]$ to the sum

$$d_v^{(n)}[y; x_1, \cdots, x_n] + d_v^{(n)}[x_n; x_1 \vee x_n, \cdots, x_{n-1} \vee x_n, x'_n]\,,$$

where we further observe

$$|x_n \setminus y| < |x'_n \setminus y|\,, \qquad |x'_n \setminus x_n| < |x'_n \setminus y|$$

and

$$|(x_i \cup x_n) \setminus x_n| \le |x_i \setminus y|\,,$$

whenever $x_i \vee x_n \ne \top$. Using Proposition 11.7 we may tidy away any mentions of $\top$. This reduces $d_v^{(n)}[y; x_1, \cdots, x'_n]$ to the sum of at most two terms, each of lesser weight. For notational simplicity we have concentrated on the nth argument in $d_v^{(n)}[y; x_1, \cdots, x'_n]$, but by Corollary 11.4 an analogous reduction is possible w.r.t. any argument.

Repeated use of the reduction, rewrites $d_v^{(n)}[y; x_1, \cdots, x_n]$ to a sum of terms of the form

$$d_v^{(k)}[u; w_1, \cdots, w_k]$$

where $k \le n$ and $u \multimap w_1, \cdots, w_k \subseteq x_1 \cup \cdots \cup x_n$. This justifies the claims of the lemma. □

11.1.3 The characterisation

Our goal is to prove that probabilistic event structures correspond to event structures with a continuous valuation. It is clear that a continuous valuation w on the Scott-open subsets of an event structure E gives rise to a configuration-valuation v on E: take $v(x) =_{\text{def}} w(\widehat{x})$, for $x \in \mathcal{C}(E)$. We will show that this construction has an inverse, that a configuration-valuation determines a continuous valuation.

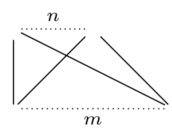
For this we need a combinatorial lemma:[1]

[1] The proof of the combinatorial lemma below is due to the author. It appears with acknowledgement as Lemma 6.App.1 in [29], the PhD thesis of my former student Daniele Varacca, whom I thank, both for the collaboration and the latex.

Lemma 11.12. *For all finite sets I, J,*

$$\sum_{\substack{\varnothing \neq K \subseteq I \times J \\ \pi_1(K)=I, \pi_2(K)=J}} (-1)^{|K|} = (-1)^{|I|+|J|-1} .$$

Proof. Without loss of generality we can take $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$. Also observe that a subset $K \subseteq I \times J$ such that $\pi_1(K) = I, \pi_2(K) = J$ is in fact a surjective and total relation between the two sets.



Let

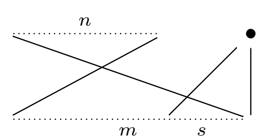
$$t_{n,m} =_{\text{def}} \sum_{\substack{\varnothing \neq K \subseteq I \times J \\ \pi_1(K)=I, \pi_2(K)=J}} (-1)^{|}K| ;$$

$$t^o_{n,m} =_{\text{def}} |\{\varnothing \neq K \subseteq I \times J \mid |K| \text{ odd}, \pi_1(K) = I, \pi_2(K) = J\}| ;$$

$$t^e_{n,m} := |\{\varnothing \neq K \subseteq I \times J \mid |K| \text{ even}, \pi_1(K) = I, \pi_2(K) = J\}| .$$

Clearly $t_{n,m} = t^e_{n,m} - t^o_{n,m}$. We want to prove that $t_{n,m} = (-1)^{n+m+1}$. We do this by induction on n. It is easy to check that this is true for $n = 1$. In this case, if m is even then $t^e_{1,m} = 1$ and $t^o_{1,m} = 0$, so that $t^e_{1,m} - t^o_{1,m} = (-1)^{1+m+1}$. Similarly if m is odd.

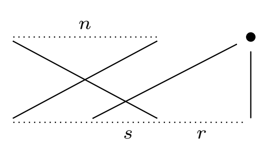
Now assume that for every p, $t_{n,p} = (-1)^{n+p+1}$ and compute $t_{n+1,m}$. To evaluate $t_{n+1,m}$ we count all surjective and total relations K between I and J together with their"sign." Consider the pairs in K of the form $(n+1, h)$ for $h \in J$. The result of removing them is a a total surjective relation between $\{1, \ldots, n\}$ and a subset J_K of $\{1, \ldots, m\}$.



Consider first the case where $J_K = \{1, \ldots, m\}$. Consider the contribution of such K's to $t_{n+1,m}$. There are $\binom{m}{s}$ ways of choosing s pairs of the form $(n+1, h)$. For every such choice there are $t_{n,m}$ (signed) relations. Adding the pairs $(n+1, h)$ possibly modifies the sign of such relations. All in all the contribution amounts to

$$\sum_{1 \leq s \leq m} \binom{m}{s} (-1)^s t_{n,m} .$$

Suppose now that J_K is a proper subset of $\{1, \ldots, m\}$ leaving out r elements.



Since K is surjective, all such elements h must be in a pair of the form $(n+1,h)$. Moreover there can be s pairs of the form $(n+1,h')$ with $h' \in J_K$. What is the contribution of such K's to $t_{n,m}$? There are $\binom{m}{r}$ ways of choosing the elements that are left out. For every such choice and for every s such that $0 \le s \le m-r$ there are $\binom{m-r}{s}$ ways of choosing the $h' \in J_K$. And for every such choice there are $t_{n,m-r}$ (signed) relations. Adding the pairs $(n+1,h)$ and $(n+1,h')$ possibly modifies the sign of such relations. All in all, for every r such that $1 \le r \le m-1$, the contribution amounts to

$$\binom{m}{r} \sum_{1 \le s \le m-r} \binom{m}{s} (-1)^{s+r} t_{n,m-n} \,.$$

The (signed) sum of all these contribution will give us $t_{n+1,m}$. Now we use the induction hypothesis and we write $(-1)^{n+p+1}$ for $t_{n,p}$.

Thus,

$$\begin{aligned}
t_{n+1,m} &= \sum_{1\le s\le m} \binom{m}{s} (-1)^s t_{n,m} \\
&\quad + \sum_{1\le r\le m-1} \binom{m}{r} \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^{s+r} t_{n,m-r} \\
&= \sum_{1\le s\le m} \binom{m}{s} (-1)^{s+n+m+1} \\
&\quad + \sum_{1\le r\le m-1} \binom{m}{r} \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^{s+n+m+1} \\
&= (-1)^{n+m+1} \Bigg(\sum_{1\le s\le m} \binom{m}{s} (-1)^s \\
&\quad + \sum_{1\le r\le m-1} \binom{m}{r} \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^s \Bigg) .
\end{aligned}$$

By the binomial formula, for $1 \le r \le m-1$ we have

$$0 = (1-1)^{m-r} = \sum_{0\le s\le m-r} \binom{m-r}{s} (-1)^s \,.$$

So we are left with

$$\begin{aligned}
t_{n+1,m} &= (-1)^{n+m+1} \left(\sum_{1\le s\le m} \binom{m}{s} (-1)^s \right) \\
&= (-1)^{n+m+1} \left(\sum_{0\le s\le m} \binom{m}{s} (-1)^s - \binom{m}{0} (-1)^0 \right) \\
&= (-1)^{n+m+1} \, (0-1) \\
&= (-1)^{n+1+m+1} \,,
\end{aligned}$$

as required. □

Theorem 11.13. *A configuration-valuation v on an event structure E extends to a unique continuous valuation w_v on the open sets of $\mathcal{C}^\infty(E)$, so that $w_v(\widehat{x}) = v(x)$, for all $x \in \mathcal{C}(E)$.*

Conversely, a continuous valuation w on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation v_w on E, assigning $v_w(x) = w(\widehat{x})$, for all $x \in \mathcal{C}(E)$.

Proof. The proof is inspired by the proofs in the appendix of [23] and the thesis [29].

First, a continuous valuation w on the open sets of $\mathcal{C}^\infty(E)$ restricts to a configuration-valuation v defined as $v(x) =_{\rm def} w(\widehat{x})$ for $x \in \mathcal{C}(E)$. Note that any extension of a configuration-valuation to a continuous valuation is bound to be unique by continuity.

To show the converse we first define a function w from the basic open sets $Bs =_{\rm def} \{\widehat{x_1} \cup \cdots \cup \widehat{x_n} \mid x_1, \cdots, x_n \in \mathcal{C}(E)\}$ to $[0,1]$ and show that it is normalised, strict, monotone and modular. Define

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) =_{\rm def}\ & 1 - d_v^{(n)}[\varnothing; x_1, \cdots, x_n] \\ = & \sum_{\varnothing \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) \end{aligned}$$

—this can be shown to be well-defined using Corollaries 11.4 and 11.6.

Clearly, w is normalised in the sense that $w(\mathcal{C}^\infty(E)) = w(\widehat{\varnothing}) = 1$ and strict in that $w(\varnothing) = 1 - v(\varnothing) = 0$.

To see that it is monotone, first observe that

$$w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \leq w(\widehat{x_1} \cup \cdots \cup \widehat{x_{n+1}})$$

as

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_{n+1}}) - w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) &= d_v^{(n)}[\varnothing; x_1, \cdots, x_n] - d_v^{(n+1)}[\varnothing; x_1, \cdots, x_{n+1}] \\ &= d_v^{(n)}[x_{n+1}; x_1 \vee x_{n+1}, \cdots, x_n \vee x_{n+1}] \geq 0\,. \end{aligned}$$

By a simple induction (on m),

$$w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \leq w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m})\,.$$

Suppose that $\widehat{x_1} \cup \cdots \cup \widehat{x_n} \subseteq \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. Then $\widehat{y_1} \cup \cdots \cup \widehat{y_m} = \widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}$. By the above,

$$\begin{aligned} w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) &\leq w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ &= w(\widehat{y_1} \cup \cdots \cup \widehat{y_m})\,, \end{aligned}$$

as required to show w is monotone.

To show modularity we require

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) + w(\widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ =& w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) + w((\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \cap (\widehat{y_1} \cup \cdots \cup \widehat{y_m}))\,. \end{aligned}$$

Note

$$(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \cap (\widehat{y_1} \cup \cdots \cup \widehat{y_m}) = (\widehat{x_1} \cap \widehat{y_1}) \cup \cdots \cup (\widehat{x_i} \cap \widehat{y_j}) \cdots \cup (\widehat{x_n} \cap \widehat{y_m})$$
$$= \widehat{x_1 \vee y_1} \cup \cdots \cup \widehat{x_i \vee y_j} \cdots \cup \widehat{x_n \vee y_m}\,.$$

From the definition of w we require

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ &= \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) + \sum_{\emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|J|+1} v(\bigvee_{j \in J} y_j) \\ & \quad - \sum_{\emptyset \neq R \subseteq \{1,\cdots,n\} \times \{1,\cdots,m\}} (-1)^{|R|+1} v(\bigvee_{(i,j) \in R} x_i \vee y_j)\,. \end{aligned} \tag{1}$$

Consider the definition of $w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m})$ as a sum. Its components are associated with indices which either lie entirely within $\{1, \cdots, n\}$, entirely within $\{1, \cdots, m\}$, or overlap both. Hence

$$\begin{aligned} & w(\widehat{x_1} \cup \cdots \cup \widehat{x_n} \cup \widehat{y_1} \cup \cdots \cup \widehat{y_m}) \\ &= \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i \in I} x_i) + \sum_{\emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|J|+1} v(\bigvee_{j \in J} y_j) \\ & \quad + \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}, \emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|I|+|J|+1} v(\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j)\,. \end{aligned} \tag{2}$$

Comparing (1) and (2), we require

$$\begin{aligned} & - \sum_{\emptyset \neq R \subseteq \{1,\cdots,n\} \times \{1,\cdots,m\}} (-1)^{|R|+1} v(\bigvee_{(i,j) \in R} x_i \vee y_j) \\ &= \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}, \emptyset \neq J \subseteq \{1,\cdots,m\}} (-1)^{|I|+|J|+1} v(\bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j)\,. \end{aligned} \tag{3}$$

Observe that

$$\bigvee_{(i,j) \in R} x_i \vee y_j = \bigvee_{i \in I} x_i \vee \bigvee_{j \in J} y_j$$

when $I = R_1 =_{\text{def}} \{i \in I \mid \exists j \in J.\ (i,j) \in R\}$ and $J = R_2 =_{\text{def}} \{j \in J \mid \exists i \in I.\ (i,j) \in R\}$ for a relation $R \subseteq \{1, \cdots, n\} \times \{1, \cdots, m\}$. With this observation we see that equality (3) follows from the combinatorial lemma, Lemma 11.12 above. This shows modularity.

Finally, we can extend w to all open sets by taking an open set U to $\sup_{b \in Bs \,\&\, b \subseteq U} w(b)$. The verification that w is indeed a continuous valuation extending v is now straightforward. □

The above theorem also holds (with the same proof) for Scott domains. Now, by [30], Corollary 4.3:

Theorem 11.14. *For a configuration-valuation v on E there is a unique probability measure μ_v on the Borel subsets of $\mathcal{C}^\infty(E)$ extending w_v.*

Example 11.15. Consider the event structure comprising two concurrent events e_1, e_2 with configuration-valuation v for which $v(\varnothing) = 1, v(\{e_1\}) = 1/3, v(\{e_2\}) = 1/2$ and $v(\{e_1, e_2\}) = 1/12$. This means in particular that there is a probability of 1/3 of a result within the Scott open set consisting of both the configuration $\{e_1\}$ and the configuration $\{e_1, e_2\}$. In other words, there is a probability of 1/3 of observing e_1 (possibly with or possibly without e_2). The induced probability measure p assigns a probability to any Borel set, in this simple case any subset of configurations, and is determined by its value on single configurations: $p(\varnothing) = 1 - 4/12 - 6/12 + 1/12 = 3/12$, $p(\{e_1\}) = 4/12 - 1/12 = 3/12$, $p(\{e_2\}) = 6/12 - 1/12 = 5/12$ and $p(\{e_1, e_2\}) = 1/12$. Thus there is a probability of 3/12 of observing neither e_1 nor e_2, and a probability of 5/12 of observing just the event e_2 (and not e_1). There is a drop $d_v^{(0)}[\varnothing; \{e_1\}, \{e_2\}] = 1-4/12-6/12+1/12 = 3/12$ corresponding to the probability of remaining at the empty configuration and not observing any event. Sometimes it's said that probability "leaks" at the empty configuration, but it's more accurate to think of this leak in probability as associated with a non-zero chance that the initial observation of no events will not improve.

Example 11.16. Consider the event structure with events $\mathbb{N}^+$ with causal dependency $n \le n + 1$, with all finite subsets consistent. It is not hard to check that all subsets of $\mathcal{C}^\infty(\mathbb{N}^+)$ are Borel sets. Consider the ensuing probability distributions w.r.t. the following configuration-valuations:
(i) $v_0(x) = 1$ for all $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 1 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$, and 0 to $\varnothing$ and all other singleton sets of configurations.
(ii) $v_1(\varnothing) = v_1(\{1\}) = 1$ and $v_1(x) = 0$ for all other $x \in \mathcal{C}(\mathbb{N}^+)$. The resulting probability distribution assigns probability 0 to $\varnothing$ and probability 1 to the singleton set $\{1\}$, and 0 to all other singleton sets of configurations.
(iii) $v_2(\varnothing) = 1$ and $v_2(\{1, \cdots, n\}) = (1/2)^n$ for all $n \in \mathbb{N}^+$. The resulting probability distribution assigns probability 1/2 to $\varnothing$ and $(1/2)^{n+1}$ to each singleton $\{\{1, \cdots, n\}\}$ and 0 to the singleton set $\{\mathbb{N}^+\}$, comprising the single infinite configuration $\mathbb{N}^+$.

When x a finite configuration has $v(x) > 0$ and $\mu_v(\{x\}) = 0$ we can understand x as being a transient configuration on the way to a final with probability $v(x)$. In general, there is a simple expression for the probability of terminating at a finite configuration.

Proposition 11.17. *Let E, v be a probabilistic event structure. For any finite configuration $y \in \mathcal{C}(E)$, the singleton set $\{y\}$ is a Borel subset with probability measure*

$$\mu_v(\{y\}) = \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \& \ y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}\,.$$

Proof. Let $y \in \mathcal{C}(E)$. Then $\{y\} = \widehat{y} \setminus U_y$ is clearly Borel as $U_y =_{\text{def}} \{x \in \mathcal{C}^\infty(E) \mid y \subsetneq x\}$ is open. Let w be the continuous valuation extending v. Then

$$w(U_y) = \sup\{w(\widehat{x_1} \cup \cdots \cup \widehat{x_n}) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}$$

as U_y is the directed union $\bigcup\{\widehat{x}_1 \cup \cdots \cup \widehat{x}_n \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}$. Hence

$$\begin{aligned}\mu_v(\{y\}) = v(y) - w(U_y) &= v(y) - \sup\{w(\widehat{x}_1 \cup \cdots \cup \widehat{x}_n) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\} \\ &= \inf\{v(y) - \sum_{\emptyset \neq I \subseteq \{1,\cdots,n\}} (-1)^{|I|+1} v(\bigvee_{i\in I} x_i) \mid y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\} \\ &= \inf\{d_v^{(n)}[y; x_1, \cdots, x_n] \mid n \in \omega \ \& \ y \subsetneq x_1, \cdots, x_n \in \mathcal{C}(E)\}\,.\end{aligned}$$

□

Example 11.18. It might be thought that probabilistic event structures could only capture discrete distributions. However consider the event structure representing streams of 0's and 1's. We saw this earlier in Example 2.1. Its finite configurations comprise the empty set and downwards-closures $[s]$ of single event occurrences s given by a finite sequence of 0's and 1's. Assign value 1 to the empty configuration and $1/2^n$ to a sequence $s = (s_1, s_2, \cdots, s_n)$. Then all finite configurations $[s]$ are transient it the sense that the probability of ending up at precisely the finite stream $[s]$ is zero; all the probabilistic measure is concentrated on the maximal configurations, the infinite streams. On the maximal configurations the probabilistic measure gives a continuous distribution with zero probability of the result being any particular infinite stream.

Remark. There is perhaps some redundancy in the definition of purely probabilistic event structures, in that there are two different ways to say, for example, that events e_1 and e_2 do not occur together at a finite configuration y where $y \stackrel{e_1}{\longrightarrow\!\!\!\subset} x_1$ and $y \stackrel{e_2}{\longrightarrow\!\!\!\subset} x_2$: either through $\{e_1, e_2\} \notin \mathrm{Con}$; or via the configuration-valuation v through $v(x_1 \cup x_2) = 0$. However, when we mix probability with nondeterminism, as we do in the next section, we shall make use of both order-consistency and the valuation.

11.2 Probability with an Opponent

Assume now that the events of the stable family or event structure carry a polarity, + or −. Imagine the event structure or stable family represents a strategy for Player. The Player cannot foresee what probabilities Opponent will ascribe to moves under Opponent's control. Nor, without information about the stochastic rates of Player and Opponent can we hope to ascribe probabilities to play outcomes in the presence of races. For this reason we shall restrict probabilistic event structures with polarity to those which are race-free.

It will be convenient, more generally, to define a probabilistic stable family in which some events are distinguished as Opponent events (where the other events may be Player events or "neutral" events due to synchronizations between Player and Opponent). Events which are not Opponent events we shall call p-events. For configurations x, y we shall write $x \subseteq^p y$ if $x \subseteq y$ and $y \setminus x$ contains no Opponent events; we write $x \longrightarrow\!\!\!\subset^p y$ when $x \longrightarrow\!\!\!\subset y$ and $x \subseteq^p y$; we continue to write $x \subseteq^- y$ if $x \subseteq y$ and $y \setminus x$ comprises solely Opponent events.

Definition 11.19. We extend the notion of configuration-valuation to the situation where events carry polarities. Let $\mathcal{F}$ be a stable family $\mathcal{F}$ together with a specified subset of its events which are Opponent events. A *configuration-valuation* is a function $v : \mathcal{F} \to [0,1]$ for which $v(\varnothing) = 1$,

$$x \subseteq^- y \implies v(x) = v(y) \tag{1}$$

for all $x, y \in \mathcal{F}$, and satisfies the "drop condition"

$$d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0 \tag{2}$$

for all $n \in \omega$ and $y, x_1, \cdots, x_n \in \mathcal{F}$ with $y \subseteq^p x_1, \cdots, x_n$.

The notion of *probabilistic stable family* thus extends to a stable family $\mathcal{F}$ together with a specified subset of Opponent events and a configuration-valuation $v : \mathcal{F} \to [0,1]$. The notion specialises to event structures with a distinguished subset of Opponent events.

In particular, a *probabilistic event structure with polarity* comprises E an event structure with polarity together with a configuration-valuation $v : \mathcal{C}(E) \to [0,1]$.

Remark There is an equivalent way of presenting a configuration-valuation for an event structure with polarity S as a family of conditional probabilities. Define a familiy of conditional probabilities over S to comprise $\text{Prob}(x \mid y)$, indexed by $y \subseteq^+ x$ in $\mathcal{C}(S)$, such that

(i) $\text{Prob}(y \mid y) = 1$ and $x \mapsto \text{Prob}(x \mid y)$ satisfies the drop condition for x s.t. $y \subseteq^+ x$ in $\mathcal{C}(S)$;

(ii) $\text{Prob}(w \mid y) = \text{Prob}(w \mid x)\text{Prob}(x \mid y)$ if $y \subseteq^+ x \subseteq^+ w$ in $\mathcal{C}(S)$;

(iii) $\text{Prob}(x \mid y) = \text{Prob}(x' \mid y')$ if $y \subseteq^+ x$, $y \subseteq^- y'$ and $x \cup y' = x'$.

Given a configuration-valuation v we define $\text{Prob}(x \mid y) = v(x)/v(y)$. Conversely, given a family of conditional probabilities, as described above, first extend it by taking $\text{Prob}(x \mid y) = 1$ for $y \subseteq^- x$. We then obtain a configuration-valuation by defining

$$v(x) =_{\text{def}} \text{Prob}(x_1 \mid x_0)\text{Prob}(x_2 \mid x_1) \cdots \text{Prob}(x_n \mid x_{n-1})$$

w.r.t. a covering chain

$$\varnothing = x_0 \longrightarrow\!\!\!\!\!\subset x_1 \longrightarrow\!\!\!\!\!\subset x_2 \longrightarrow\!\!\!\!\!\subset \cdots \longrightarrow\!\!\!\!\!\subset x_{n-1} \longrightarrow\!\!\!\!\!\subset x_n = x\,;$$

by (ii) and (iii) the choice of covering chain does not affect the value assigned to x. The two operations provide mutual inverses between configuration-valuations and families of conditional probabilities as described above. There is an analogous result for configuration-valuations for a stable family $\mathcal{F}$ together with a specified subset of Opponent events.

As indicated above, the extra generality in the definition of a probabilistic stable family with polarity is to cater for a situation later in which we shall ascribe probabilities not only to results of Player moves but also to events arising as synchronizations between Player and Opponent moves. As earlier, by Lemma 11.11(i), it suffices to verify the "drop condition" for p-covering intervals.

Definition 11.20. Let A be a race-free event structure with polarity. A *probabilistic strategy* in A comprises a probabilistic event structure S, v and a strategy $\sigma : S \to A$. [By Lemma 5.5, S will also be race-free.]

Let A and B be a race-free event structures with polarity. A *probabilistic strategy* from A to B comprises a probabilistic event structure S, v and a strategy $\sigma : S \to A^\perp \| B$.

We extend the usual composition of strategies to probabilistic strategies. Assume probabilistic strategies $\sigma : S \to A^\perp \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^\perp \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$. We first tentatively define their composition on stable families, taking $v : \mathcal{C}(T) \circledast \mathcal{C}(S) \to [0,1]$ to be

$$v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$$

for $x \in \mathcal{C}(T) \circledast \mathcal{C}(S)$.

Proposition 11.21. *Let $v : \mathcal{C}(T) \circledast \mathcal{C}(S) \to [0,1]$ be defined as above. Then, $v(\varnothing) = 0$. If $x \subseteq^- y$ in $\mathcal{C}(T) \circledast \mathcal{C}(S)$ then $v(x) = v(y)$.*

Proof. Clearly,

$$v(\varnothing) = v_S(\pi_1 \varnothing) \times v_T(\pi_2 \varnothing) = 1 \times 1 = 1\,.$$

Assuming $x \longrightarrow\!\!\subset^- y$ in $\mathcal{C}(T) \circledast \mathcal{C}(S)$, then either $x \overset{(s,*)}{\longrightarrow\!\!\subset} y$, with s a −ve event of S, or $x \overset{(*,t)}{\longrightarrow\!\!\subset} y$, with t a −ve event of T. Suppose $x \overset{(s,*)}{\longrightarrow\!\!\subset} y$, with s −ve. Then $\pi_1 x \overset{s}{\longrightarrow\!\!\subset} \pi_1 y$, where as s is −ve, $v_S(\pi_1 x) = v_S(\pi_1 y)$. In addition, $\pi_2 x = \pi_2 y$ so certainly $v_T(\pi_2 x) = v_T(\pi_2 y)$. Combined these two facts yield $v(x) = v(y)$. Similarly, $x \overset{(*,t)}{\longrightarrow\!\!\subset} y$, with t −ve, implies $v(x) = v(y)$. As $x \subseteq^- y$ is obtained via the reflexive transitive closure of $\longrightarrow\!\!\subset^-$ it entails $v(x) = v(y)$, as required. □

But of course we need to check that v is indeed a configuration-valuation. For this it remains to show that v satisfies the "drop condition." For this we need only consider covering intervals, by Lemma 11.11(i).

Lemma 11.22. *Let $y, x_1, \cdots, x_n \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ with $y \longrightarrow\!\!\subset^p x_1, \cdots, x_n$. Assume that $\pi_1 y \longrightarrow\!\!\subset^+ \pi_1 x_i$ when $1 \le i \le m$ and $\pi_2 y \longrightarrow\!\!\subset^+ \pi_2 x_i$ when $m+1 \le i \le n$. Then in $\mathcal{C}(T) \circledast \mathcal{C}(S), v$,*

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_{v_S}^{(m)}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d_{v_T}^{(n-m)}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n]\,.$$

Proof. Under the assumptions of the lemma, by proposition 11.3,

$$d^{(m)}_{v_S}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] = v_S(\pi_1 y) - \sum_{I_1} (-1)^{|I_1|+1} v_S(\bigcup_{i \in I_1} \pi_1 x_i)\,,$$

where I_1 ranges over sets satisfying $\varnothing \neq I_1 \subseteq \{1, \cdots, m\}$ s.t. $\{\pi_1 x_i \mid i \in I_1\}\uparrow$. Similarly,

$$d^{(n-m)}_{v_T}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n] = v_T(\pi_2 y) - \sum_{I_2} (-1)^{|I_2|+1} v_T(\bigcup_{i \in I_2} \pi_2 x_i)\,,$$

where I_2 ranges over sets satisfying $\varnothing \neq I_2 \subseteq \{m+1, \cdots, n\}$ s.t. $\{\pi_2 x_i \mid i \in I_2\}\uparrow$.

Note, by strong receptivity of τ, that when $\varnothing \neq I_1 \subseteq \{1, \cdots, m\}$,

$$\{\pi_1 x_i \mid i \in I_1\}\uparrow \text{ in } \mathcal{C}(S) \text{ iff } \{x_i \mid i \in I_1\}\uparrow \text{ in } \mathcal{C}(T) \circledast \mathcal{C}(S)$$

and, similarly by strong receptivity of σ, when $\varnothing \neq I_2 \subseteq \{m+1, \cdots, n\}$,

$$\{\pi_2 x_i \mid i \in I_2\}\uparrow \text{ in } \mathcal{C}(T) \text{ iff } \{x_i \mid i \in I_2\}\uparrow \text{ in } \mathcal{C}(T) \circledast \mathcal{C}(S)\,.$$

Hence

$$\bigcup_{i \in I_1} \pi_1 x_i = \pi_1 \bigcup_{i \in I_1} x_i \quad \text{and} \quad \bigcup_{i \in I_2} \pi_2 x_i = \pi_2 \bigcup_{i \in I_2} x_i\,.$$

Making these rewrites and taking the product

$$d^{(m)}_{v_S}[\pi_1 y; \pi_1 x_1, \cdots, \pi_1 x_m] \times d^{(n-m)}_{v_T}[\pi_2 y; \pi_2 x_{m+1}, \cdots, \pi_2 x_n]\,,$$

we obtain

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) & - \sum_{I_2} (-1)^{|I_2|+1}\ v_S(\pi_1 y) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \\ & - \sum_{I_1} (-1)^{|I_1|+1}\ v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 y) \\ & + \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|}\ v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i)\,. \end{aligned}$$

But at each index I_2,

$$v_S(\pi_1 y) = v_S(\pi_1 \bigcup_{i \in I_2} x_i)$$

as $\pi_1 y \subseteq^- \pi_1 \bigcup_{i \in I_2} x_i$. Similarly, at each index I_1,

$$v_T(\pi_2 y) = v_T(\pi_2 \bigcup_{i \in I_1} x_i)\,.$$

Hence the product becomes

$$\begin{aligned} v_S(\pi_1 y) \times v_T(\pi_2 y) & - \sum_{I_2} (-1)^{|I_2|+1}\ v_S(\pi_1 \bigcup_{i \in I_2} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i) \\ & - \sum_{I_1} (-1)^{|I_1|+1}\ v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_1} x_i) \\ & + \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|}\ v_S(\pi_1 \bigcup_{i \in I_1} x_i) \times v_T(\pi_2 \bigcup_{i \in I_2} x_i)\,. \end{aligned}$$

To simplify this further, we observe that

$$\{x_i \mid i \in I_1\}\uparrow \ \& \ \{x_i \mid i \in I_2\}\uparrow \iff \{x_i \mid i \in I_1 \cup I_2\}\uparrow \ .$$

The "$\Leftarrow$" direction is clear. We show "$\Rightarrow$." Assume $\{x_i \mid i \in I_1\}\uparrow$ and $\{x_i \mid i \in I_2\}\uparrow$. We obtain $\{\pi_1 x_i \mid i \in I_1\}\uparrow$ and $\{\pi_1 x_i \mid i \in I_2\}\uparrow$ as the projection map π_1 preserves consistency. Hence $\bigcup_{i\in I_1} \pi_1 x_i$ and $\bigcup_{i\in I_2} \pi_1 x_i$ are configurations of S. Furthermore, by assumption,

$$\pi_1 y \subseteq^+ \bigcup_{i\in I_1} \pi_1 x_i \quad \text{and} \quad \pi_1 y \subseteq^- \bigcup_{i\in I_2} \pi_1 x_i \, .$$

As S, a strategy over the race-free game $A^\perp \| B$, is automatically race-free—Lemma 5.5—we obtain

$$\bigcup_{i\in I_1\cup I_2} \pi_1 x_i \in \mathcal{C}(S)$$

by Proposition 5.4. Similarly, because T is race-free, we obtain

$$\bigcup_{i\in I_1\cup I_2} \pi_2 x_i \in \mathcal{C}(T)\, .$$

Together these entail

$$\bigcup_{i\in I_1\cup I_2} x_i \in \mathcal{C}(T) \circledast \mathcal{C}(S)\, ,$$

i.e. $\{x_i \mid i \in I_1 \cup I_2\}\uparrow$, as required. Notice too that

$$\pi_1 \bigcup_{i\in I_1} x_i \subseteq^- \pi_1 \bigcup_{i\in I_1\cup I_2} x_i \quad \text{and} \quad \pi_2 \bigcup_{i\in I_2} x_i \subseteq^- \pi_2 \bigcup_{i\in I_1\cup I_2} x_i\, ,$$

which ensure

$$v_S(\pi_1 \bigcup_{i\in I_1} x_i) = v_S(\pi_1 \bigcup_{i\in I_1\cup I_2} x_i) \quad \text{and} \quad v_T(\pi_2 \bigcup_{i\in I_2} x_i) = v_T(\pi_2 \bigcup_{i\in I_1\cup I_2} x_i)\, ,$$

so that

$$v(\bigcup_{i\in I_1\cup I_2} x_i) = v_S(\pi_1 \bigcup_{i\in I_1} x_i) \times v_T(\pi_2 \bigcup_{i\in I_2} x_i)\, .$$

We can now further simplify the product to

$$\begin{aligned} v(y) &- \sum_{I_2} (-1)^{|I_2|+1}\ v(\bigcup_{i\in I_2} x_i) \\ &- \sum_{I_1} (-1)^{|I_1|+1}\ v(\bigcup_{i\in I_1} x_i) \\ &+ \sum_{I_1, I_2} (-1)^{|I_1|+|I_2|}\ v(\bigcup_{i\in I_1\cup I_2} x_i)\, . \end{aligned}$$

Noting that any subset I for which $\varnothing \neq I \subseteq \{1, \cdots, n\}$ either lies entirely within $\{1, \cdots, m\}$, entirely within $\{m+1, \cdots, n\}$, or properly intersects both, we have finally reduced the product to

$$v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_I x_i)\, ,$$

with indices those I which satisfy $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow$, *i.e.* the product reduces to $d_v^{(n)}[y; x_1 \cdots, x_n]$ as required. □

Corollary 11.23. *The assignment* $(v_T \circledast v_S)(x) =_{\text{def}} v_S(\pi_1 x) \times v_T(\pi_2 x)$ *to* $x \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ *yields a configuration-valuation on the stable family* $\mathcal{C}(T) \circledast \mathcal{C}(S)$.

Proof. From Proposition11.21 we have requirement (1); by Lemma 11.11(i) we need only verify requirement (2), the 'drop condition,' for p-covering intervals, which we can always permute into the form covered by Lemma 11.22—any p-event of $\mathcal{C}(T) \circledast \mathcal{C}(S)$ has a +ve component on one and only one side. □

Example 11.24. The assumption that games are race-free is needed for Corollary 11.23. Consider the composition of strategies $\sigma : \varnothing \rightarrow\!\!\!\!\!\!+\!\!> B$ and $\tau : B \rightarrow\!\!\!\!\!\!+\!\!> \varnothing$ where B is the game comprising the two moves $\oplus$ and $\ominus$ in conflict with each other—a game with a race. Suppose σ assigns probability 1 to playing $\oplus$ and τ assigns probability 1 to playing $\ominus$, in the dual game. Then the "drop condition" required for the corollary fails.

We can now complete the definition of the composition of probabilistic strategies:

Lemma 11.25. *Let A, B and C be race-free event structure with polarity. Let $\sigma : S \to A^\perp \| B$, with configuration-valuation $v_S : \mathcal{C}(S) \to [0,1]$, and $\tau : T \to B^\perp \| C$ with configuration-valuation $v_T : \mathcal{C}(T) \to [0,1]$ be probabilistic strategies. Assigning $(v_T \odot v_S)(x) =_{\text{def}} v_S(\Pi_1 x) \times v_T(\Pi_2 x)$ to $x \in \mathcal{C}(T \odot S)$ yields a configuration-valuation on $T \odot S$ which with $\tau \odot \sigma : T \odot S \to A^\perp \| C$ forms a probabilistic strategy from A to C.*

Proof. We need to show that the assignment $w(x) =_{\text{def}} v_S(\Pi_1 x) \times v_T(\Pi_2 x)$ to $x \in \mathcal{C}(T \odot S)$ is a configuration-valuation on $T \odot S$. We use that $v(z) =_{\text{def}} v_S(\pi_1 z) \times v_T(\pi_2 z)$, for $z \in \mathcal{C}(T) \circledast \mathcal{C}(S)$, is a configuration-valuation on $\mathcal{C}(T) \circledast \mathcal{C}(S)$

Recalling, for $x \in \mathcal{C}(T \odot S)$, that $\bigcup x \in \mathcal{C}(T) \circledast \mathcal{C}(S)$ with $\Pi_1 x = \pi_1 \bigcup x$ and $\Pi_2 x = \pi_2 \bigcup x$, we obtain

$$w(x) =_{\text{def}} v_S(\Pi_1 x) \times v_T(\Pi_2 x) = v_S(\pi_1 \bigcup x) \times v_T(\pi_2 \bigcup x) = v(\bigcup x)\,.$$

Consequently,

$$w(\varnothing) = v(\bigcup \varnothing) = v(\varnothing) = 1\,.$$

The function w inherits requirement (1) to be a configuration-valuation from v because

$x \stackrel{p}{\longrightarrow\!\!\!\subset} y$ with p −ve in $T \odot S$ implies $\bigcup x \stackrel{top(p)}{\longrightarrow\!\!\!\subset} \bigcup y$ with $top(p)$ −ve in $\mathcal{C}(T) \circledast \mathcal{C}(S)$.

To see this observe that $top(p)$ either has the form $(s, *)$ or $(*, t)$. Suppose $top(p) = (*, t)$. Suppose $e \rightarrowtriangle_{\bigcup y} (*, t)$. Then, by Lemma 3.21,

either (i) $e = (s', t')$ and $t' \rightarrowtriangle_T t$ or (ii) $e = (*, t')$ and $t' \rightarrowtriangle_T t$.

But (i) would violate the −-innocence of τ. Hence (ii) and being 'visible' the prime $[e]_{\bigcup y} \in x$ ensuring $e \in \bigcup x$. As all $\rightarrowtriangle_{\bigcup y}$-predecessors of $(*, t)$ are in $\bigcup x$ we obtain $\bigcup x \stackrel{(*,t)}{\longrightarrow\!\!\!\subset} \bigcup y$. The proof in the case where $top(p) = (s, *)$ is similar.

Similarly, w inherits requirement (2) from v, as w.r.t. w,

$$\begin{aligned} d_w^{(n)}[y; x_1, \cdots, x_n] &= w(y) - \sum_I (-1)^{|I|+1} w(\bigcup_{i \in I} x_i) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v(\bigcup \bigcup_{i \in I} x_i) \\ &= v(\bigcup y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i \in I} (\bigcup x_i)) \\ &\geq 0 \,, \end{aligned}$$

whenever $y \subseteq^+ x_1, \cdots, x_n$ in $\mathcal{C}(T \odot S)$. (Above, the index I ranges over sets satisfying $\emptyset \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\} \uparrow$.) □

A copy-cat strategy is easily turned into a probabilistic strategy, as is any deterministic strategy:

Lemma 11.26. *Let S be a deterministic event structure with polarity. Defining $v_S : \mathcal{C}(S) \to [0,1]$ to satisfy $v_S(x) = 1$ for all $x \in \mathcal{C}(S)$, we obtain a probabilistic event structure with polarity.*

Proof. Clearly

$$x \subseteq^- y \implies v_S(x) = v_S(y) = 1$$

for all $x, y \in \mathcal{C}(S)$. As S is deterministic,

$$y \subseteq^+ x \ \& \ y \subseteq^+ x' \implies x \cup x' \in \mathcal{C}(S) \,,$$

for all $y, x, x' \in \mathcal{C}(S)$. For the remaining requirement, a simple induction shows that for all $n \geq 1$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = 0$$

whenever $y \subseteq^+ x_1, \cdots, x_n$. The basis, when $n = 1$, is clear as

$$d_v^{(1)}[y; x] = v_S(y) - v_S(x) = 1 - 1 = 0$$

when $y \subseteq^+ x$. For the induction step, assuming $y \subseteq^+ x_1, \cdots, x_n$ with $n > 1$,

$$d_v^{(n)}[y; x_1, \cdots, x_n] = d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] = 0 - 0 = 0 \,,$$

from the induction hypothesis. □

Definition 11.27. We say a probabilistic event structure with polarity is *deterministic* when its configuration valuation assigns 1 to every finite configuration (provided it is race-free it will necessarily also be deterministic as an event structure with polarity—see the proposition immediately below). We say a probabilistic strategy $\sigma : S \to A$ with configuration-valuation v on $\mathcal{C}(S)$ is *deterministic* when the probabilistic event structure S, v is deterministic.

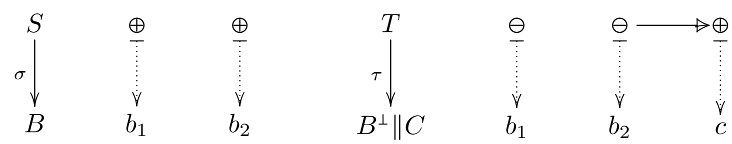
Proposition 11.28. *If a race-free probabilistic event structure with polarity is deterministic, as defined above, then the event structure with polarity itself is deterministic.*

Proof. Assume S, v, a race-free probabilistic event structure with polarity, is deterministic, as defined above. Suppose $y \stackrel{+}{-\!\!\!\subset} x_1$ and $y \stackrel{+}{-\!\!\!\subset} x_2$. We must have $x_1 \uparrow x_2$ as otherwise the drop condition would be violated. This with race-freeness implies that the event structure with polarity S itself is deterministic by Lemma 5.1. $\square$

Recall that race-freeness of a game A ensures that $\mathbb{C}_A$ is deterministic. Hence as a direct corollary of Lemma 11.26:

Corollary 11.29. *Let A be a race-free game. The copy-cat strategy from A to A comprising $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ with configuration-valuation $v_{\mathbb{C}_A} : \mathcal{C}(\mathbb{C}_A) \to [0,1]$ satisfying $v_{\mathbb{C}_A}(x) = 1$, for all $x \in \mathcal{C}(\mathbb{C}_A)$, forms a probabilistic strategy.*

Example 11.30. Let A be the empty game $\varnothing$, B be the game consisting of two concurrent +ve events b_1 and b_2, and C the game with a single +ve event c. We illustrate the composition of two probabilistic strategies $\sigma : \varnothing \mathrel{\rightarrow\!\!\!\!\!+} B$ and $\tau : B \mathrel{\rightarrow\!\!\!\!\!+} C$.



The strategy σ plays b_1 with probability $2/3$ and b_2 with probability $1/3$ (and plays both with probability 0). The strategy τ does nothing if just b_1 is played and plays the single +ve event c of C with probabilty $1/2$ if b_2 is played. Their composition yields the strategy $\tau \odot \sigma : \varnothing \mathrel{\rightarrow\!\!\!\!\!+} C$ which plays c with probability $1/6$, so has a $5/6$ chance of doing nothing.

The example illustrates how through probability we can track the presence of terminal configurations within a set of results despite their not being $\subseteq$-maximal. The empty configuration is such a terminal configuration; it could be the final result of the composition as could the configuration $\{c\}$. Such terminal but incomplete results can appear in a composition of strategies through the strategies being partial, in that one or both strategies do not respond in all cases—the example above. Such partial strategies can appear as the composition of two strategies through the occurrence of deadlocks because the two strategies impose incompatible causal dependencies on moves in game at which they interact. $\square$

Remark on schedulers Often in compositional treatments of probabilistic processes one sees a use of "schedulers" to "resolve the nondeterminism" due to openness to the environment. Here the use of schedulers is replaced by that of counterstrategy to resolve the nondeterminism. The counterstrategy may be deterministic (so straightforwardly a deterministic probabilistic strategy), in which case it resolves the nondeterminism by selecting at most one play for Opponent.

11.3 2-cells, a bicategory

We have thus extended composition of strategies to composition of probabilistic strategies. This doesn't yet yield a bicategory of probabilistic strategies. The extra structure of configuration-valuations in strategies has to be respected in our choice of 2-cell. The investigation of a suitable notion of 2-cell is the subject of the next section.

We first look for an analogue of the well-known result allowing a probability distribution to be pushed forward across an continuous (or measurable) function. This is not immediate as the configuration-valuations associated with strategies take account of Opponent moves so do not correspond to traditional probability distributions.

Proposition 11.31. *Let $\sigma : S \to A$ be a strategy in A and $\sigma' : S' \to A$ a total map of event structures with polarity. Let $f : S \to S'$ be a total map of event structures with polarity s.t. $\sigma' f = \sigma$. Then, f is receptive and innocent. A fortiori if f is 2-cell from strategy σ to strategy σ' in the bicategory of games and strategies, then f is receptive and innocent.*

Proof. The map f inherits receptivity and innocence from σ, in the case of innocence using the fact the σ' locally reflects causally dependency. □

Example 11.32. It seems impossible to push forward configuration valuations across arbitrary 2-cells. For example, consider the game A comprising two conflicting Opponent move and one Player move:

$$\begin{array}{c} \oplus \\ \\ \ominus_1 \sim\sim \ominus_2 \,. \end{array}$$

Let one probabilistic strategy comprise

$$\begin{array}{ccc} \oplus_1 & & \oplus_2 \\ \uparrow & & \uparrow \\ \ominus_1 & \sim\sim & \ominus_2 \end{array}$$

with obvious map σ, where the left Player move occurs with probability p_1 and the Player move on the right with probability p_2 according to a configuratiopn-valuation v, *i.e.* $v(\{\ominus_1, \oplus_1\}) = p_1$ and $v(\{\ominus_2, \oplus_2\}) = p_2$. Take another strategy to be the identity map A to A. It seems compelling to make the push forward of v across σ assign p_1 to the configuration $\{\ominus_1, \oplus\}$ and p_2 to the configuration $\{\ominus_2, \oplus\}$. What value should the push forward of v assign to the configuration $\{\oplus\}$? Because configuration-valuations are invariant under Opponent moves, it has to be simultaneously p_1 and p_2 —impossible if $p_1 \neq p_2$.

We shall now show the following theorem showing how to push forward configuration valuations across maps which are both rigid and receptive; in particular it will allow us to push forward a configuration valuation across a rigid map between strategies.

Theorem11.35. Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Let v be a configuration-valuation on S. Then, taking

$$v'(y) =_{\text{def}} \sum_{x:fx=y} v(x)$$

for $y \in \mathcal{C}(S')$, defines a configuration-valuation, written fv, on S'. (An empty sum gives 0 as usual.)

The proof of the theorem proceeds in the following steps, needed to cope with the fact sums can be infinite while also involving negative terms.

Lemma 11.33. *Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Let v be a configuration-valuation on S. Then, taking*

$$v'(y) =_{\text{def}} \sum_{x:fx=y} v(x)$$

we have $v'(y) \in [0,1]$, for $y \in \mathcal{C}(S')$. Moreover, $v'(\varnothing) = 1$ and $y \subseteq^- y'$ in $\mathcal{C}(S')$ implies $v'(y) = v'(y')$.

Proof. We check that for $y \in \mathcal{C}(S')$ the assignment $v'(y)$ is in $[0,1]$. Choose a covering chain

$$\varnothing \overset{t_1}{\longrightarrow\!\!\subset} y_1 \overset{t_2}{\longrightarrow\!\!\subset} \cdots \overset{t_n}{\longrightarrow\!\!\subset} y_n = y$$

up to y. As f is rigid for each $x \in \mathcal{C}(S)$ s.t. $fx = y$ there is a corresponding covering chain

$$\varnothing \overset{s_1}{\longrightarrow\!\!\subset} x_1 \overset{s_2}{\longrightarrow\!\!\subset} \cdots \overset{s_n}{\longrightarrow\!\!\subset} x_n = x$$

with $f(s_i) = t_i$ for $0 < i \leq n$. Consider the tree with sub-branches all initial sub-chains of covering chains up to each x s.t. $fx = y$; the tree has the empty covering chain as its root and configurations x, where $fx = y$, as its maximal nodes. Because f is receptive the tree only branches at its +ve coverings, associated with different, possibly infinitely many, s_i which map to a +ve event t_i. The corresponding configurations x_i are pairwise incompatible. Although such configurations x_i may form an infinite set, by the drop condition for v, the values of any finite subset will have sum less than or equal to $v(x_{i-1})$, a property which must therefore also hold for the sum of values of all the x_i. The value remains constant across any −ve event. Hence, working up the tree from the root we obtain that $\sum_{x:fx=y} v(x) \leq 1$.

Clearly, $v'(\varnothing) = v(\varnothing) = 1$. Suppose $y \subseteq^- y'$ in $\mathcal{C}(S')$. From the properties of f, x s.t. $fx = y$ determines a unique x' s.t. $x \subseteq^- x'$ and $fx' = y'$, and *vice versa*; in this correspondence $v(x) = v(x')$, as v is a configuration-valuation. Consequently, the sums yielding $v'(y)$ and $v'(y')$ have the same component values and are the same. □

For v' to be a configuration valuation it remains to verify that v' satisfies the +ve drop condition. We first show this for a special case:

Lemma 11.34. *Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Assume that S has only finitely many +ve events. Then, v' as defined above in Lemma 11.33 is a configuration valuation.*

Proof. Suppose $y \stackrel{+}{-\!\!\!\subset} y_1, \cdots, y_n$. We claim that

$$d_{v'}^{(n)}[y; y_1, \cdots, y_n] = \sum_{x : fx = y} d_v^{(n)}[x; X(x)]$$

so is non-negative, where

$$X(x) =_{\mathrm{def}} \{x' \mid x -\!\!\!\subset x' \ \&\ fx' \in \{y_1, \cdots, y_n\}\}\,.$$

The notation $d_v^{(n)}[x; X(x)]$ is justifiable as the drop function is invariant under permutation and repetition of arguments. Recall

$$d_{v'}^{(n)}[y; y_1, \cdots, y_n] =_{\mathrm{def}} v'(y) - \sum_{\emptyset \neq I \subseteq \{1, \cdots, n\}} (-1)^{|I|+1} v'(\bigvee_{i \in I} y_i)\,.$$

The claim follows because by the rigidity of f any non-zero contribution

$$(-1)^{|I|+1} v'(\bigcup_{i \in I} y_i)$$

is the sum of contributions

$$(-1)^{|I|+1} v(\bigcup_{i \in I} x_i)\,,$$

a summand of $d_v^{(n)}[x; X(x)]$, over x s.t. there are $x_i \in X(x)$ with $fx_i = y_i$ for all $i \in I$. □

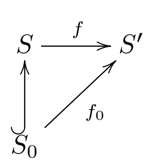
We can now complete the proof of the theorem.

Theorem 11.35. *Let $f : S \to S'$ be a receptive and rigid map between event structures with polarity. Let v be a configuration-valuation on S. Then, taking*

$$v'(y) =_{\mathrm{def}} \sum_{x : fx = y} v(x)$$

for $y \in \mathcal{C}(S')$, defines a configuration-valuation, written fv, on S'.

Proof. We use a slight variation on the $\trianglelefteq$ approximation order between event structures from [5, 3]. We write $S_0 \trianglelefteq S_1$ to mean there is a *receptive* rigid inclusion map between event structures with polarity from S_0 to S_1. Together all $S_0 \trianglelefteq S$ where S_0 has finitely many +-events form a directed subset of approximations to S; their $\trianglelefteq$-least upper bound is S got as their union. Such S_0 are associated with receptive rigid maps $f_0 : S_0 \to S'$ got as restrictions of f,



and configuration-valuations v_{S_0} got as restrictions v.

Let $y \overset{+}{\longrightarrow\!\!\subset} y_1, \cdots, y_n$ in $\mathcal{C}(S')$. We claim that

$$d_v[y; y_1, \cdots, y_n] = \lim_{S_0 \trianglelefteq S} d^{S_0}[y; y_1, \cdots, y_n] \qquad (\dagger)$$

i.e., that $d_v[y; y_1, \cdots, y_n]$ is the limit of $d^{S_0}[y; y_1, \cdots, y_n]$, the drop functions got by pushing forward v_{S_0} along f_0 to a configuration-valuation for S'—justified by Lemma 11.34.

Let $\epsilon > 0$. For each $I \subseteq \{1, \cdots, n\}$ there is large enough $S_I \trianglelefteq S$ s.t. for all $\trianglelefteq$-larger S_0,

$$0 \leq v(\bigvee_{i \in I} y_i) - v_{S_0}(\bigvee_{i \in I} y_i) \leq \epsilon / 2^n\,.$$

(When $I = \emptyset$ take $\bigvee_{i \in I} y_i = y$.) Taking S_1 to be $\trianglelefteq$-larger than all S_I where $I \subseteq \{1, \cdots, n\}$, we get for all S_2 with $S_1 \trianglelefteq S_2$ that

$$|d_v[y; y_1, \cdots, y_n] - d^{S_2}[y; y_1, \cdots, y_n]| < 2^n \epsilon / 2^n = \epsilon\,.$$

As ϵ was arbitrary we deduce ($\dagger$), ensuring $d_v[y; y_1, \cdots, y_n] \geq 0$, as required. □

Consequently, we can push forward a configuration-valuation across a rigid 2-cell between strategies—recall that 2-cells are automatically receptive. Given this it is sensible to adopt the following definition of 2-cell between probabilistic strategies. A 2-cell from a probabilistic strategy $v, \sigma : S \to A^\perp \| B$ to a probabilistic strategy $v', \sigma' : S' \to A^\perp \| B$ is a rigid map $f : S \to S'$ for which both $\sigma = \sigma' f$ and the push-forward $fv \leq v'$, *i.e.* for any finite configuration of S' the value $(fv)(x) \leq v'(fx)$.

Such 2-cells include receptive rigid embeddings f which preserve the value assigned by configuration-valuations, so $(fv)(x) = v'(fx)$ when $x \in \mathcal{C}(S)$; notice that the push-forward fv will assign value 0 to any configuration not in the image of f, so not impose any additional constraint on the values v' takes outside the image of f. Rigid embeddings, first introduced by Kahn and Plotkin [31] provide a method for defining strategies recursively. One way to characterize those maps $f : S \to S'$ of event structures which are rigid embeddings is as injective functions on events for which the inverse relation f^{op} is a (partial) map of event structures $f^{\text{op}} : S' \to S$.

In turn, 2-cells based on rigid embeddings include as special case that in which the function f is an inclusion. Receptive rigid embeddings which are inclusions give a (slight variant on a) well-known approximation order $\trianglelefteq$ on event structures. The order $\trianglelefteq$ forms a 'large cpo' and is useful when defining event structures recursively [5, 3]. With some care in choosing the precise construction of composition it provides an enrichment of probabilistic strategies and an elementary technique for defining probabilistic strategies recursively. Spelt out, when $v, \sigma : S \to A^\perp \| B$ and $v', \sigma' : S' \to A^\perp \| B$ are probabilistic strategies, we write

$$(v, \sigma) \trianglelefteq (v', \sigma')$$

iff $S \trianglelefteq S'$, the associate inclusion map $i : S \hookrightarrow S'$ makes $\sigma = \sigma' i$ and $v(x) = v'(x)$ for all $x \in \mathcal{C}(S)$. There can be many different, though isomorphic, $\trianglelefteq$-minimal probabilistic strategies, differing only in their choices of initial −-events; to be receptive they must start with copies of initial −-events of the game. Any chain

$$(v_0, \sigma_0) \trianglelefteq (v_1, \sigma_1) \trianglelefteq \cdots \trianglelefteq (v_n, \sigma_n) \trianglelefteq \cdots$$

has a least upper bound got by taking the union of the event structures.

To show that 2-cells compose functorially we use the following lemma. For probabilistic strategies $v_S, \sigma : S \to A^\perp \| B$ and $v_T, \tau : T \to B^\perp \| C$ we write $v_T \odot v_S$, respectively, $v_T \circledast v_S$ for the configuration-valuations on $T \odot S$ and $T \circledast S$ in the composition $(v_T, \tau) \odot (v_S, \sigma)$ and the composition without hiding $(v_T, \tau) \circledast (v_S, \sigma)$.

Lemma 11.36. *Let $f : \sigma \to \sigma'$ be a rigid 2-cell between strategies $\sigma : S \to A^\perp \| B$ and $\sigma' : S' \to A^\perp \| B$. Let $g : \tau \to \tau'$ be a rigid 2-cell between strategies $\tau : T \to B^\perp \| C$ and $\tau' : T' \to B^\perp \| C$. Let v_S be a configuration-valuation for S and v_T a configuration-valuation for T. Then,*

$$(g \odot f)(v_T \odot v_S) = (g v_T) \odot (f v_S)$$

and

$$(g \circledast f)(v_T \circledast v_S) = (g v_T) \circledast (f v_S)\,.$$

Proof. Omitted—see [1] □

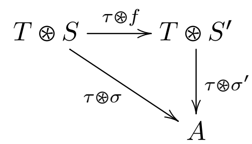
Corollary 11.37. *Composition of probabilistic strategies is functorial w.r.t. 2-cells, and functorial w.r.t. those 2-cells which are rigid embeddings.*

Combining:

Theorem 11.38. *Race-free games with probabilistic strategies with composition and copy-cat defined as in Lemma 11.25 and Corollary 11.29 inherit the structure of a a bicategory from that of games with strategies. 2-cells between probabilistic strategies are now restricted to rigid maps satisfying the conditions explained above. The bicategory restricts to one in which the cells are rigid embeddings.*

The order-enriched category $\mathbf{Games}_0$ of rigid-image strategies supports probability to give us an order-enriched category of probabilistic rigid-image strategies. A probabilistic rigid-image strategy over a game A comprises a rigid-image strategy $\sigma : S \to A$ together with a configuration-evaluation v for S. Given probabilistic rigid image strategies $v_S, \sigma : S \to A^\perp \| B$ and $v_T, \tau : T \to B^\perp \| C$ their composition comprises $(\tau \odot \sigma)_0 : (T \odot S)_0 \to A^\perp \| C$, the rigid image of $\tau \odot \sigma$, with configuration-valuation the push-forward along the map $T \odot S \to (T \odot S)_0$ to the rigid image of the configuration valuation $x \mapsto v_S(\Pi_S x) \times v_T(\Pi_T x)$. Is anything lost in moving to probabilistic rigid-image strategies? No, in the sense that a probabilistic strategy and its probabilistic rigid-image will always induce the same probability distribution on the game whenever they are composed with a probabilistic counterstrategy [1]:

Proposition 11.39. *Let $f : (\sigma, \upsilon) \Rightarrow (\sigma', \upsilon')$ be a 2-cell between probabilistic strategies $\upsilon, \sigma : S \to A$ and $\upsilon', \sigma' : S' \to A$ for which the push-forward $f\upsilon = \upsilon'$. Let $\upsilon_T, \tau : T \to A^\perp$ be a probabilistic counterstrategy. Then*



commutes and the push-forward $(\tau \circledast f)(\upsilon_T \circledast \upsilon) = \upsilon_T \circledast \upsilon'$. Moreover, $T \circledast S$ with $\upsilon_T \circledast \upsilon$ and $T \circledast S'$ with $\upsilon_T \circledast \upsilon'$ are probabilistic event structures determining continuous valuations w and w' respectively. The push-forwards of w and w' across the maps $\tau \circledast \sigma$ and $\tau \circledast \sigma'$ respectively to continuous valuations on the open sets of $\mathcal{C}^\infty(A)$ are the same.

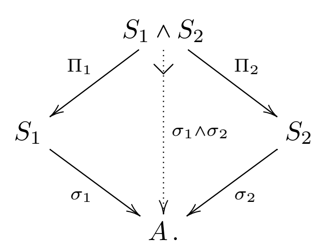
11.4 Probabilistic processes

As an indication of the expressivity of probabilistic strategies we sketch how they straightforwardly include a simple language of probabilistic processes, reminiscent of a higher-order CCS. For this section only, write $\sigma : A$ to mean σ is a probabilistic strategy in game A. Probabilistic strategies are closed under the following operations.[2]

Composition $\sigma \odot \tau : A \| C$, if $\sigma : A \| B$ and $\tau : B^\perp \| C$. Hiding is automatic in a synchronized composition directly based on the composition of strategies.

Simple parallel composition $\sigma \| \tau : A \| B$, if $\sigma : A$ and $\tau : B$. Note that simple parallel composition can be regarded as a special case of synchronized composition: via the identification of $\sigma \| \tau$ with $\tau \odot \sigma$, taking $\sigma : A^\perp \relbar\joinrel\mapsto \varnothing$ and $\tau : \varnothing \relbar\joinrel\mapsto B$, the operation $\sigma \| \tau$ yields a probabilistic strategy. Supposing $\sigma : S \to A$ and $\tau : T \to B$ and S and T have configuration valuations υ_S and υ_T, respectively, then the configuration valuation υ for $S \| T$ satisfies $\upsilon(x) = \upsilon_S(x_1) \times \upsilon_T(x_2)$, for $x \in \mathcal{C}(S \| T)$.

Conjunction if $\sigma_1 : A$ and $\sigma_2 : A$ we can conjoin the strategies by forming their pullback:



[2]For a richer language of probabilistic strategies see [32].

If σ_1 and σ_2 are associated with configuration-valuations v_1 and v_2 respectively then we tentatively take the configuration-valuation of the pullback to be $v(x) = v_1(\Pi_1 x) \times v_2(\Pi_2 x)$ for $x \in \mathcal{C}(S_1 \wedge S_2)$.

To check that v is indeed a configuration-valuation we embed configurations of $S_1 \wedge S_2$ in those of $S_1 \| S_2$ as described in the next lemma, so inheriting the conditions required of v from those of the configuration-valuation of $\sigma_1 \| \sigma_2$.

Lemma 11.40. *Define*

$$\psi : \mathcal{C}(S_1 \wedge S_2) \to \mathcal{C}(S_1 \| S_2)$$

by $\psi(x) = \Pi_1 x \| \Pi_2 x$ *for* $x \in \mathcal{C}(S_1 \wedge S_2)$. *Then,*

(i) ψ *is injective,*

(ii) ψ *preserves unions, and*

(iii) ψ *reflects compatibility, and in particular +-compatibility: if* $x \subseteq^+ y$ *and* $x \subseteq^+ z$ *in* $\mathcal{C}(S_1 \wedge S_2)$ *and* $\psi(y) \cup \psi(z) \in \mathcal{C}(S_1 \| S_2)$, *then* $y \cup z \in \mathcal{C}(S_1 \wedge S_2)$.

Proof. Consider the pullback $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$, π_1, π_2 in stable families of σ_1 and σ_2, regarded as maps between families of configurations. Configurations $\mathcal{C}(S_1 \wedge S_2)$ are order isomorphic, under inclusion, to configurations $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$. See the end of Section 3.3.4 for the detailed construction of pullbacks of stable families. It is thus sufficient to show that $\phi : \mathcal{C}(S_1) \wedge \mathcal{C}(S_2) \to \mathcal{C}(S_1 \| S_2)$, where $\phi(x) = \pi_1 x \| \pi_2 x$ for $x \in \mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$, satisfies conditions (i), (ii) and (iii) in place of ψ. (i) Injectivity follows because configurations in the pullback of stable families are determined by their projections; the nature of events of the pullback fixes their synchronisations. (ii) is obvious. (iii) To show ϕ reflects compatibility, assume $x \subseteq y$ and $x \subseteq z$ in $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$ and $\phi(y) \cup \phi(z) \in \mathcal{C}(S_1 \| S_2)$. Inspecting the construction of the pullback $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$ it is now easy to check that $y \cup z$ satisfies the conditions needed to be in $\mathcal{C}(S_1) \wedge \mathcal{C}(S_2)$, as required. □

Corollary 11.41. *Taking* $v(x) = v_1(\Pi_1 x) \times v_2(\Pi_2 x)$ *for* $x \in \mathcal{C}(S_1 \wedge S_2)$ *defines a configuration-valuation of* $S_1 \wedge S_2$.

Proof. The assignment $x \mapsto v_1(x_1) \times v_2(x_2)$, for $x \in \mathcal{C}(S_1 \| S_2)$ determines a configuration-valuation of $S_1 \| S_2$. The one non-obvious condition required of v to be a configuration-valuation is the +-drop condition. This follows directly from the +-drop condition holding in $\mathcal{C}(S_1 \| S_2)$ because ψ reflects +-compatibility. □

Input prefixing $\sum_{i \in I} \ominus.\sigma_i : \sum_{i \in I} \ominus.A_i$, if $\sigma_i : A_i$, for $i \in I$, where I is countable.

Output prefixing $\sum_{i \in I} p_i \oplus.\sigma_i : \sum_{i \in I} \oplus.A_i$, if $\sigma_i : A_i$, for $i \in I$, where I is countable, and $p_i \in [0,1]$ for $i \in I$ with $\sum_{i \in I} p_i \leq 1$. If $\sum_{i \in I} p_i < 1$, there is non-zero probability of terminating without any action. By design $(\sum_{i \in I} \oplus.A_i)^\perp = \sum_{i \in I} \ominus.A_i^\perp$.

General probabilistic sum More generally we can define $\bigoplus_{i\in I} p_i\sigma_i : A$, for $\sigma_i : A$ and I countable with sub-probability distribution $p_i, i \in I$. The operation makes the +-events of different components conflict and re-weights the configuration-valuation on the components according to the sub-probability distribution. In order for the sum to remain receptive, the initial −ve events of the components over a common event in the game A must be identified.

Relabelling, the composition $f_*\sigma : B$, if $\sigma : A$ and $f : A \to B$, possibly partial on +ve events but always defined on −ve events, is receptive and innocent in the sense of Definition 4.6. Then the composition of maps $f\sigma : S \to B$ is receptive and innocent. Its defined part, taken to be $f_*\sigma : B$, is given by the factorization

$$\begin{array}{ccc} S & \longrightarrow & S \downarrow D \\ & {\scriptstyle \sigma}\searrow & \downarrow {\scriptstyle f_*\sigma} \\ & & A\,, \end{array}$$

where D is the subset of S at which $f\sigma$ is defined, is a strategy over B. If the configuration-valuation on S is v then that on $S \downarrow D$ is given by $x \mapsto v([x])$, for $x \in \mathcal{C}(S \downarrow D)$, where $[x]$ is the down-closure of x in S. The map $f_*\sigma : B$ is a strategy because, directly from the definition of innocence of partial maps, the projection $S \to S \downarrow D$ reflects immediate causal dependencies *from* +ve events and *to* −ve events. The function $x \mapsto v([x])$, for $x \in \mathcal{C}(S \downarrow D)$, is a configuration valuation: First, clearly $v[\varnothing]) = v(\varnothing) = 0$. Second, if $x \subseteq^- y$ in $\mathcal{C}(S \downarrow D)$, then $[x] \subseteq^- [y]$ in $\mathcal{C}(S)$ directly from the −-innocence of f, ensuring $v([x]) = v([y])$. Third, the drop condition is inherited from v. Assuming $y \stackrel{+}{-\!\!\!\subset} x_1, \cdots, x_n$ in $\mathcal{C}(S \downarrow D)$ we obtain $[y] \subseteq^+ [x_1], \cdots, [x_n]$ in $\mathcal{C}(S)$ because f is only undefined on +ve events. Hence, by the drop condition for v,

$$v([y]) - \sum_I (-1)^{|I|+1} v(\bigcup_{i\in I} [x_i]) \geq 0\,,$$

where I ranges over subsets $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{[x_i] \mid i \in I\}\uparrow_S$. But,

$$\{[x_i] \mid i \in I\}\uparrow_S \iff \{x_i \mid i \in I\}\uparrow_{S\downarrow V}\,,$$

and down-closure commutes with unions. So

$$v([y]) - \sum_I (-1)^{|I|+1} v(\bigcup_{i\in I}[x_i]) = v([y]) - \sum_I (-1)^{|I|+1} v([\bigcup_{i\in I} x_i])\,,$$

where in the latter expression I ranges over subsets $\varnothing \neq I \subseteq \{1, \cdots, n\}$ s.t. $\{x_i \mid i \in I\}\uparrow_{S\downarrow V}$.

In particular, the composition $f\sigma : B$, if $\sigma : A$ and $f : A \to B$ is itself a strategy, *i.e.* total, receptive and innocent.

Pullback $f^*\sigma : A$, if $\sigma : B$ and $f : A \to B$ is a map of event structures, possibly partial, which reflects +-consistency in the sense that

$$y \stackrel{+}{-\!\!\!\subset} x_1, \cdots, x_n \;\&\; \{fx_i \mid 1 \leq i \leq n\}\uparrow \implies \{x_i \mid 1 \leq i \leq n\}\uparrow\ .$$

The strategy $f^*\sigma$ is got by the pullback

$$\begin{array}{ccc} S' & \xrightarrow{f'} & S \\ {\scriptstyle f^*\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow[f]{} & B\,. \end{array}$$

Then, the map f' also reflects $+$-consistency. This fact ensures we define a configuration-valuation $v_{S'}$ on S' by taking $v_{S'}(x) = v_S(f'x)$, for $x \in \mathcal{C}(S')$. If $\sigma : S \to B$ is a strategy then so is $f^*\sigma : S' \to A$. Pullback along $f : A \to B$ may introduce events and causal links, present in A but not in B. The pullback operation subsumes the operations of prefixing $\ominus.\sigma$ and $\oplus.\sigma$ and we can recover the previous prefix sums if we also have have sum types—see below.

Sum types If A_i, $i \in I$, is a countable family of games, we can form their sum, the game $\sum_{i\in I} A_i$ as the sum of event structures. If $\sigma : A_j$, for $j \in I$, we can create the probabilistic strategy $j\,\sigma : \sum_{i\in I} A_i$ in which we extend σ with those initial $-$ve events needed to maintain receptivity. A probabilistic strategy of sum type $\sigma : \sum_{i\in I} A_i$ projects to a probabilistic strategy $(\sigma)_j : A_j$ where $j \in I$.

Abstraction $\lambda x : A.\sigma : A \multimap B$. Because probabilistic strategies form a monoidal-closed bicategory, with tensor $A\|B$ and function space $A \multimap B =_{\text{def}} A^\perp\|B$, they support an (linear) λ-calculus, which in this context permits process-passing as in [33].

Recursive types and probabilistic processes can be dealt with along standard lines [5].

The types as they stand are somewhat inflexible. For example, that maps of event structures are locally injective would mean that simple labelling of events as in say CCS could not be directly captured through typing. However, this can be remedied by introducing monads, but doing this in sufficient generality would involve the introduction of symmetry.

In the pullback operations we have relied on certain maps being stable under pullback. The following two propositions make good our debt, and use techniques from open maps [34].

Proposition 11.42. *If $\sigma : S \to B$ is a strategy then so is $f^*\sigma : S' \to A$.*

Proof. Define an *étale* map (w.r.t. to a path category $\mathcal{P}$) to be like an open map, but where the lifting is unique. It is straightforward to show that the pullback of an étale map is étale. In fact, strategies can be regarded as étale maps, from which the proposition follows. Within the category of event structures with polarity and partial maps, take the path subcategory $\mathcal{P}$ to comprise all finite elementary event structures with polarity and take a typical map $f : p \to q$ in $\mathcal{P}$ to be a map such that:

(i) if $e \rightarrowtriangle_p e'$ with e −ve and e' +ve and both $f(e)$ and $f(e')$ defined, then $f(e) \rightarrowtriangle_q f(e')$; and

(ii) all events in q not in the image fp are −ve.

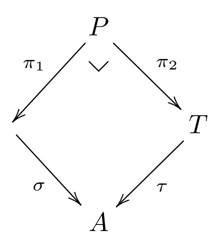
It can be checked that w.r.t. this choice of $\mathcal{P}$ the étale maps are precisely those maps which are strategies. □

Proposition 11.43. *If $f : A \to B$ reflects +-consistency, then so does $f' : S' \to S$.*

Proof. As +-consistency-reflecting maps are special kinds of open maps, known to be stable under pullback. An appropriate path category comprises: all finite event structures with polarity for which there is a subset M of ≤-maximal +-events s.t. a subset X is consistent iff $X \cap M$ contains at most one event of M—all finite elementary event structures with polarity are included as M, the chosen subset of ≤-maximal +-events, may be empty; maps in the path category are rigid maps of event structures with polarity whose underlying functions are bijective on events. □

11.4.1 Payoff

Given a probabilistic strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^\perp$ we obtain



with valuation $v(x) = v_S(\pi_1 x) \times v_T(\pi_2 x)$, for $x \in \mathcal{C}(P)$, on the pullback P—a probabilistic event structure, with probability measure $\mu_{\sigma,\tau}$. Define $f =_{\mathrm{def}} \sigma\pi_1 = \tau\pi_2$. Adding *payoff* as a Borel measurable function $X : \mathcal{C}^\infty(A) \to \mathbb{R}$ the *expected payoff* is obtained as the Lebesgue integral

$$\begin{aligned}\mathbf{E}_{\sigma,\tau}(X) &=_{\mathrm{def}} \int_{x\in\mathcal{C}^\infty(P)} X(f(x))\ d\mu_{\sigma,\tau}(x)\\ &= \int_{y\in\mathcal{C}^\infty(A)} X(y)\ d\mu_{\sigma,\tau}f^{-1}(y)\,,\end{aligned}$$

where we can choose either to integrate over $\mathcal{C}^\infty(P)$ with measure $\mu_{\sigma,\tau}$, or over $\mathcal{C}^\infty(A)$ with measure $\mu_{\sigma,\tau}f^{-1}$.

11.4.2 A simple value-theorem

Let A be a game with payoff X. Its dual is the game $A^\perp$ with payoff $-X$. If A, X and B, Y are two games with payoff, their parallel composition $(A, X) \parr (B, Y)$ is the game with payoff $(A \| B, X + Y)$.

Let A be a game with payoff X. Define

$$\mathrm{val}(\mathrm{A}, \mathrm{X}) =_{\mathrm{def}} \sup_\sigma \inf_\tau \mathrm{E}_{\sigma,\tau}(\mathrm{X})$$
$$\mathrm{val}(\mathrm{A}^\perp, -\mathrm{X}) =_{\mathrm{def}} \sup_\tau \inf_\sigma \mathrm{E}_{\tau,\sigma}(-\mathrm{X}) = -\inf_\tau \sup_\sigma \mathrm{E}_{\sigma,\tau}(\mathrm{X}) .$$

The game A, X is said to have a value if

$$\mathrm{val}(\mathrm{A}, \mathrm{X}) = -\mathrm{val}(\mathrm{A}^\perp, -\mathrm{X}) = \mathrm{E}_{\sigma_0,\tau_0}(\mathrm{X}) ,$$

its value then being val(A, X).

The following proposition says that a Nash equiibrium—expressed in properties (1) and (2)—determines a value for a game with payoff.

Theorem 11.44. *Let A be a game with payoff X. Suppose there are strategy σ_0 and counterstrategy τ_0 s.t.*

$$(1) \forall \tau, \text{ a counterstrategy. } E_{\sigma_0,\tau}(X) \geq E_{\sigma_0,\tau_0}(X) \text{ and}$$
$$(2) \forall \sigma, \text{ a strategy. } E_{\sigma,\tau_0}(X) \leq E_{\sigma_0,\tau_0}(X) .$$

Then, the game A, X has a value and $E_{\sigma_0,\tau_0}(X)$ is the value of the game.

Proof. Letting σ stand for strategies and τ for counterstrategies, we have

$$\mathrm{val}(\mathrm{A}) =_{\mathrm{def}} \sup_\sigma \inf_\tau \mathrm{E}_{\sigma,\tau}(\mathrm{X})$$
$$\mathrm{val}(\mathrm{A}^\perp) =_{\mathrm{def}} \sup_\tau \inf_\sigma \mathrm{E}_{\tau,\sigma}(-\mathrm{X}) = -\inf_\tau \sup_\sigma \mathrm{E}_{\sigma,\tau}(\mathrm{X}) .$$

We require

$$\mathrm{val}(\mathrm{A}) = -\mathrm{val}(\mathrm{A}^\perp) = \mathrm{E}_{\sigma_0,\tau_0}(\mathrm{X}) .$$

For all strategies σ,

$$\inf_\tau E_{\sigma,\tau}(X) \leq E_{\sigma,\tau_0}(X) \leq E_{\sigma_0,\tau_0}(X)$$

by (2). Therefore

$$\sup_\sigma \inf_\tau E_{\sigma,\tau}(X) \leq E_{\sigma_0,\tau_0}(X) .$$

Also

$$\sup_\sigma \inf_\tau E_{\sigma,\tau}(X) \geq \inf_\tau E_{\sigma_0,\tau}(X) \geq E_{\sigma_0,\tau_0}(X)$$

by (1). Hence

$$\sup_\sigma \inf_\tau E_{\sigma,\tau}(X) = E_{\sigma_0,\tau_0}(X) . \tag{3}$$

Dually,

$$\sup_\sigma E_{\sigma,\tau}(X) \geq E_{\sigma_0,\tau}(X) \geq E_{\sigma_0,\tau_0}(X)$$

by (1). Therefore

$$\inf_\tau \sup_\sigma E_{\sigma,\tau}(X) \geq E_{\sigma_0,\tau_0}(X)\,.$$

Also,

$$\inf_\tau \sup_\sigma E_{\sigma,\tau}(X) \leq \sup_\sigma E_{\sigma,\tau_0}(X) \leq E_{\sigma_0,\tau_0}(X)$$

by (2). Hence

$$\inf_\tau \sup_\sigma E_{\sigma,\tau}(X) = E_{\sigma_0,\tau_0}(X)\,. \tag{4}$$

From (3) and (4) it follows that

$$\mathrm{val}(\mathrm{A}) = -\mathrm{val}(\mathrm{A}^\perp) = \mathrm{E}_{\sigma_0,\tau_0}(\mathrm{X})\,,$$

the value of the game, as required. □

Chapter 12

Quantum strategies

We first explore a definition of quantum event structure in which events are associated with projection or unitary operators. It is shown how this structure induces configuration-valuations, and hence probability measures, on compatible parts of the domain of configurations of the event structure. We conclude with a brief exploration of quantum games and strategies. A quantum game is taken to be a quantum event structure in which events carry polarities and a strategy in a quantum game as a probabilistic strategy in its event structure.

12.1 Quantum event structures

Event structures are a model of distributed computation in which the causal dependence and independence of events is made explicit. By associating events with the most basic operators on a Hilbert space, *viz.* projection and unitary operators, so that independent (*i.e.* concurrent) events are associated with independent (*i.e.* commuting) operators, we obtain quantum event structures.

An event associated with a projection is thought of as an elementary positive test; its occurrence leaves the system in the eigenspace associated with eigenvalue 1 (rather than 0) of the projection. An event associated with a unitary operator is an event of preparation; the preparation might be a change of the direction in which to make a measurement, or the undisturbed evolution of the system over a time interval. A configuration is thought of as specifying a distributed quantum experiment. As we shall see, w.r.t. an initial state given as a density operator, each configuration w of a quantum event structure determines a probabilistic event structure, giving a probability distribution on its sub-configurations—the possible results of the experiment w.

Throughout let $\mathcal{H}$ be a separable Hilbert space over the complex numbers. For operators A, B on $\mathcal{H}$ we write $[A, B] =_{\mathrm{def}} AB - BA$.

12.1.1 Events as operators

Formally, we obtain a quantum event structure from an event structure by interpreting its events as unitary or projection operators which must commute when events are concurrent.

Definition 12.1. A *quantum event structure* (over $\mathcal{H}$) comprises an event structure $(E, \leq, \mathrm{Con})$ together with an assignment Q_e of projection or unitary operators on $\mathcal{H}$ to events $e \in E$ such that for all $e_1, e_2 \in E$,

$$e_1 \; co \; e_2 \implies [Q_{e_1}, Q_{e_2}] = 0 \, .$$

Given a finite configuration, $x \in \mathcal{C}(E)$, define the operator A_x to be the composition $Q_{e_n} Q_{e_{n-1}} \cdots Q_{e_2} Q_{e_1}$ for some covering chain

$$\varnothing \overset{e_1}{\longrightarrow\!\!\!\subset} x_1 \overset{e_2}{\longrightarrow\!\!\!\subset} x_2 \cdots \overset{e_n}{\longrightarrow\!\!\!\subset} x_n = x$$

in $\mathcal{C}(E)$. This is well-defined as for any two covering chains up to x the sequences of events are Mazurkiewicz trace equivalent, *i.e.* obtainable, one from the other, by successively interchanging concurrent events. In particular $A_\varnothing$ is the identity operator on $\mathcal{H}$. An *initial state* is given by a density operator ρ on $\mathcal{H}$.

Interpretation

Consider first the simpler situation where in a quantum event structure E, Q the event structure E is elementary (*i.e.* all finite subsets are consistent). We regard E, Q as specifying a, possibly distributed, quantum experiment. The experiment says which unitary operators (events of preparation) and projection operators (elementary positive tests) to apply and in which order. The order being partial permits commuting operators to be applied concurrently, independently of each other, perhaps in a distributed fashion.

For a quantum event structure, E, Q, in general, an individual configuration $w \in \mathcal{C}^\infty(E)$ inherits the order of the ambient event structure E to become an elementary event structure, and can itself be regarded as a quantum experiment. The quantum event structure E, Q represents a collection of quantum experiments which may extend or overlap each other: when $w \subseteq w'$ in $\mathcal{C}^\infty(E)$ the experiment w' extends the experiment w, or equivalently w is a restriction of the experiment w'. In this sense a quantum event structure in general represents a nondeterministic quantum experiment. The extra generality will be crucial later in interpreting probabilistic quantum experiments.

12.1.2 From quantum to probabilistic

Consider a quantum event structure with initial state. A configuration w stands for an experiment and specifies which tests and preparations to try and in which order. In general, not all the tests in w need succeed, yielding as final result a possibly proper sub-configuration x of w. Theorem 12.2 below explains how there is an inherent probability distribution q_w over such final results. So an experiment provides a context for measurement w.r.t. which there is an intrinsic probability distribution over the possible outcomes. In particular, when the event structure is elementary it itself becomes a probabilistic event structure. (Below, by an unnormalised density operator we mean a positive, self-adjoint operator with trace less than or equal to one.)

Theorem 12.2. *Let E, Q be a quantum event structure with initial state ρ. Each configuration $x \in \mathcal{C}(E)$ is associated with an unnormalised density operator $\rho_x =_{\rm def} A_x \rho A_x^\dagger$ and a value in $[0,1]$ given by $v(x) =_{\rm def} \mathrm{Tr}(\rho_x) = \mathrm{Tr}(A_x^\dagger A_x \rho)$. For any $w \in \mathcal{C}^\infty(E)$, the function v restricts to a configuration-valuation v_w on the elementary event structure w (viz. the event structure with events w, and causal dependency and (trivial) consistency inherited from E); hence v_w extends to a probability measure q_w on $\mathcal{F}_w =_{\rm def} \{x \in \mathcal{C}^\infty(E) \mid x \subseteq w\}$.*

Proof. We show v restricts to a configuration-valuation on $\mathcal{F}_w$. As $A_\varnothing = \mathrm{id}_\mathcal{H}$, $v(\varnothing) = \mathrm{Tr}(\rho) = 1$. By Lemma 11.11, we need only to show $d_v^{(n)}[y; x_1, \cdots, x_n] \geq 0$ when $y \overset{e_1}{\longrightarrow\!\!\subset} x_1, \cdots, y \overset{e_n}{\longrightarrow\!\!\subset} x_n$ in $\mathcal{F}_w$.

First, observe that if for some event e_i the operator Q_{e_i} is unitary, then $d_v^{(n)}[y; x_1, \cdots, x_n] = 0$. W.l.o.g. suppose e_n is assigned the unitary operator U. Then, $A_{x_n} = U A_y$ so

$$v(x_n) = \mathrm{Tr}(A_{x_n}^\dagger A_{x_n} \rho) = \mathrm{Tr}(A_y^\dagger U^\dagger U A_y \rho) = \mathrm{Tr}(A_y^\dagger A_y \rho) = v(y)\,.$$

Let $\varnothing \neq I \subseteq \{1, \cdots, n\}$. Then, either $\bigcup_{i\in I} x_i = \bigcup_{i \in I} x_i \cup x_n$ or $\bigcup_{i\in I} x_i \overset{e_n}{\longrightarrow\!\!\subset} \bigcup_{i\in I} x_i \cup x_n$. In the either case—in the latter case by an argument similar to that above,

$$v(\bigcup_{i\in I} x_i) = v(\bigcup_{i\in I} x_i \cup x_n)\,.$$

Consequently,

$$\begin{aligned} d_v^{(n)}[y; x_1, \cdots, x_n] =& d_v^{(n-1)}[y; x_1, \cdots, x_{n-1}] - d_v^{(n-1)}[x_n; x_1 \cup x_n, \cdots, x_{n-1} \cup x_n] \\ =& v(y) - \sum_I (-1)^{|I|+1} v(\bigcup_{i\in I} x_i) - v(x_n) + \sum_I (-1)^{|I|+1} v(\bigcup_{i\in I} x_i \cup x_n) \\ =& \; 0 \end{aligned}$$

—above index I is understood to range over sets for which $\varnothing \neq I \subseteq \{1, \cdots, n\}$.

It remains to consider the case where all events e_i are assigned projection operators P_{e_i}. As $x_1, \cdots, x_n \subseteq w$ we must have that all the projection operators $P_{e_1}, \cdots, P_{e_n}$ commute.

As $[P_{e_i}, P_{e_j}] = 0$, for $1 \leq i, j \leq n$, we can assume an orthonormal basis which extends the sub-basis of eigenvectors of all the projection operators P_{e_i}, for $1 \leq i \leq n$. Let $y \subseteq x \subseteq \bigcup_{1\leq i\leq n} x_i$. Define P_x to be the projection operator got as the composition of all the projection operators P_e for $e \in x \setminus y$—this is a projection operator, well-defined irrespective of the order of composition as the relevant projection operators commute. Define B_x to be the set of those basis vectors fixed by the projection operator P_x. In particular, P_y is the identity operator and B_y the set of all basis vectors. When $x, x' \in \mathcal{C}(E)$ with $y \subseteq x \subseteq \bigcup_{1\le i\le n} x_i$ and $y \subseteq x' \subseteq \bigcup_{1\le i\le n} x_i$,

$$B_{x\cup x'} = B_x \cap B_{x'} .$$

Also,

$$P_x|\psi\rangle = \sum_{i\in B_x} \langle i|\psi\rangle\, |i\rangle ,$$

so

$$\langle\psi|P_x|\psi\rangle = \sum_{i\in B_x} \langle i|\psi\rangle\langle\psi|i\rangle = \sum_{i\in B_x} |\langle i|\psi\rangle|^2 ,$$

for all $|\psi\rangle \in \mathcal{H}$.

Assume $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$, where the ψ_k are normalised and all the p_k are positive with sum $\sum_k p_k = 1$. For x with $y \subseteq x \subseteq \bigcup_{1\le i\le n} x_i$,

$$\begin{aligned} v(x) &= \mathrm{Tr}(A_x^\dagger A_x \rho) \\ &= \mathrm{Tr}(A_y^\dagger P_x^\dagger P_x A_y \rho) \\ &= \mathrm{Tr}(A_y^\dagger P_x A_y \sum_k p_k|\psi_k\rangle\langle\psi_k|) \\ &= \sum_k p_k\, \mathrm{Tr}(A_y^\dagger P_x A_y|\psi_k\rangle\langle\psi_k|) \\ &= \sum_k p_k\langle A_y\psi_k|P_x|A_y\psi_k\rangle \\ &= \sum_{i\in B_x}\sum_k p_k|\langle i|A_y\psi_k\rangle|^2 = \sum_{i\in B_x} r_i , \end{aligned}$$

where we define $r_i =_{\mathrm{def}} \sum_k p_k|\langle i|A_y\psi_k\rangle|^2$, necessarily a non-negative real for $i \in B_x$.

We now establish that

$$d_v^{(n)}[y; x_1, \cdots, x_n] = \sum_{i\in B_y \smallsetminus B_{x_1}\cup\cdots\cup B_{x_n}} r_i ,$$

for all $n \in \omega$, by mathematical induction—it then follows directly that its value is non-negative.

The base case of the induction, when $n = 0$, follows as

$$d_v^{(0)}[y;] = v(y) = \sum_{i\in B_y} r_i ,$$

a special case of the result we have just established.

For the induction step, assume $n > 0$. Observe that

$$B_y \smallsetminus B_{x_1} \cup\cdots\cup B_{x_{n-1}} = (B_y \smallsetminus B_{x_1}\cup\cdots\cup B_{x_n}) \,\dot\cup\, (B_{x_n} \smallsetminus B_{x_1\cup x_n} \cup\cdots\cup B_{x_{n-1}\cup x_n}) ,$$

where as signified the outer union is disjoint. Hence,

$$\sum_{i\in B_y\smallsetminus B_{x_1}\cup\cdots\cup B_{x_{n-1}}} r_i = \sum_{i\in B_y\smallsetminus B_{x_1}\cup\cdots\cup B_{x_n}} r_i + \sum_{i\in B_{x_n}\smallsetminus B_{x_1\cup x_n}\cup\cdots\cup B_{x_{n-1}\cup x_n}} r_i ,$$

By definition,

$$d_v^{(n)}[y;x_1,\cdots,x_n] =_{\text{def}} d_v^{(n-1)}[y;x_1,\cdots,x_{n-1}] - d_v^{(n-1)}[x_n;x_1\cup x_n,\cdots,x_{n-1}\cup x_n]$$

—making use of the fact that we are only forming unions of compatible configurations. From the induction hypothesis,

$$d_v^{(n-1)}[y;x_1,\cdots,x_{n-1}] = \sum_{i\in B_y\setminus B_{x_1}\cup\cdots\cup B_{x_{n-1}}} r_i$$

$$\text{and } d_v^{(n-1)}[x_n;x_1\cup x_n,\cdots,x_{n-1}\cup x_n] = \sum_{i\in B_{x_n}\setminus B_{x_1\cup x_n}\cup\cdots\cup B_{x_{n-1}\cup x_n}} r_i\,.$$

Hence

$$d_v^{(n)}[y;x_1,\cdots,x_n] = \sum_{i\in B_y\setminus B_{x_1}\cup\cdots\cup B_{x_n}} r_i\,,$$

ensuring $d_v^{(n)}[y;x_1,\cdots,x_n]\geq 0$, as required.

By Theorem 11.14, the configuration-valuation v_w extends to a unique probability measure on $\mathcal{F}_w$. □

Corollary 12.3. *Let E,Q be a quantum event structure in which E is elementary. Assume an initial state ρ. Then, $x\mapsto \mathrm{Tr}(A_x^\dagger A_x\rho)$, for $x\in\mathcal{C}(E)$, is a configuration-valuation on E. It extends to a probability measure on the Borel sets of $\mathcal{C}^\infty(E)$.*

Theorem 12.2 is reminiscent of the consistent-histories approach to quantum theory [35] once we understand configurations as partial-order histories. The traditional decoherence/consistency conditions on histories, saying when a family of histories supports a probability distribution, have been replaced by $\subseteq$-compatibility.

Example 12.4. Let E comprise the quantum event structure with two concurrent events e_0 and e_1 associated with projectors P_0 and P_1, where necessarily $[P_0,P_1]=0$. Assume an initial state $|\psi\rangle\langle\psi|$, corresponding to the pure state $|\psi\rangle$. The configuration $\{e_0,e_1\}$ is associated with the following probability distribution. The probability that e_0 succeeds is $\|P_0|\psi\rangle\|^2$, that e_1 succeeds $\|P_1|\psi\rangle\|^2$, and that both succeed is $\|P_1P_0|\psi\rangle\|^2$.

In the case where P_0 and P_1 commute because $P_0P_1=P_1P_0=0$, the events e_0 and e_1 are mutually exclusive in the sense that there is probability zero of both events e_0 and e_1 succeeding, probability $\|P_0|\psi\rangle\|^2$ of e_0 succeeding, $\|P_1|\psi\rangle\|^2$ of e_1 succeeding, and probability $1-\|P_0|\psi\rangle\|^2-\|P_1|\psi\rangle\|^2$ of getting stuck at the empty configuration where neither event succeeds.

A special case of this is the measurement of a qubit in state ψ, the measurement of 0 where $P_0=|0\rangle\langle 0|$, and the measurement of 1 where $P_1=|1\rangle\langle 1|$, though here $\|P_0|\psi\rangle\|^2+\|P_1|\psi\rangle\|^2=1$, as a measurement of the qubit will determine a result of either 0 or 1. □

Example 12.5. Let E comprise the event structure with three events e_1, e_2, e_3 with trivial causal dependency and consistency relation generated by taking $\{e_1, e_2\} \in \text{Con}$ and $\{e_2, e_3\} \in \text{Con}$—so $\{e_1, e_3\} \notin \text{Con}$. To be a quantum event structure we must have $[Q_{e_1}, Q_{e_2}] = 0$, $[Q_{e_2}, Q_{e_3}] = 0$. The maximal configurations are $\{e_1, e_2\}$ and $\{e_2, e_3\}$. Assume an initial state $|\psi\rangle\langle\psi|$. The first maximal configuration is associated with a probability distribution where e_1 occurs with probability $\|Q_{e_1}|\psi\rangle\|^2$ and e_2 occurs with probability $\|Q_{e_2}|\psi\rangle\|^2$. The second maximal configuration is associated with a probability distribution where e_2 occurs with probability $\|Q_{e_2}|\psi\rangle\|^2$ and e_3 occurs with probability $\|Q_{e_3}|\psi\rangle\|^2$. □

12.1.3 Measurement

To support measurements yielding values we associate values with configurations of a quantum event structure E, Q, in the form of a measurable function, $V : \mathcal{C}^\infty(E) \to \mathbb{R}$. If the experiment results in $x \in \mathcal{C}^\infty(E)$ we obtain $V(x)$ as the measurement value resulting from the experiment. By Theorem 12.2, assuming an initial state given by a density operator ρ, we obtain a probability measure q_w on the sub-configurations of $w \in \mathcal{C}^\infty(E)$. This is interpreted as giving a probability distribution on the final results of an experiment w. Accordingly, w.r.t. an experiment $w \in \mathcal{C}^\infty(E)$, the expected value is

$$\mathbf{E}_w(V) =_{\text{def}} \int_{x \in \mathcal{F}_w} V(x)\ dq_w(x)\,.$$

Traditionally quantum measurement is associated with an Hermitian operator A on $\mathcal{H}$ where the possible values of a measurement are eigenvalues of A. How is this realized by a quantum event structure? Suppose the Hermitian operator has spectral decomposition

$$A = \sum_{i \in I} \lambda_i P_i$$

where orthogonal projection operators P_i are associated with eigenvalue λ_i. The projection operators satisfy $\sum_{i \in I} P_i = \text{id}_{\mathcal{H}}$ and $P_i P_j = 0$ if $i \neq j$.

Form the quantum event structure with concurrent events e_i, for $i \in I$, and $Q(e_i) = P_i$. Because the projection operators are orthogonal, $[P_i, P_j] = 0$ when $i \neq j$, so we do indeed obtain a quantum event structure. Let $V(\{e_i\}) = \lambda_i$, and take arbitrary values on all other configurations. The event structure has a single, maximum configuration $w =_{\text{def}} \{e_i \mid i \in I\}$. It is the experiment w which will correspond to traditional measurement via A. Assume an initial state $|\psi\rangle\langle\psi|$. It can be checked that the probability ascribed to each of the singleton configurations $\{e_i\}$ is $\langle\psi|P_i|\psi\rangle$, and is zero elsewhere. Consequently,

$$\mathbf{E}_w(V) = \sum_{i \in I} \lambda_i \langle\psi|P_i|\psi\rangle = \langle\psi|A|\psi\rangle$$

—the well-known expression for the expected value of the measurement A on pure state $|\psi\rangle$.

Example 12.6. The spin state of a spin-1/2 particle is an element of two-dimensional Hilbert space, $\mathcal{H}_2$. Traditionally the Hermitian operator for measuring spin in a particular fixed direction is

$$|\uparrow\rangle\langle\uparrow| \;-\; |\downarrow\rangle\langle\downarrow|\,.$$

It has eigenvectors $|\uparrow\rangle$ (spin up) with eigenvalue +1 and $|\downarrow\rangle$ (spin down) with eigenvalue −1. Accordingly, its quantum event structure comprises the two concurrent events u associated with projector $|\uparrow\rangle\langle\uparrow|$ and d with projector $|\downarrow\rangle\langle\downarrow|$. Its configurations are: $\varnothing$, $\{u\}$, $\{d\}$ and $\{u, d\}$. The value associated with the configuration $\{u\}$ is +1, and that with $\{d\}$ is −1. Given an initial pure state $a|\uparrow\rangle + b|\downarrow\rangle$, the probability of the experiment $\{u, d\}$ yielding value +1 is $|a|^2$ and that of yielding −1 is $|b|^2$. The probability that the experiment ends in configurations $\varnothing$ or $\{u, d\}$ is zero. Its expected value is $|a|^2 - |b|^2$. This would be the average value resulting from measuring the spin of a large number of particles initially in the pure state. □

An event logic

One way to assign values to configurations is via logic of which the assertions will be true (taken as 1) or false (0) at a configuration. Given a countable event structure E, we can build terms for events and assertions in a straightforward way. Event terms are given by $\epsilon ::= e \in E \mid v \in \mathrm{Var}$, where Var is a set of variables over events, and assertions by

$$L ::= \epsilon \mid \mathrm{T} \mid \mathrm{F} \mid L_1 \wedge L_2 \mid L_1 \vee L_2 \mid \neg L \mid \forall v.L \mid \exists v.L\,.$$

W.r.t. an environment $\zeta : \mathrm{Var} \to E$, an assertion L denotes $[\![L]\!]\zeta$, a Borel subset of $\mathcal{C}^\infty(E)$, for example:

$$[\![e]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid e \in x\} \qquad [\![v]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid \zeta(v) \in x\}$$
$$[\![\forall v.L]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid \forall e \in x.\ x \in [\![L]\!]\zeta[e/v]\}$$
$$[\![\exists v.L]\!]\zeta = \{x \in \mathcal{C}^\infty(E) \mid \exists e \in x.\ x \in [\![L]\!]\zeta[e/v]\}$$

with T, F, $\wedge$, $\vee$ and $\neg$ interpreted standardly by the set of all configurations, the emptyset, intersection, union and complement. In this logic, for example, $\neg(a{\downarrow} \wedge b{\downarrow}) \wedge \neg(a{\uparrow} \wedge b{\uparrow})$ could express the anti-correlation of the spin of particles a and b.

W.r.t. a quantum event structure with initial state, for an experiment the configuration w, the probability of the result of the quantum experiment satisfying L, a closed assertion of the logic with denotation U, is

$$q_w(U \cap \mathcal{F}_w)$$

which coincides with the expected value of the characteristic function for U.

12.1.4 Probabilistic quantum experiments

It can be useful, or even necessary, to allow the choice of which quantum measurements to perform to be made probabilistically. For example, experiments to invalidate the Bell inequalities, to demonstrate the non-locality of quantum physics, may make use of probabilistic quantum experiments.

Formally, a probability distribution over quantum experiments can be realized by a total map of event structures $f : P \to E$ where P, v is a probabilistic event structure and E, Q is a quantum event structure; the configurations of E correspond to quantum experiments assigned probabilities through P. Through the map f we can integrate the probabilistic and quantum features. Via the map f, the event structure E inherits a configuration valuation, making it itself a probabilistic event structure; we can see this indirectly by noting that if w_o is a continuous valuation on the open sets of P then $w_o f^{-1}$ is a continuous valuation on the open sets of E. On the other hand, via f the event structure P becomes a quantum event structure; an event $p \in P$ is interpreted as operation $Q(f(p))$. Of course, f can be the identity map, as is so in Example 12.7 below.

Suppose E, Q is a quantum event structure with initial state ρ and a measurable value function $V : \mathcal{C}^\infty(E) \to \mathbb{R}$. Recall, from Section 12.1.3, that the expected value of a quantum experiment $w \in \mathcal{C}^\infty(E)$ is

$$\mathbf{E}_w(V) =_{\text{def}} \int_{x \in \mathcal{F}_w} V(x)\ dq_w(x)\,,$$

where q_w is the probability measure induced on $\mathcal{F}_w$ by Q and ρ. The expected value of a probabilistic quantum experiment $f : P \to E$, where P, v is a probabilistic event structure is

$$\int_{w \in \mathcal{C}^\infty(E)} \mathbf{E}_w(V)\ d\mu f^{-1}(w)\,,$$

where μ is the probability measure induced on $\mathcal{C}^\infty(P)$ by the configuration-valuation v. Specialising the value function to the characteristic function of a Borel subset $U \subseteq \mathcal{C}^\infty(E)$, perhaps given by an assertion of the event logic of Section 12.1.3, the probability of the result of the probabilistic experiment satisfying U is

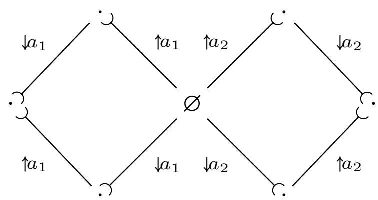
$$\int_{w \in \mathcal{C}^\infty(E)} q_w(U \cap \mathcal{F}_w)\ d\mu f^{-1}(w)\,.$$

The following example illustrates how a very simple form of probabilistic quantum experiment (in which the event structure has a discrete partial order of causal dependency) provides a basis for the analysis of Bell and EPR experiments.

Example 12.7. Imagine an observer who randomly chooses between measuring spin in a first fixed direction $\mathbf{a_1}$ or in a second fixed direction $\mathbf{a_2}$. Assume that the probability of measuring in the $\mathbf{a_1}$-direction is p_1 and in the $\mathbf{a_2}$-direction is p_2, where $p_1 + p_2 = 1$. The two directions $\mathbf{a_1}$ and $\mathbf{a_2}$ correspond to choices of bases $|{\uparrow} a_1\rangle$, $|{\downarrow} a_1\rangle$ and $|{\uparrow} a_2\rangle$, $|{\downarrow} a_2\rangle$ in $\mathcal{H}_2$. We describe this scenario as a probabilistic quantum experiment. The quantum event structure has four events, $\uparrow a_1, \downarrow a_1, \uparrow a_2, \downarrow a_2$, in which $\uparrow a_1, \downarrow a_1$ are concurrent, as are $\uparrow a_2, \downarrow a_2$; all other pairs of events are in conflict. The event $\uparrow a_1$ is associated with measuring spin up in direction $\mathbf{a_1}$ and the event $\downarrow a_1$ with measuring spin down in direction $\mathbf{a_1}$. Similarly, events $\uparrow a_2$ and $\downarrow a_2$ correspond to measuring spin up and down, respectively, in direction $\mathbf{a_2}$. Correspondingly, we associate events with the following projection operators:

$$Q(\uparrow a_1) = |\uparrow a_1\rangle\langle\uparrow a_1| , \qquad Q(\downarrow a_1) = |\downarrow a_1\rangle\langle\downarrow a_1| ,$$
$$Q(u_2) = |\uparrow a_2\rangle\langle\uparrow a_2| , \qquad Q(d_2) = |\downarrow a_2\rangle\langle\downarrow a_2| .$$

The configurations of the event structure take the form



where we have taken the liberty of inscribing the events just on the covering intervals. Measurement in the $\mathbf{a_1}$-direction corresponds to the configuration $\{\uparrow a_1, \downarrow a_1\}$—the configuration to the far left in the diagram—and in the $\mathbf{a_2}$-direction to the configuration $\{\uparrow a_2, \downarrow a_2\}$—that to the far right. To describe that the probability of the measurement in the $\mathbf{a_1}$-direction is p_1 and that in the $\mathbf{a_2}$-direction is p_2, we assign a configuration valuation v for which

$$v(\{\uparrow a_1, \downarrow a_1\}) = v(\{\uparrow a_1\}) = v(\{\downarrow a_1\}) = p_1 ,$$
$$v(\{\uparrow a_2, \downarrow a_2\}) = v(\{\uparrow a_2\}) = v(\{\downarrow a_2\}) = p_2 \quad \text{and} \quad v(\varnothing) = 1 .$$

Such a probabilistic quantum experiment is not very interesting on its own. But imagine that there are two similar observers A and B measuring the spins in directions $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{b_1}$, $\mathbf{b_2}$, respectively, of two particles created so that together they have zero angular momentum, ensuring they have a total spin of zero in any direction. Then quantum mechanics predicts some remarkable correlations between the observations of A and B, even at distances where their individual choices of what directions to perform their measurements could not possibly be communicated from one observer to another. For example, were both observers to choose the same direction to measure spin, then if one measured spin up then other would have to measure spin down even though the observers were light years apart.

We can describe such scenarios by a probabilistic quantum experiment which is essentially a simple parallel composition of two versions of the (single-observer) experiment above. In more detail, make two copies of the single-observer event structure: that for A, the event structure E_A, has events $\uparrow a_1, \downarrow a_1, \uparrow a_2, \downarrow a_2$, while that for B, the event structure E_B, has events $\uparrow b_1, \downarrow b_1, \uparrow b_2, \downarrow b_2$. Assume they possess configuration valuations v_A and v_B, respectively, determined by the probabilistic choices of directions made by A and B. Write Q_A and Q_B for the respective assignments of projection operators to events of E_A and E_B. The probabilistic event structure for the two observers together is got as $E_A \| E_B$, their simple parallel composition got by juxtaposition, with configuration valuation $v(x) = v_A(x_A) \times v_B(x_B)$, for $x \in \mathcal{C}(E_A \| E_B)$, where x_A and x_B are projections of x to configurations of A and B. In this compound system an event such as *e.g.* $\uparrow a_1$ is interpreted as the projection operator $Q_A(\uparrow a_1) \otimes \mathrm{id}_{\mathcal{H}_2}$ on the Hilbert space $\mathcal{H}_2 \otimes \mathcal{H}_2$, where the combined state of the two particles belongs. We can capture the correlation or anti-correlation of the observers' measurements of spin through a value function on configurations, given by

$$V(\{\uparrow a_i, \uparrow b_j\}) = V(\{\downarrow a_i, \downarrow b_j\}) = 1\,, \quad V(\{\uparrow a_i, \downarrow b_j\}) = V(\{\downarrow a_i, \uparrow b_j\}) = -1\,, \text{ and}$$
$$V(x) = 0 \text{ otherwise,}$$

and study their expectations under various initial states and choices of measurement. In this way probabilistic quantum experiments, as formalised through probabilistic and quantum event structures, provide a basis for the analysis of Bell or EPR experiments. □

The ideas of probabilistic and quantum event structures carry over to probabilistic and quantum games and their strategies; the result of the play of quantum strategy against a counterstrategy is a probabilistic event structure. This is yielding operations and languages which should be helpful in a structured development and analysis of experiments on quantum systems.

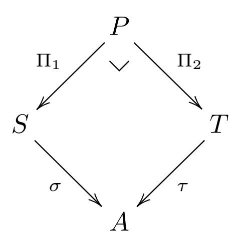
12.2 A simple form of quantum strategy

We present a simple form of quantum game and strategy.

Define a *quantum game* to comprise $A, pol, \mathcal{H}_A, Q, \rho$ where A, pol is a race-free event structure with polarity and A, Q is a quantum event structure, with Hilbert space $\mathcal{H}_A$; its *initial state* is a quantum game with ρ a density operator.

A *strategy* in a quantum game A, pol, Q, ρ comprises a probabilistic strategy in A, so a strategy $\sigma : S \to A$ together with configuration-valuation v on $\mathcal{C}(S)$.

Given a strategy $v_S, \sigma : S \to A$ and counter-strategy $v_T, \tau : T \to A^\perp$ in a quantum game A, Q we obtain a probabilistic event structure P via pull-back, *viz.*



with a configuration-valuation $v(x) =_{\mathrm{def}} v_S \Pi_1(x) \times v_T \Pi_2(x)$ on finite configurations $x \in \mathcal{C}(P)$. This induces a probabilistic measure μ on the event structure

P. Write $f =_{\text{def}} \sigma\Pi_1 = \tau\Pi_2$. We can interpret $f : P \to A$ as the probabilistic quantum experiment which results from the interaction of the strategy σ and the counter-strategy τ. We can investigate the probability the interaction of σ with τ produces a result in a Borel subset U of $\mathcal{C}^\infty(A)$ —that the probabilistic experiment induced by the interaction succeeds in U. Recall from Section 12.1.4 that the probability of the result of the probabilistic experiment satisfying U is

$$\int_{w \in \mathcal{C}^\infty(A)} q_w(U \cap \mathcal{F}_w)\ d\mu f^{-1}(w)\,.$$

We examine some special cases.

Consider the case where σ and τ are deterministic, with configuration valuations assigning one to each finite configuration. Then, P will also be deterministic in the sense that all its finite subsets will be consistent. It will thus have a single maximal configuration $x_0 \in \mathcal{C}^\infty(P)$. The configuration-valuation v will assign one to each finite configuration of P. In this case the probability measure on Borel subsets V of $\mathcal{C}^\infty(P)$ is simple to describe:

$$\mu(V) = \begin{cases} 1 & \text{if } x_0 \in V\,, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to

$$\int_{w \in \mathcal{C}^\infty(A)} q_w(U \cap \mathcal{F}_w)\ d\mu f^{-1}(w) = q_{fx_0}(U \cap \mathcal{F}_{fx_0})\,.$$

Consider now the case where Opponent initially offers $n \in \{1, \cdots, N\}$ mutually-inconsistent alternatives to Player and resumes with a deterministic strategy. Suppose too that initially Player chooses amongst the alternatives probabilistically, choosing option n with probability p_n, and then resumes deterministically. This will result in an event structure P taking the form of a prefixed sum $\sum_{1 \le n \le N} e_n.P_n$ in which all the events of P_n causally depend on event e_n. In this situation,

$$\int_{w \in \mathcal{C}^\infty(E)} q_w(U \cap \mathcal{F}_w)\ d\mu f^{-1}(w) = \sum_{1 \le n \le N} p_n \cdot q_{fx_n}(U \cap \mathcal{F}_{fx_n})\,,$$

where x_n is the maximal configuration in the component $e_n.P_n$ for $1 \le n \le N$.

Quantum games inherit the structure of a bicategory from probabilistic games. A strategy *from* a quantum game A *to* a quantum game B is a strategy in the quantum game $A^\perp \| B$. For this to make sense we have to extend the definitions of simple parallel composition and dual to quantum games. Assume A and B are quantum games. In defining their simple parallel composition $A\|B$ and dual $A^\perp$ we take:

$$\mathcal{H}_{A\|B} = \mathcal{H}_A \otimes \mathcal{H}_B, \quad Q_{A\|B}(1,a) = Q_A \otimes \mathrm{id}_{\mathcal{H}_B}, \quad Q_{A\|B}(2,b) = \mathrm{id}_{\mathcal{H}_A} \otimes Q_B,$$
$$\text{and} \quad \rho_{A\|B} = \rho_A \otimes \rho_B\,;$$

$$\mathcal{H}_{A^\perp} = \mathcal{H}_A, \quad \rho_{A^\perp} = \rho_A \quad \text{and} \quad Q_{A^\perp} = Q_A\,.$$

Although we do obtain a bicategory of quantum games in this way, it is not the final story. It presently lacks an operation to introduce entanglement across parallel components. There are limitations in all the quantum structure of a strategy being inherited from that of the game; in a more liberal notion of quantum strategy one would expect quantum structure to be possessed directly by the strategy. There is also the issue of adjoining value functions (*cf.* Section 12.1.3) to quantum games in a way that respects their bicategorical structure. Providing a structured account and analysis of quantum experiments, as in the simple experiment discussed in Example 12.7, should provide guidelines.

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Appendix A

Exercises

On event structures and stable families

Recommended exercises: 1, 3, 4, 5 (Harder), 6, 7, 10.

Exercise A.1. *Let* $(A, \leq_A, \mathrm{Con}_A), (B, \leq_B, \mathrm{Con}_B)$ *be event structures. Let* $f : A \rightharpoonup B$. *Show* f *is a map of event structures,* $f : (A, \leq_A, \mathrm{Con}_A) \to (B, \leq_B, \mathrm{Con}_B)$*, iff*

$$(i)\ \forall a \in A, b \in B.\ b \leq_B f(a) \implies \exists a' \in A.\ a' \leq_A a\ \&\ f(a') = b\,,\ \text{and}$$
$$(ii)\ \forall X \in \mathrm{Con}_A.\ fX \in \mathrm{Con}_B\ \&\ \forall a_1, a_2 \in X.\ f(a_1) = f(a_2) \implies a_1 = a_2\,.$$

□

Exercise A.2. *Show a map* $f : A \rightharpoonup B$ *of* $\mathcal{E}$ *is mono if the function* $\mathcal{C}(A) \to \mathcal{C}(B)$ *taking configuration* x *to its direct image* fx *is injective. [Recall a map* $f : A \to B$ *is mono iff for all maps* $g, h : C \to A$ *if* $fg = fh$ *then* $g = h$*.] Show the converse does not hold, that it is possible for a map to be mono but not injective on configurations. Taking* B *to be the event structure comprising two concurrent events, can you find an event structure* A *and an example of a total map* $f : A \to B$ *of event structures which is both mono and where* f *is not injective as a function on events?* □

Exercise A.3. *Verify that the finite configurations of an event structure form a stable family.* □

Exercise A.4. *Say an event structure* A *is* tree-like *when its concurrency relation is empty (so two events are either causally related or inconsistent). Suppose* B *is tree-like and* $f : A \to B$ *is a total map of event structures. Show* A *must also be tree-like, and moreover that the map* f *is rigid, i.e. preserves causal dependency.*

Exercise A.5. *Let $\mathcal{F}$ be a nonempty family of finite sets satisfying the Completeness axiom in the definition of stable families. Show $\mathcal{F}$ is coincidence-free iff*

$$\forall x, y \in \mathcal{F}.\ x \subsetneq y \implies \exists x_1, e_1.\ x \overset{e_1}{\longrightarrow\!\!\subset} x_1 \subseteq y\,.$$

[Hint: For 'only if' use induction on the size of $y \setminus x$.] □

Exercise A.6. *Prove Proposition 3.10: Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. Show if $f(e) \leq_{fx} f(e')$ (with both $f(e)$ and $f(e')$ defined) then $e \leq_x e'$.*

Exercise A.7. *Prove the two propositions 3.6 and 3.7.* □

Exercise A.8. *(From Section 3.2) For an event structure E, show $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$.* □

Exercise A.9. *(From Section 3.2) Let $\mathcal{F}$ be a stable family. Show $\mathcal{F}^\infty$ satisfies:*

Completeness: $\forall Z \subseteq \mathcal{F}^\infty.\ Z\!\uparrow \implies \bigcup Z \in \mathcal{F}^\infty$ *;*
Stability: $\forall Z \subseteq \mathcal{F}^\infty.\ Z \neq \emptyset \ \&\ Z\!\uparrow \implies \bigcap Z \in \mathcal{F}^\infty$*;*
Coincidence-freeness: *For all $x \in \mathcal{F}^\infty$, $e, e' \in x$ with $e \neq e'$,*

$$\exists y \in \mathcal{F}^\infty.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y)\,;$$

Finiteness: *For all $x \in \mathcal{F}^\infty$,*

$$\forall e \in x \exists y \in \mathcal{F}.\ e \in y \ \&\ y \subseteq x \ \&\ y \textit{ is finite}\,.$$

Show that $\mathcal{F}$ consists of precisely the finite sets in $\mathcal{F}^\infty$. □

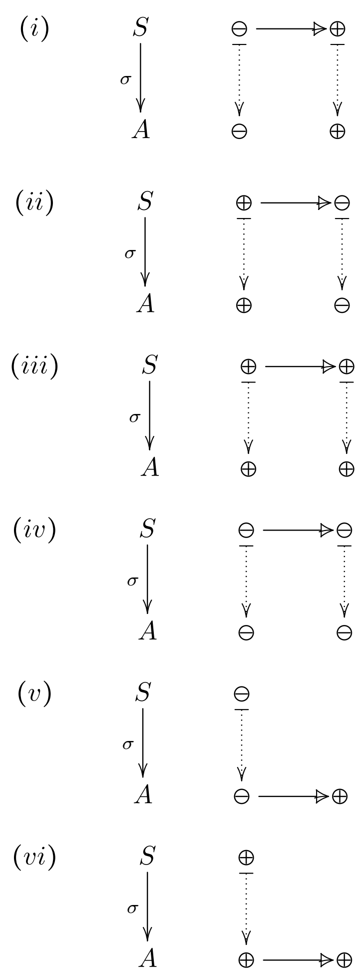
Exercise A.10. *Let A be the event structure consisting of two distinct events $a_1 \leq a_2$ and B the event structure with a single event b. Following the method of Section 3.3.1 describe the product of event structures $A \times B$.* □

On strategies

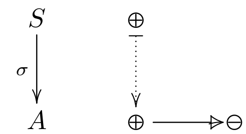
Recommended exercises: 11, 12, 13, 14, 15, 17.

Exercise A.11. *Consider the empty map of event structures with polarity* $\varnothing \to A$. *Is it a strategy? Is it a deterministic strategy? Consider now the identity map* $\mathrm{id}_A : A \to A$ *on an event structure with polarity* A. *Is it a strategy? Is it a deterministic strategy?* □

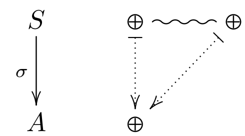
Exercise A.12. *For each instance of total map* σ *of event structures with polarity below say whether* σ *is a strategy and whether it is deterministic. In each case give a short justification for your answer. (Immediate causal dependency within the event structures is represented by an arrow* $\rightarrowtriangle$ *and inconsistency, or conflict, by a wiggly line* $\sim\!\sim\!\sim$ *.)*



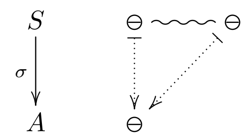
(vii)



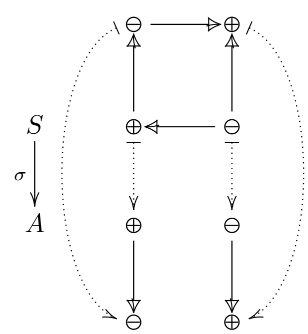
(viii)



(ix)



(x)



□

Exercise A.13. *Let* $\mathrm{id}_A : A \to A$ *be the identity map of event structures, sending an event to itself. Show the identity map forms a strategy in the game A. Is it deterministic in general?* □

Exercise A.14. *Show any strategy* $\sigma : A \dashrightarrow\!\!\!\!\!\!\;\; B$ *has a dual strategy* $\sigma^\perp : B^\perp \dashrightarrow\!\!\!\!\!\!\;\; A^\perp$*. In more detail, supposing* $\sigma : S \to A^\perp \| B$ *is a strategy show* $\sigma^\perp : S \to (B^\perp)^\perp \| A^\perp$ *is a strategy where*

$$\sigma^\perp(s) = \begin{cases} (1, b) & \text{if } \sigma(s) = (2, b) \\ (2, a) & \text{if } \sigma(s) = (1, a)\,. \end{cases}$$

□

Exercise A.15. *Let B be the event structure consisting of the two concurrent events b_1, assumed −ve, and b_2, assumed +ve in B . Let C consist of a single +ve event c. Let the strategy* $\sigma : \varnothing \dashrightarrow\!\!\!\!\!\!\;\; B$ *comprise the event structure* $s_1 \rightarrowtriangle s_2$

with s_1 *−ve and* s_2 *+ve,* $\sigma(s_1) = b_1$ *and* $\sigma(s_2) = b_2$*. In* $B^\perp$ *the polarities are reversed so there is a strategy* $\tau : B \nrightarrow C$ *comprising the map* $\tau : T \to B^\perp \| C$ *from the event structure* T*, with three events* t_1 *and* t_3 *both +ve and* t_2 *−ve so* $t_2 \rightarrowtriangle t_1$ *and* $t_2 \rightarrowtriangle t_3$*, which acts so* $\tau(t_1) = \overline{b}_1$*,* $\tau(t_2) = \overline{b}_2$ *and* $\tau(t_3) = c$*. Describe the composition* $\tau \odot \sigma$*.* □

Exercise A.16. *Say an event structure is* set-like *if its causal dependency relation is the identity relation and all pairs of distinct events are inconsistent. Let* A *and* B *be games with underlying event structures which are set-like event structures. In this case, can you see a simpler way to describe deterministic strategies* $A \nrightarrow B$*? What does composition of deterministic strategies between set-like games corresponds to? What do strategies in general between set-like games correspond to? What does composition of strategies between set-like games corresponds to? [No proofs are required.]* □

Exercise A.17. *By considering the game* A *comprising two concurrent events, one +ve and one −ve, show there is a nondeterministic pre-strategy* $\sigma : S \to A$ *such that* $s \rightarrowtriangle s'$ *in* S *without* $\sigma(s) \rightarrowtriangle \sigma(s')$*. Could you find such a counterexample were* σ *deterministic? Explain.* □

Exercise A.18. *Let* $G =_{\text{def}} (A, W)$ *be a game with winning conditions. Say a pre-strategy* $\sigma : S \to A$ *is winning iff* $\sigma x \in W$ *for all +-maximal configurations* $x \in \mathcal{C}^\infty(S)$*. Show that if* G *has a winning* receptive *pre-strategy, then the dual game* $G^\perp$ *has no winning strategy (use Corollary 8.3.) Show that* G *may have a winning pre-strategy (necessarily not receptive) while* $G^\perp$ *has a winning strategy.* □