For the ACS course
Advanced Topics in Concurrency

Event Structures, Stable Families and Concurrent Games

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Preface

These lecture notes introduce a theory of two-party games still under development. A lot can be said for a general theory to unify all manner of games found in the literature. But this has not been the original motivation. That has been the development of a generalized domain theory, to lift the methodology of domain theory and denotational semantics to address the highly interactive nature of computation we find today. There are several arguments why the next generation of domain theory should be an intensional theory, one which pays careful attention to the ways in which output is computed from input. One is that if the theory is to be able to reason about operational concerns it had better address them, albeit abstractly. Another is that sometimes the demands of compositionality force denotations to be more intensional than one would at first expect; this occurs for example with nondeterministic dataflow—see the Introduction. These notes take seriously the idea that intensional aspects be described by strategies, and to fit computational needs adequately try to understand the concept of strategy very broadly.

This idea comes from game semantics where the domains and continuous functions of traditional domain theory and denotational semantics are replaced by games and strategies. Strategies supercede functions because they give a much better account of interaction extended in time. (Functions, if you like, have too clean a separation of interaction into input and output.) In traditional denotational semantics a program phrase or process term denotes a continuous function, whereas in game semantics a program phrase or process term denotes a strategy. However, traditional game semantics is not always general enough, for instance in accounting for nondeterministic or concurrent computation.

Rather than extending traditional game semantics with various bells and whistles, these notes attempt to carve out a general theory of games within a general model of nondeterministic, concurrent computation. The model chosen is the partial-order model of event structures, and for technical reasons, its enlargement to stable families. Event structures have the advantage of occupying a central position within models for concurrency, and the development here should suggest analogous developments for other 'partial-order' models such as Mazurkiewicz trace languages, Petri nets and asynchronous transition systems, and even 'interleaving' models based on transition systems or sequences.

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Chapter 1

Introduction

Games and strategies are everywhere, in logic, philosophy, computer science, economics, in leisure and in life.

Slogan for this course: Processes are nondeterministic concurrent strategies.

1.1 Motivation

We summarise some reasons for developing a theory of nondeterministic concurrent games and strategies.

1.1.1 What is a process?

In the earliest days of computer science it became accepted that a computation was essentially an (effective) partial function $f : \mathbb{N} \to \mathbb{N}$ between the natural numbers. This view underpins the Church-Turing thesis on the universality of computability.

As computer science matured it demanded increasingly sophisticated mathematical representations of processes. The pioneering work of Strachey and Scott in the denotational semantics of programs assumed a view of a process still as a function $f : D \to D'$, but now acting in a continuous fashion between datatypes represented as special topological spaces, 'domains' D and D'; reflecting the fact that computers can act on complicated, conceptually-infinite objects, but only by virtue of their finite approximations.

In the 1960's, around the time that Strachey started the programme of denotational semantics, Petri advocated his radical view of a process, expressed in terms of its events and their effect on local states—a model which addressed directly the potentially distributed nature of computation, but which, in common with many other current models, ignored the distinction between data and process implicit in regarding a process as a function. Here it seems that an adequate notion of process requires a marriage of Petri's view of a process and the vision of Scott and Strachey. An early hint in this direction came in answer to the following question.

What is the information order in domains? There are essentially two answers in the literature, the '*topological*,' the most well-known from Scott's work, and the '*temporal*,' arising from the work of Berry:

- *Topological*: the basic units of information are *propositions* describing finite properties; more information corresponds to more propositions being true. Functions are ordered pointwise.

- *Temporal*: the basic units of information are *events*; more information corresponds to more events having occurred over time. Functions are restricted to 'stable' functions and ordered by the intensional 'stable order,' in which common output has to be produced for the same minimal input. Berry's specialized domains 'dI-domains' are represented by event structures.

In truth, Berry developed 'stable domain theory' by a careful study of how to obtain a suitable category of domains with stable rather than all continuous functions. He arrived at the axioms for his 'dI-domains' because he wanted function spaces (so a cartesian-closed category). The realization that dI-domains were precisely those domains which could be represented by event structures, came a little later.

1.1.2 From models for concurrency

Causal models are alternatively described as: causal-dependence models; independence models; non-interleaving models; true-concurrency models; and partial-order models. They include Petri nets, event structures, Mazurkiewicz trace languages, transition systems with independence, multiset rewriting, and many more. The models share the central feature that they represent processes in terms of the events they can perform, and that they make explicit the causal dependency and conflicts between events.

Causal models have arisen, and have sometimes been rediscovered as *the* natural model, in many diverse and often unexpected areas of application:
Security protocols: for example, forms of event structure, strand spaces, support reasoning about secrecy and authentication through causal relations and the freshness of names;
Systems biology: ideas from Petri nets and event structures are used in taming the state-explosion in the stochastic simulation of biochemical processes and in the analysis of biochemical pathways;
Hardware: in the design and analysis of asynchronous circuits;
Types and proof: event structures appear as representations of propositions as types, and of proofs;
Nondeterministic dataflow: where numerous researchers have used or rediscovered causal models in providing a compositional semantics to nondeterministic dataflow;
Network diagnostics: in the patching together local of fault diagnoses of com-

munication networks;
Logic of programs: in concurrent separation logic where artificialities in Brookes' pioneering soundness proof are obviated through a Petri-net model;
Partial order model checking: following the seminal work of McMillan the unfolding of Petri nets (described below) is exploited in recent automated analysis of systems;
Distributed computation: event structures appear both classically,*e.g.* in early work of Lamport, and recently in the Bayesian analysis of trust and modelling multicore memory.

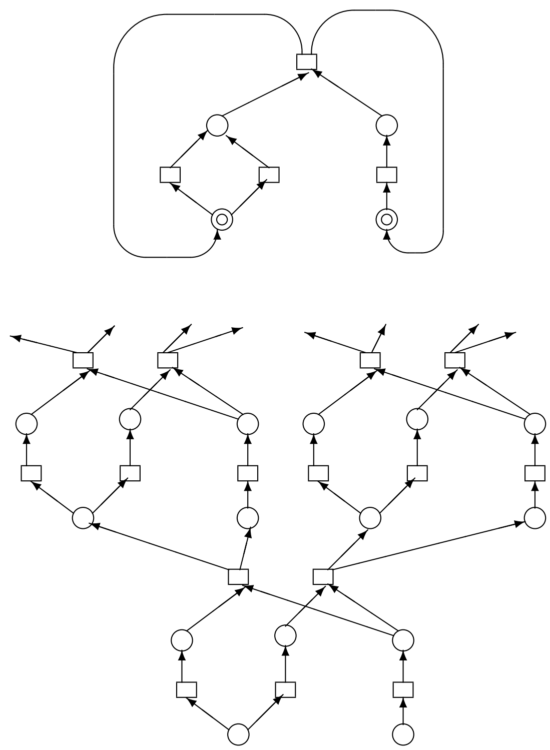
To illustrate the close relationship between Petri nets and the 'partial-order models' of occurrence nets and event structures, we sketch how a (1-safe) Petri net can be unfolded first to a net of occurrences and from there to an event structure [1]. The unfolding construction is analogous to the well-known method of unfolding a transition system to a tree, and is central to several analysis tools in the applications above. In the figure, the net on top has loops. The net below it is its *occurrence-net unfolding.* It consists of all the occurrences of conditions and events of the original net, and is infinite because of the original repetitive behaviour. The occurrences keep track of what enabled them. The simplest form of event structure, the one we shall consider here, arises by abstracting away the conditions in the occurrence net and capturing their role in relations of causal dependency and conflict on event occurrences.

The relations between the different forms of causal models are well understood [2]. Despite this and their often very successful, specialized applications, causal models lack a *comprehensive* theory which would support their systematic use in giving semantics to a broad range of programming and process languages, in particular we lack an expressive form of '*domain theory*' for causal models with rich higher-order type constructions needed by mathematical semantics.

1.1.3 From semantics

Denotational semantics and domain theory of Scott and Strachey set the standard for semantics of computation. The theory provided a global mathematical setting for sequential computation, and thereby placed programming languages in connection with each other; connected with the mathematical worlds of algebra, topology and logic; and inspired programming languages, type disciplines and methods of reasoning. Despite the many striking successes it has become very clear that many aspects of computation do not fit within the traditional framework of denotational semantics and domain theory. In particular, classical domain theory has not scaled up to the more intricate models used in interactive/distributed computation. Nor has it been as operationally informative as one could hope.

While, as Kahn was early to show, deterministic dataflow is a shining application of simple domain theory, nondeterministic dataflow is beyond its scope. The compositional semantics of nondeterministic dataflow needs a form of generalized relation which specifies the *ways* input-output pairs are realized.A com-

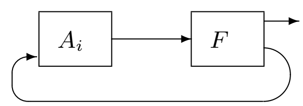


A Petri net and its occurrence-net unfolding

pelling example comes from the early work of Brock and Ackerman who were the first to emphasize the difficulties in giving a compositional semantics to nondeterministic dataflow, though our example is based on simplifications in the later work of Rabinovich and Trakhtenbrot, and Russell.

Nondeterministic dataflow—Brock-Ackerman anomaly

$C[A_i]$ =



There are two simple nondeterministic processes A_1 and A_2, which have the same input-output relation, and yet behave differently in the common feedback context $C[-]$, illustrated above. The context consists of a fork process F (a process that copies every input to two outputs), through which the output of the automata A_i is fed back to the input channel, as shown in the figure. Process A_1 has a choice between two behaviours: either it outputs a token and stops, *or* it outputs a token, waits for a token on input and then outputs another token. Process A_2 has a similar nondeterministic behaviour: Either it outputs a token and stops, *or* it waits for an input token, then outputs two tokens. For both automata, the input-output relation relates empty input to the eventual output of one token, and non-empty input to one or two output tokens. But $C[A_1]$ can output two tokens, whereas $C[A_2]$ can only output a single token. Notice that A_1 has two ways to realize the output of a single token from empty input, while A_2 only has one. It is this extra way, not caught in a simple input-output relation, that gives A_1 the richer behaviour in the feedback context.

Over the years there have been many solutions to giving a compositional semantics to nondeterministic dataflow. But they all hinge on some form of generalized relation, to distinguish the different ways in which output is produced from input. A compositional semantics can be given using *stable spans* of event structures, an extension of Berry's stable functions to include nondeterminism [3]—see Section 6.2.1.

How are we to extend the methodology of denotational semantics to the much broader forms of computational processes we need to design, understand and analyze today? How are we to maintain clean algebraic structure and abstraction alongside the operational nature of computation?

Game semantics advanced the idea of replacing the traditional continuous functions of domain theory and denotational semantics by strategies. The reason for doing this was to obtain a representation of interaction in computation that was more faithful to operational reality. It is not always convenient or mathematically tractable to assume that the environment interacts with a computation in the form of an input argument. It is built into the view of a process as a strategy that the environment can direct the course of evolution of a process throughout its duration. Game semantics has had many dramatic successes. But it has developed from simple well-understood games, based on alternating sequences of player and opponent moves, to sometimes arcane extensions and generalizations designed to fit the demands of a succession of additional programming or process features. It is perhaps time to stand back and see how games fit within a very general model of computation, to understand better what current features of games in computer science are simply artefacts of the particular history of their development.

1.1.4 From logic

An informal understanding of games and strategies goes back at least as far as the ancient Greeks where truth was sought through debate using the dialectic method; a contention being true if there was an argument for it that could survive all counter-arguments. Formalizing this idea, logicians such as Lorenzen and Blass investigated the meaning of a logical assertion through strategies in a game built up from the assertion. These ideas were reinforced in game semantics which can provide semantics to proofs as well as programs. The study of the mathematics and computational nature of proof continues. There are several strands of motivation for games in logic. Along with automata games constitute one of the tools of logic and algorithmics. Games are used in verification and, for example, the central equivalence of bisimulation on processes has a reading in terms of strategies.

Chapter 2

Event structures

Event structures are a fundamental model of concurrent computation and, along with their extension to stable families, provide a mathematical foundation for the course.

2.1 Event structures

Event structures are a model of computational processes. They represent a process, or system, as a set of event occurrences with relations to express how events causally depend on others, or exclude other events from occurring. In one of their simpler forms they consist of a set of events on which there is a consistency relation expressing when events can occur together in a history and a partial order of causal dependency—writing $e' \leq e$ if the occurrence of e depends on the previous occurrence of e'.

An *event structure* comprises $(E, \leq, \text{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \text{Con for all } e \in E,\\
&Y \subseteq X \in \text{Con} \implies Y \in \text{Con}, \quad \text{and}\\
&X \in \text{Con} \ \&\ e \leq e' \in X \implies X \cup \{e\} \in \text{Con}.
\end{aligned}$$

The events are to be thought of as event occurrences without significant duration; in any history an event is to appear at most once. We say that events e, e' are *concurrent*, and write $e \, co \, e'$ if $\{e, e'\} \in \text{Con} \ \&\ e \not\leq e' \ \&\ e' \not\leq e$. Concurrent events can occur together, independently of each other. The relation of *immediate* dependency $e \rightarrowtriangle e'$ means e and e' are distinct with $e \leq e'$ and no event in between.

An event structure represents a process. A configuration is the set of all events which may have occurred by some stage, or history, in the evolution of the process. According to our understanding of the consistency relation and causal dependency relations a configuration should be consistent and such that if an event appears in a configuration then so do all the events on which it causally depends.

The *configurations* of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and

Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

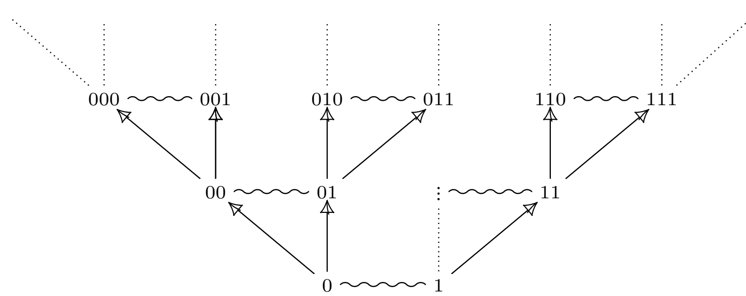
We shall largely work with *finite* configurations, written $\mathcal{C}(E)$. Write $\mathcal{C}^\infty(E)$ for the set of *finite and infinite* configurations of the event structure E.

The configurations of an event structure are ordered by inclusion, where $x \subseteq x'$, *i.e.* x is a sub-configuration of x', means that x is a sub-history of x'. Note that an individual configuration inherits an order of causal dependency on its events from the event structure so that the history of a process is captured through a partial order of events. The finite configurations correspond to those events which have occurred by some finite stage in the evolution of the process, and so describe the possible (finite) states of the process.

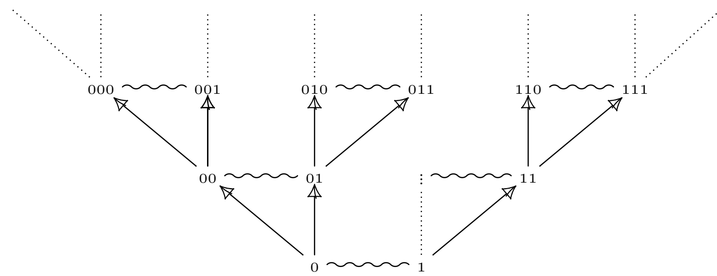
For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X. The axioms on the consistency relation ensure that the down-closure of any finite set in the consistency relation s a finite configuration, and that any event appears in a configuration: given $X \in \mathrm{Con}$ its down-closure $\{e' \in E \mid \exists e \in X.\ e' \leq e\}$ is a finite configuration; in particular, for an event e, the set $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event e.

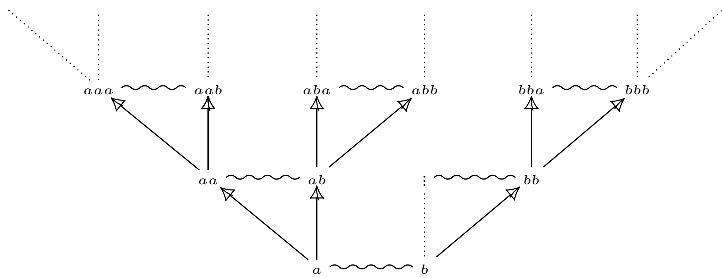
When the consistency relation is determined by the pairwise consistency of events we can replace it by a binary relation or, as is more usual, by a complementary binary conflict relation on events (written as $\#$ or $\smile$).

Example 2.1. *The diagram below illustrates an event structure representing streams of 0s and 1s:*



Above we have indicated conflict (or inconsistency) between events by $\sim\!\sim\!\sim$ *. The event structure representing pairs of* 0/1*-streams and a/b-streams is represented* *by the juxtaposition of two event structures:*





Exercise 2.2. Draw the event structure of the occurrence net unfolding in the introduction. □

2.1.1 Maps of event structures

Let E and E' be event structures. A *(partial) map* of event structures $f : E \rightharpoonup E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and

$$\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2) \text{ (with both defined), then } e_1 = e_2.$$

The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. The map f respects the instantaneous nature of events: two distinct event occurrences which are consistent with each other cannot both coincide with the occurrence of a common event in the image. Partial maps of event structures compose as partial functions, with identity maps given by identity functions.

For any event e a map of event structures $f : E \rightharpoonup E'$ must send the configuration $[e]$ to the configuration $f[e]$. Partial maps preserve the concurrency relation, when defined.

We will say the map is *total* if the function f is total. Notice that for a total map f the condition on maps now says it is *locally injective*, in the sense that w.r.t. any configuration x of the domain the restriction of f to a function from x is injective; the restriction of f to a function from x to fx is thus bijective. Say a total map of event structures is *rigid* when it preserves causal dependency.

Definition 2.3. Write $\mathcal{E}$ for the category of event structures with (partial) maps. Write $\mathcal{E}_t$ for the category of event structures with total maps.

Exercise 2.4. Show a map of event structures $f : E \rightharpoonup E'$ locally reflects causal dependency: for all $x \in \mathcal{C}^\infty(E)$, for all $e_1, e_2 \in x$ whenever $f(e_1)$ and $f(e_2)$ are both defined with $f(e_1) \le f(e_2)$, then $e_1 \le e_2$.

Exercise 2.5. Show a map $f : A \rightharpoonup B$ of $\mathcal{E}$ is mono iff the function $\mathcal{C}(A) \to \mathcal{C}(B)$ taking configuration x to its direct image fx is injective. [Recall a map $f : A \to B$ is mono iff for all maps $g, h : C \to A$ if $fg = fh$ then $g = h$.] □

Proposition 2.6. *Let E and E' be event structures. Suppose*

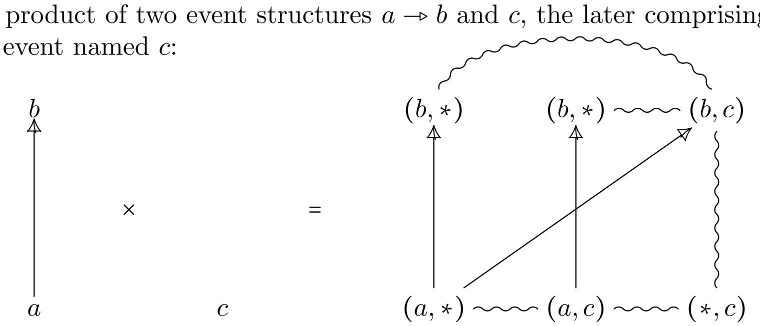
$$\theta_x : x \cong \theta_x x, \text{ indexed by } x \in \mathcal{C}(E),$$

is a family of bijections such that whenever $\theta_y : y \cong \theta_y y$ is in the family then its restriction $\theta_z : z \cong \theta_z z$ is also in the family, whenever $z \in \mathcal{C}(E)$ and $z \subseteq y$. Then, $\theta =_{\text{def}} \bigcup_{x \in \mathcal{C}(E)} \theta_x$ is the unique total map of event structures from E to E' such that $\theta\, x = \theta_x x$ for all $x \in \mathcal{C}(E)$.

Proof. The conditions ensure that $\theta =_{\text{def}} \bigcup_{x \in \mathcal{C}(A)} \theta_x$ is a function $\theta : A \rightharpoonup B$ such that the image of any finite configuration x of A under θ is a configuration of B and local injectivity holds. □

2.2 Products of event structures

The category of event structures has products, which essentially allow arbitrary synchronizations between their components. For example, here is an illustration of the product of two event structures $a \rightarrowtriangle b$ and c, the later comprising just a single event named c:



The original event b has split into three events, one a synchronization with c, another b occurring unsynchronized after an unsynchronized a, and the third b occurring unsynchronized after a synchronizes with c. The splittings correspond to the different histories of the event.

It can be awkward to describe operations such as products, pullbacks and synchronized parallel compositions directly on the simple event structures here, essentially because an event determines its whole causal history. One closely related and more versatile, though perhaps less intuitive and familiar, model is that of stable families. Stable families will play an important technical role in establishing and reasoning about constructions on event structures.

Chapter 3

Stable families

Stable families, their basic properties and relations to event structures are developed.[1]

3.1 Stable families

The notion of stable family extends that of finite configurations of an event structure to allow an event can occur in several incompatible ways.

Notation 3.1. *Let $\mathcal{F}$ be a family of subsets. Let $X \subseteq \mathcal{F}$. We write $X\uparrow$ for $\exists y \in \mathcal{F}.\ \forall x \in X. x \subseteq y$ and say X is* compatible. *When $x, y \in \mathcal{F}$ we write $x \uparrow y$ for $\{x, y\}\uparrow$.*

A *stable family* comprises $\mathcal{F}$, a nonempty family of finite subsets, satisfying:
Completeness: $\forall Z \subseteq \mathcal{F}.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}$;
Stability: $\forall Z \subseteq \mathcal{F}.\ Z \neq \varnothing \ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}$;
Coincidence-freeness: For all $x \in \mathcal{F}$, $e, e' \in x$ with $e \neq e'$,

$$\exists y \in \mathcal{F}.\ y \subseteq x \ \&\ (e \in y \iff e' \notin y)\,.$$

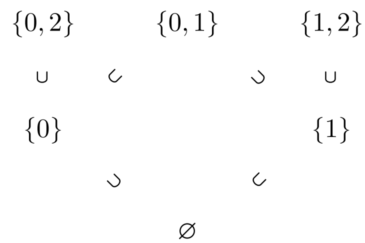
Proposition 3.2. *A stable family comprises $\mathcal{F}$, a family of finite subsets, for which: $\varnothing \in \mathcal{F}$; if $x \uparrow y$ in $\mathcal{F}$, then $x \cup y$ and $x \cap y$ are in $\mathcal{F}$; coincidence-freeness holds.*

Proposition 3.3. *The family of finite configurations of an event structure forms a stable family.*

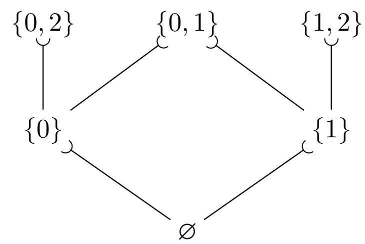
On the other hand stable families are more general than finite configurations of an event structure, as the following example shows.

[1] A useful reference for stable families is the report "Event structure semantics for CCS and related languages," a full version of the article [4], available from `www.cl.cam.ac.uk/~gw104`, though its terminology can differ from that here.

Example 3.4. *Let F be the stable family, with events $E = \{0,1,2\}$,*



or equivalently



where $\mathrel{-\!\!\!\subset}$ is the covering relation representing an occurrence of one event. The events 0 and 1 are concurrent, neither depends on the occurrence or non-occurrence of the other to occur. The event 2 can occur in two incompatible ways, either through event 0 having occurred or event 1 having occurred. This possibility can make stable families more flexible to work with than event structures.

A (partial) map of stable families $f : \mathcal{F} \to \mathcal{G}$ is a partial function f from the events of $\mathcal{F}$ to the events of $\mathcal{G}$ such that for all $x \in \mathcal{F}$,

$$fx \in \mathcal{G} \ \& \ (\forall e_1, e_2 \in x.\ f(e_1) = f(e_2) \implies e_1 = e_2)\,.$$

Maps of stable families compose as partial functions, with identity maps given by identity functions.

Proposition 3.5. *Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. If $f(e)$ and $f(e')$ are defined and $f(e) \leq_{fx} f(e')$ then $e \leq_x e'$.*

Definition 3.6. Let $\mathcal{F}$ be a stable family. We use $x \mathrel{-\!\!\!\subset} y$ to mean y covers x in $\mathcal{F}$, *i.e.* $x \subset y$ in $\mathcal{F}$ with nothing in between, and $x \stackrel{e}{\mathrel{-\!\!\!\subset}} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{F}$ and event $e \notin x$. We sometimes use $x \stackrel{e}{\mathrel{-\!\!\!\subset}}$, expressing that event e is enabled at configuration x, when $x \stackrel{e}{\mathrel{-\!\!\!\subset}} y$ for some y.

Exercise 3.7. Let $\mathcal{F}$ be a nonempty family of sets satisfying the Completeness axiom in the definition of stable families. Show $\mathcal{F}$ is coincidence-free iff

$$\forall x, y \in \mathcal{F}.\ x \subsetneq y \implies \exists x_1, e_1.\ x \stackrel{e_1}{\mathrel{-\!\!\!\subset}} x_1 \subseteq y\,.$$

[Hint: For 'only if' use induction on the size of $y \setminus x$.] □

3.1.1 Stable families and event structures

Finite configurations of an event structure form a stable family. Conversely, a stable family determines an event structure:

Proposition 3.8. *Let x be a configuration of a stable family $\mathcal{F}$. For $e, e' \in x$ define*

$$e' \leq_x e \text{ iff } \forall y \in \mathcal{F}.\ y \subseteq x \ \&\ e \in y \implies e' \in y.$$

When $e \in x$ define the prime *configuration*

$$[e]_x = \bigcap \{y \in \mathcal{F} \mid y \subseteq x \ \&\ e \in y\}\ .$$

Then $\leq_x$ is a partial order and $[e]_x$ is a configuration such that

$$[e]_x = \{e' \in x \mid e' \leq_x e\}.$$

Moreover the configurations $y \subseteq x$ are exactly the down-closed subsets of $\leq_x$.

Proposition 3.9. *Let $\mathcal{F}$ be a stable family. Then,* $\mathrm{Pr}(\mathcal{F}) =_{\mathrm{def}} (P, \mathrm{Con}, \leq)$ *is an event structure where:*

$$\begin{aligned} &P = \{[e]_x \mid e \in x \ \&\ x \in \mathcal{F}\}\ , \\ &Z \in \mathrm{Con} \textit{ iff } Z \subseteq P \ \&\ \bigcup Z \in \mathcal{F} \quad \textit{and}, \\ &p \leq p' \textit{ iff } p, p' \in P \ \&\ p \subseteq p'\ . \end{aligned}$$

Exercise 3.10. Prove the two propositions 3.8 and 3.9. □

The operation Pr is right adjoint to the "inclusion" functor, taking an event structure E to the stable family $\mathcal{C}(E)$. The unit of the adjunction $E \to \mathrm{Pr}(\mathcal{C}(E))$ takes an event e to the prime configuration $[e] =_{\mathrm{def}} \{e' \in E \mid e' \leq e\}$. The counit $max : \mathcal{C}(\mathrm{Pr}(\mathcal{F})) \to \mathcal{F}$ takes prime configuration $[e]_x$ to e.

Definition 3.11. Let $\mathcal{F}$ be a stable family. W.r.t. $x \in \mathcal{F}$, write $[e)_x =_{\mathrm{def}} \{e' \in E \mid e' \leq_x e \ \&\ e' \neq e\}$. The relation of *immediate* dependence of event structures generalizes: with respect to $x \in \mathcal{F}$: the relation $e \rightarrowtriangle_x e'$ means $e \leq_x e'$ with $e \neq e'$ and no event in between.

3.2 Infinite configurations

We can extend a stable family to include infinite configurations, by constructing its "ideal completion."

Definition 3.12. Let $\mathcal{F}$ be a stable family. Define $\mathcal{F}^\infty$ to comprise all $\bigcup I$ where $I \subseteq \mathcal{F}$ is an ideal (*i.e.*, I is a nonempty subset of $\mathcal{F}$ closed downwards w.r.t. $\subseteq$ in $\mathcal{F}$ and such that if $x, y \in I$ then $x \cup y \in I$).

Exercise 3.13. For an event structure E, show $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$. □

Exercise 3.14. Let $\mathcal{F}$ be a stable family. Show $\mathcal{F}^\infty$ satisfies:

Completeness: $\forall Z \subseteq \mathcal{F}^\infty.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}^\infty$;
Stability: $\forall Z \subseteq \mathcal{F}^\infty.\ Z \neq \emptyset\ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}^\infty$;
Coincidence-freeness: For all $x \in \mathcal{F}^\infty$, $e, e' \in x$ with $e \neq e'$,

$$\exists y \in \mathcal{F}^\infty.\ y \subseteq x\ \&\ (e \in y \iff e' \notin y)\,;$$

Finiteness: For all $x \in \mathcal{F}^\infty$,

$$\forall e \in x \exists y \in \mathcal{F}.\ e \in y\ \&\ y \subseteq x\ \&\ y \text{ is finite}\ .$$

Show that $\mathcal{F}$ consists of precisely the finite sets in $\mathcal{F}^\infty$. □

3.3 Process constructions

3.3.1 Products

Let $\mathcal{A}$ and $\mathcal{B}$ be stable families with events A and B, respectively. Their product, the stable family $\mathcal{A} \times \mathcal{B}$, has events comprising pairs in $A \times_* B =_{\mathrm{def}} \{(a, *) \mid a \in A\} \cup \{(a,b) \mid a \in A\ \&\ b \in B\} \cup \{(*, b) \mid b \in B\}$, the product of sets with partial functions, with (partial) projections π_1 and π_2—treating $*$ as 'undefined'—with configurations

$$\begin{aligned}
&x \in \mathcal{A} \times \mathcal{B} \text{ iff}\\
&x \text{ is a finite subset of } A \times_* B \text{ such that } \pi_1 x \in \mathcal{A}\ \&\ \pi_2 x \in \mathcal{B},\\
&\forall e, e' \in x.\ \pi_1(e) = \pi_1(e') \text{ or } \pi_2(e) = \pi_2(e') \Rightarrow e = e'\,, \&\\
&\forall e, e' \in x.\ e \neq e' \Rightarrow \exists y \subseteq x.\ \pi_1 y \in \mathcal{A}\ \&\ \pi_2 y \in \mathcal{B}\ \&\\
&\qquad\qquad\qquad\qquad (e \in y \iff e' \notin y)\,.
\end{aligned}$$

Right adjoints preserve products. Consequently we obtain a product of event structures A and B by first regarding them as stable families $\mathcal{C}(A)$ and $\mathcal{C}(B)$, forming their product $\mathcal{C}(A) \times \mathcal{C}(B), \pi_1, \pi_2$, and then constructing the event structure

$$A \times B =_{\mathrm{def}} \Pr(\mathcal{C}(A) \times \mathcal{C}(B))$$

and its projections as $\Pi_1 =_{\mathrm{def}} \pi_1 max$ and $\Pi_2 =_{\mathrm{def}} \pi_2 max$.

Exercise 3.15. Let A be the event structure consisting of two distinct events $a_1 \leq a_2$ and B the event structure with a single event b. Following the method above describe the product of event structures $A \times B$. □

Later we shall use the following properties of $\rightarrowtriangle$ in a product of stable families or event structures.

Lemma 3.16. *Suppose $e \rightarrowtriangle_x e'$ in a product of stable families $\mathcal{A} \times \mathcal{B}, \pi_1, \pi_2$.*
(i) If $e = (a, *)$ then $e' = (a', b)$ or $e' = (a', *)$ with $a \rightarrowtriangle_{\pi_1 x} a'$ in $\mathcal{A}$.
(ii) If $e' = (a', *)$ then $e = (a, b)$ or $e = (a, *)$ with $a \rightarrowtriangle_{\pi_1 x} a'$ in $\mathcal{A}$.
(iii) If $e = (a, b)$ and $e' = (a', b')$ then $a \rightarrowtriangle_{\pi_1 x} a'$ in $\mathcal{A}$ or $b \rightarrowtriangle_{\pi_2 x} b'$ in $\mathcal{B}$.

3.3.2 Restriction

The *restriction* of $\mathcal{F}$ to a subset of events R is the stable family $\mathcal{F} \upharpoonright R =_{\mathrm{def}} \{x \in \mathcal{F} \mid x \subseteq R\}$. Defining $E \upharpoonright R$, the restriction of an event structure E to a subset of events R, to have events $E' = \{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency induced by E, we obtain $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$.

Proposition 3.17. *Let $\mathcal{F}$ be a stable family and R a subset of its events. Then,* $\mathrm{Pr}(\mathcal{F} \upharpoonright R) = \mathrm{Pr}(\mathcal{F}) \upharpoonright max^{-1} R$.

3.3.3 Synchronized compositions

Synchronized parallel compositions are obtained as restrictions of products to those events which are allowed to synchronize or occur asynchronously. For example, the synchronized composition of Milner's CCS on stable families $\mathcal{A}$ and $\mathcal{B}$ (with labelled events) is defined as $\mathcal{A} \times \mathcal{B} \upharpoonright R$ where R comprises events which are pairs $(a, *)$, $(*, b)$ and (a, b), where in the latter case the events a of $\mathcal{A}$ and b of $\mathcal{B}$ carry complementary labels. Similarly, synchronized compositions of event structures A and B are obtained as restrictions $A \times B \upharpoonright R$. By Proposition 3.17, we can equivalently form a synchronized composition of event structures by forming the synchronized composition of their stable families of configurations, and then obtaining the resulting event structure—this has the advantage of eliminating superfluous events earlier.

3.3.4 Pullbacks

The construction of pullbacks can be viewed as a special case of synchronized composition. Once we have products of event structures pullbacks are obtained by restricting products to the appropriate equalizing set. Pullbacks of even structures can also be constructed via pullbacks of stable families, in a similar manner to the way we have constructed products of event structures. We obtain pullbacks of stable families as restrictions of products. Suppose $f_1 : \mathcal{F}_1 \to \mathcal{G}$ and $f_2 : \mathcal{F}_2 \to \mathcal{G}$ are maps of stable families. Let E_1, E_2 and C be the sets of events of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$, respectively. The set $P =_{\mathrm{def}} \{(e_1, e_2) \mid f(e_1) = f(e_2)\}$ with projections π_1, π_2 to the left and right, forms the pullback, in the category of sets, of the functions $f_1 : E_1 \to C$, $f_2 : E_2 \to C$. We obtain the pullback in stable families of f_1, f_2 as the stable family $\mathcal{P}$, consisting of those subsets of P which are also configurations of the product $\mathcal{F}_1 \times \mathcal{F}_2$—its associated maps are the projections π_1, π_2 from the events of $\mathcal{P}$.

3.3.5 Projection

Event structures support a simple form of hiding. Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.

Consider a partial map of event structures $f : E \to E'$. Let

$$V =_{\text{def}} \{e \in E \mid f(e) \text{ is defined}\}\,.$$

Then f clearly factors into the composition

$$E \xrightarrow{f_0} E{\downarrow}V \xrightarrow{f_1} E'$$

of f_0, a partial map of event structures taking $e \in E$ to itself if $e \in V$ and undefined otherwise, and f_1, a total map of event structures acting like f on V.

3.3.6 Prefixes and sums

The prefix of an event structure A, written $\bullet.A$, comprises the event structure in which all the events of A are made to causally depend on an event $\bullet$. The category of event structures has sums given as coproducts; a coproduct $\sum_{i\in I} E_i$ is obtained as the disjoint juxtaposition of an indexed collection of event structures, making events in distinct components inconsistent. We shall use prefixed sums $\sum_{i\in I} \bullet.A_i$ in games for modelling first-order logical quantifiers.

Chapter 4

Games and strategies

Very general nondeterministic concurrent games and strategies are presented. The intention is to formalize distributed games in which both Player (or a team of players) and Opponent (or a team of opponents) can interact in highly distributed fashion, without, for instance, enforcing that their moves alternate. Strategies, those nondeterministic plays which compose well with copy-cat strategies, are characterized.[1]

4.1 Event structures with polarities

We shall represent both a game and a strategy in a game as an event structure with polarity, comprising an event structure together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ or $-$ to its events E. The events correspond to (occurrences of) moves. The two polarities $+/-$ express the dichotomy: Player/Opponent; Process/Environment; or Ally/Enemy. Maps of event structures with polarity are maps of event structures which preserve polarity.

4.2 Operations

4.2.1 Dual

The *dual*, $E^\perp$, of an event structure with polarity E comprises a copy of the event structure E but with a reversal of polarities. It obviously extends to a functor. Write $\overline{e} \in E^\perp$ for the event complementary to $e \in E$ and *vice versa*.

4.2.2 Simple parallel composition

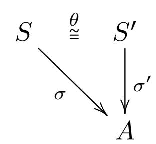
This operation simply juxtaposes two event structures with polarity. Let $(A, \le_A, \mathrm{Con}_A, pol_A)$ and $(B, \le_B, \mathrm{Con}_B, pol_B)$ be event structures with polarity. The events of $A \| B$ are $(\{1\} \times A) \cup (\{2\} \times B)$, their polarities unchanged, with: the only

[1] This key chapter is the result of joint work with Silvain Rideau [5].

relations of causal dependency given by $(1,a) \leq (1,a')$ iff $a \leq_A a'$ and $(2,b) \leq (2,b')$ iff $b \leq_B b'$; a subset of events C is consistent in $A \| B$ iff $\{a \mid (1,a) \in C\} \in \mathrm{Con}_A$ and $\{b \mid (2,b) \in C\} \in \mathrm{Con}_B$. The operation extends to a functor—put the two maps in parallel. The empty event structure with polarity $\varnothing$ is the unit w.r.t. $\|$.

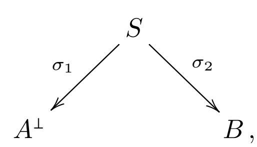
4.3 Pre-strategies

Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A *pre-strategy* in A is a total map $\sigma : S \to A$ from an event structure with polarity S. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy* (and *winning strategy* in Section 7.1). We regard two pre-strategies $\sigma : S \to A$ and $\sigma' : S' \to A$ as essentially the same when they are isomorphic, and write $\sigma \cong \sigma'$, *i.e.* when there is an isomorphism of event structures $\theta : S \cong S'$ such that

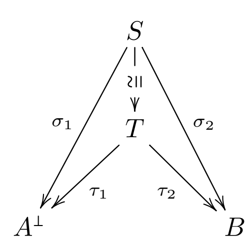


commutes.

Let A and B be event structures with polarity. Following Joyal [6], a pre-strategy from A to B is a pre-strategy in $A^\perp \| B$, so a total map $\sigma : S \to A^\perp \| B$. It thus determines a span



of event structures with polarity where σ_1, σ_2 are *partial* maps. In fact, a pre-strategy from A to B corresponds to such spans where for all $s \in S$ either, but not both, $\sigma_1(s)$ or $\sigma_2(s)$ is defined. Two pre-strategies will be essentially the same when they are isomorphic as spans. Two pre-strategies σ and τ from A to B are isomorphic, $\sigma \cong \tau$, when their spans are isomorphic, *i.e.*



commutes. We write $\sigma : A \twoheadrightarrow B$ to express that σ is a pre-strategy from A to B. Note a pre-strategy in a game A coincides with a pre-strategy from the empty game $\sigma : \varnothing \twoheadrightarrow A$.

4.3.1 Concurrent copy-cat

Identities on games are given by copy-cat strategies—strategies for Player based on copying the latest moves made by Opponent.

Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\!\mathbb{C}_A \to A^\perp \| A$. It describes a concurrent, or distributed, strategy based on the idea that Player moves, of +ve polarity, always copy previous corresponding moves of Opponent, of −ve polarity.

For $c \in A^\perp \| A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component, *i.e.*

$$\overline{(1,a)} = (2,\overline{a}) \text{ and } \overline{(2,a)} = (1,\overline{a})\,.$$

Proposition 4.1. *Let A be an event structure with polarity. There is an event structure with polarity $\mathbb{C}\!\mathbb{C}_A$ having the same events, consistency and polarity as $A^\perp \| A$ but with causal dependency $\le_{\mathbb{C}\!\mathbb{C}_A}$ given as the transitive closure of the relation*

$$\le_{A^\perp\|A} \cup \{(\overline{c}, c) \mid c \in A^\perp\|A \ \&\ pol_{A^\perp\|A}(c) = +\}\,.$$

Moreover,
(i) $c \rightarrowtriangle c'$ in $\mathbb{C}\!\mathbb{C}_A$ iff

$$c \rightarrowtriangle c' \text{ in } A^\perp\|A \text{ or } pol_{A^\perp\|A}(c') = + \ \&\ \overline{c} = c'\,;$$

(ii) $x \in \mathcal{C}(\mathbb{C}\!\mathbb{C}_A)$ iff

$$x \in \mathcal{C}(A^\perp\|A) \ \&\ \forall c \in x.\ pol_{A^\perp\|A}(c) = + \implies \overline{c} \in x\,.$$

Proof. It can first be checked that defining

$$\begin{aligned} c \le_{\mathbb{C}\!\mathbb{C}_A} c' \text{ iff } &(i)\ c \le_{A^\perp\|A} c' \text{ or} \\ &(ii)\ \exists c_0 \in A^\perp\|A.\ pol_{A^\perp\|A}(c_0) = + \ \& \\ &\qquad c \le_{A^\perp\|A} \overline{c_0} \ \&\ c_0 \le_{A^\perp\|A} c'\,, \end{aligned}$$

yields a partial order. Note that

$$c \le_{A^\perp\|A} d \text{ iff } \overline{c} \le_{A^\perp\|A} \overline{d}\,,$$

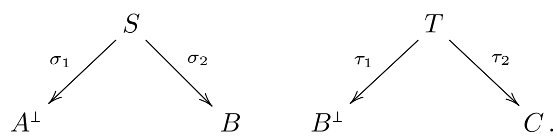
used in verifying transitivity and antisymmetry. The relation $\le_{\mathbb{C}\!\mathbb{C}_A}$ is clearly the transitive closure of $\le_{A^\perp\|A}$ together with all extra causal dependencies $(\overline{c}, c)$ where $pol_{A^\perp\|A}(c) = +$. The remaining properties required for $\mathbb{C}\!\mathbb{C}_A$ to be an event structure follow routinely.
(i) From the above characterization of $\le_{\mathbb{C}\!\mathbb{C}_A}$.

(ii) From $\mathbb{C}_A$ and $A^\perp \| A$ sharing the same consistency relation and the extra causal dependency adjoined to $\mathbb{C}_A$. □

Based on Proposition 4.1, define the *copy-cat* pre-strategy from A to A to be the pre-strategy $\gamma_A : \mathbb{C}_A \to A^\perp \| A$ where $\mathbb{C}_A$ comprises the event structure with polarity $A^\perp \| A$ together with extra causal dependencies $\overline{c} \leq_{\mathbb{C}_A} c$ for all events c with $pol_{A^\perp \| A}(c) = +$, and γ_A is the identity on the set of events common to both $\mathbb{C}_A$ and $A^\perp \| A$.

4.3.2 Composing pre-strategies

Consider two pre-strategies $\sigma : A \rightarrow\!\!\!\!+\!\!\!> B$ and $\tau : B \rightarrow\!\!\!\!+\!\!\!> C$ as spans:



We show how to define their composition $\tau \odot \sigma : A \rightarrow\!\!\!\!+\!\!\!> C$. If we ignore polarities the partial maps of event structures σ_2 and τ_1 have a common codomain, the underlying event structure of B and $B^\perp$. The composition $\tau \odot \sigma$ will be constructed as a synchronized composition of S and T, in which output events of S synchronize with input events of T, followed by an operation of hiding 'internal' synchronization events. Only those events s from S and t from T for which $\sigma_2(s) = \overline{\tau_1(t)}$ synchronize; note that then s and t must have opposite polarities as this is so for their images $\sigma_2(s)$ in B and $\tau_1(t)$ in $B^\perp$. The event resulting from the synchronization of s and t has indeterminate polarity and will be hidden in the composition $\tau \odot \sigma$.

Formally, we use the construction of synchronized composition and projection of Section 3.3.3. Via projection we hide all those events with undefined polarity.

We first define the composition of the families of configurations of S and T as a synchronized composition of stable families. We form the product of stable families $\mathcal{C}(S) \times \mathcal{C}(T)$ with projections π_1 and π_2, and then form a restriction:

$$\mathcal{C}(T) \odot \mathcal{C}(S) =_{\text{def}} \mathcal{C}(S) \times \mathcal{C}(T) \restriction R$$

where

$$\begin{aligned} R \;=\; &\{(s,*) \mid s \in S \;\&\; \sigma_1(s) \text{ is defined}\} \cup \\ &\{(s,t) \mid s \in S \;\&\; t \in T \;\&\; \sigma_2(s) = \overline{\tau_1(t)} \text{ with both defined}\} \cup \\ &\{(*,t) \mid t \in T \;\&\; \tau_2(t) \text{ is defined}\}\,. \end{aligned}$$

The stable family $\mathcal{C}(T) \odot \mathcal{C}(S)$ is the synchronized composition of the stable families $\mathcal{C}(S)$ and $\mathcal{C}(T)$ in which synchronizations are between events of S and T which project, under σ_2 and τ_1 respectively, to complementary events in B and $B^\perp$. The stable family $\mathcal{C}(T) \odot \mathcal{C}(S)$ represents all the configurations of the composition of pre-strategies, including internal events arising from synchronizations. We obtain the synchronized composition as an event structure by forming $\Pr(\mathcal{C}(T)\odot\mathcal{C}(S))$, in which events are the primes of $\mathcal{C}(T)\odot\mathcal{C}(S)$. This synchronized composition still has internal events.

To obtain the composition of pre-strategies we hide the internal events due to synchronizations. The event structure of the composition of pre-strategies is defined to be

$$T\odot S =_{\mathrm{def}} \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \downarrow V\,,$$

the projection onto "visible" events,

$$\begin{aligned} V = \{p \in \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \mid \exists s \in S.\ max(p) = (s,*)\} \cup \\ \{p \in \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \mid \exists t \in T.\ max(p) = (*,t)\}\,. \end{aligned}$$

Finally, the composition $\tau\odot\sigma$ is defined by the span

$$\begin{array}{ccc} & T\odot S & \\ {\scriptstyle \upsilon_1}\swarrow & & \searrow{\scriptstyle \upsilon_2} \\ A^{\perp} & & C \end{array}$$

where υ_1 and υ_2 are maps of event structures, which on events p of $T\odot S$ act so $\upsilon_1(p) = \sigma_1(s)$ when $max(p) = (s,*)$ and $\upsilon_2(p) = \tau_2(t)$ when $max(p) = (*,t)$, and are undefined elsewhere.

Proposition 4.2. *Above, υ_1 and υ_2 are partial maps of event structures with polarity, which together define a pre-strategy $\upsilon : A \to\!\!\!\!\!+\!\!\!\!> C$. For $x \in \mathcal{C}(T\odot S)$,*

$$\upsilon_1 x = \sigma_1\pi_1 \bigcup x \ \textit{and}\ \ \upsilon_2 x = \tau_2\pi_2 \bigcup x\,.$$

Proof. Consider the two maps of event structures

$$\begin{aligned} u_1 : \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \xrightarrow{\Pi_1} S \xrightarrow{\sigma_1} A^{\perp}\,, \\ u_2 : \Pr(\mathcal{C}(T)\odot\mathcal{C}(S)) \xrightarrow{\Pi_2} T \xrightarrow{\tau_2} C\,, \end{aligned}$$

where Π_1, Π_2 are (restrictions of) projections of the product of event structures. *E.g.* for $p \in \Pr(\mathcal{C}(T)\odot\mathcal{C}(S))$, $\Pi_1(p) = s$ precisely when $max(p) = (s,*)$, so $\sigma_1(s)$ is defined, or when $max(p) = (s,t)$, so $\sigma_1(s)$ is undefined. The partial functions υ_1 and υ_2 are restrictions of the two maps u_1 and u_2 to the projection set V. But V consists exactly of those events in $\Pr(\mathcal{C}(T)\odot\mathcal{C}(S))$ where u_1 or u_2 is defined. It follows that υ_1 and υ_2 are maps of event structures.

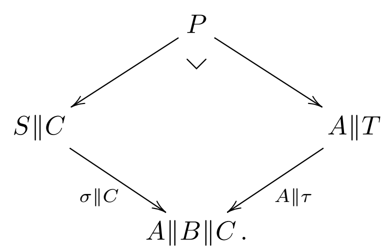
Clearly one and only one of υ_1, υ_2 are defined on any event in $T\odot S$ so they form a pre-strategy. Their effect on $x \in \mathcal{C}(T\odot S)$ follows directly from their definition. □

Proposition 4.3. *Let $\sigma : A \to\!\!\!\!\!+\!\!\!\!> B$, $\tau : B \to\!\!\!\!\!+\!\!\!\!> C$ and $\upsilon : C \to\!\!\!\!\!+\!\!\!\!> D$ be pre-strategies. The two compositions $\upsilon\odot(\tau\odot\sigma)$ and $(\upsilon\odot\tau)\odot\sigma$ are isomorphic.*

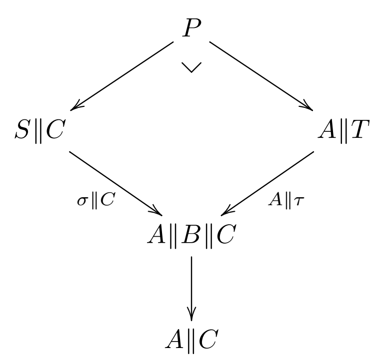
Proof. The natural isomorphism $S \times (T \times U) \cong (S \times T) \times U$, associated with the product of event structures S, T, U, restricts to the required isomorphism of spans as the synchronizations involved in successive compositions are disjoint. □

4.3.3 Composition via pullback

We can alternatively present the composition of pre-strategies via pullbacks.[2] For this section assume that the correspondence $a \leftrightarrow \overline{a}$ between the events of A and its dual $A^\perp$ is the identity, so A and $A^\perp$ share the same events, though assign opposite polarities to them. Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in $\mathcal{E}$ (or $\mathcal{E}_t$, as the maps along which we are pulling back are total)



There is an obvious partial map of event structures $A \| B \| C \to A \| C$ undefined on B and acting as identity on A and C. The partial map from P to $A \| C$ given by following the diagram (either way round the pullback square)



factors through the projection of P to V, those events at which the partial map is defined:

$$P \to P \downarrow V \to A \| C \,.$$

The resulting total map $\upsilon : P \downarrow V \to A \| C$ gives us the composition $\tau \odot \sigma : P \downarrow V \to A^\perp \| C$ once we reinstate polarities.

[2] I'm grateful to Nathan Bowler for the observations of this section.

4.3.4 Duality

A pre-strategy $\sigma : A \mathrel{\rightarrow\!\!\!\!+} B$ corresponds to a dual pre-strategy $\sigma^{\perp} : B^{\perp} \mathrel{\rightarrow\!\!\!\!+} A^{\perp}$. This duality arises from the correspondence

$$\begin{array}{ccccccc} & S & & \longleftrightarrow & & S & \\ {}^{\sigma_1}\swarrow & & \searrow^{\sigma_2} & & {}^{\sigma_2}\swarrow & & \searrow^{\sigma_1} \\ A^{\perp} & & B & & (B^{\perp})^{\perp} & & A^{\perp}\,. \end{array}$$

It is easy to check that the dual of copy-cat, $\gamma_A^{\perp}$, is isomorphic, as a span, to the copy-cat of the dual, $\gamma_{A^{\perp}}$, for A an event structure with polarity. It is also straightforward, though more involved, to show that the dual of a composition of pre-strategies $(\tau \odot \sigma)^{\perp}$ is isomorphic as a span to the composition $\sigma^{\perp} \odot \tau^{\perp}$. Duality, as usual, will save us work.

4.4 Strategies

This section is devoted to the main result of this chapter: that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient in order for copy-cat to behave as identity w.r.t. the composition of pre-strategies. It becomes compelling to define a *(nondeterministic) concurrent strategy*, in general, as a pre-strategy which is receptive and innocent.

4.4.1 Necessity of receptivity and innocence

The properties of *receptivity* and *innocence* of a pre-strategy, described below, will play a central role.

Receptivity. Say a pre-strategy $\sigma : S \to A$ is *receptive* when $\sigma x \overset{a}{\longrightarrow\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists ! s \in S.\ x \overset{s}{\longrightarrow\!\!\subset}$ & $\sigma(s) = a$, for all $x \in \mathcal{C}(S)$, $a \in A$. Receptivity ensures that no Opponent move which is possible is disallowed.

Innocence. Say a pre-strategy σ is *innocent* when it is both +-innocent and −-innocent:

+-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s) = +$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

−-*Innocence:* If $s \rightarrowtriangle s'$ & $pol(s') = -$ then $\sigma(s) \rightarrowtriangle \sigma(s')$.

The definition of a pre-strategy $\sigma : S \to A$ ensures that the moves of Player and Opponent respect the causal constraints of the game A. Innocence restricts Player further. Locally, within a configuration, Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$. Thus innocence gives Player the freedom to await Opponent moves before making their move, but prevents Player having any influence on the moves of Opponent beyond those stipulated in the game A; more surprisingly, innocence also disallows any immediate causality of the form $\oplus \rightarrowtriangle \oplus$, purely between Player moves, not already stipulated in the game A.

Two important consequences of −-innocence:

Lemma 4.4. *Let $\sigma : S \to A$ be a pre-strategy. Suppose, for $s, s' \in S$, that*

$$[s) \uparrow [s') \;\&\; pol_S(s) = pol_S(s') = - \;\&\; \sigma(s) = \sigma(s')\,.$$

(i) If σ is $-$-innocent, then $[s) = [s')$.
(ii) If σ is receptive and $-$-innocent, then $s = s'$.
[$x \uparrow y$ expresses the compatibility of $x, y \in \mathcal{C}(S)$.]

Proof. (i) Assume the property above holds of $s, s' \in S$. Assume σ is $-$-innocent. Suppose $s_1 \rightarrowtriangle s$. Then by $-$-innocence, $\sigma(s_1) \rightarrowtriangle \sigma(s)$. As $\sigma(s') = \sigma(s)$ and σ is a map of event structures there is $s_2 < s'$ such that $\sigma(s_2) = \sigma(s_1)$. But s_1, s_2 both belong to the configuration $[s) \cup [s')$ so $s_1 = s_2$, as σ is a map, and $s_1 < s'$. Symmetrically, if $s_1 \rightarrowtriangle s'$ then $s_1 < s$. It follows that $[s) = [s')$. (ii) Now both $[s)\stackrel{s}{-\!\!\!\subset}$ and $[s)\stackrel{s'}{-\!\!\!\subset}$ with $\sigma(s) = \sigma(s')$ where both s, s' have $-$ve polarity. If, further, σ is receptive, $s = s'$. □

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^- x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{-\}$, *i.e.* the configuration x' extends the configuration x solely by events of $-$ve polarity. In the presence of $-$-innocence, receptivity strengthens to the following useful property:

Lemma 4.5. *Let $\sigma : S \to A$ be a $-$-innocent pre-strategy. The pre-strategy σ is* receptive *iff whenever $\sigma x \subseteq^- y$ in $\mathcal{C}(A)$ there is a* unique *$x' \in \mathcal{C}(S)$ so that $x \subseteq x'$ & $\sigma x' = y$. Diagrammatically,*

$$\begin{array}{ccc} x & \cdots\subseteq\cdots & x' \\ {\scriptstyle\sigma}\downarrow & & \vdots\,{\scriptstyle\sigma} \\ \sigma x & \subseteq^- & y\,. \end{array}$$

[It will necessarily be the case that $x \subseteq^- x'$.]

Proof. *"if":* Clear. *"Only if":* Assuming $\sigma x \subseteq^- y$ we can form a covering chain

$$\sigma x \stackrel{a_1}{-\!\!\!\subset} y_1 \cdots \stackrel{a_n}{-\!\!\!\subset} y_n = y\,.$$

By repeated use of receptivity we obtain the existence of x' where $x \subseteq x'$ and $\sigma x' = y$. To show the uniqueness of x' suppose $x \subseteq z, z'$ and $\sigma z = \sigma z' = y$. Suppose that $z \neq z'$. Then, without loss of generality, suppose there is a $\leq_S$-minimal $s' \in z'$ with $s' \notin z$. Then $[s') \subseteq z$. Now $\sigma(s') \in y$ so there is $s \in z$ for which $\sigma(s) = \sigma(s')$. We have $[s), [s') \subseteq z$ so $[s) \uparrow [s')$. By Lemma 4.4(ii) we deduce $s = s'$ so $s' \in z$, a contradiction. Hence, $z = z'$. □

It is useful to define innocence and receptivity on partial maps of event structures with polarity.

Definition 4.6. Let $f : S \to A$ be a partial map of event structures with polarity. Say f is *receptive* when

$$f(x)\stackrel{a}{-\!\!\!\subset} \;\&\; pol_A(a) = - \implies \exists! s \in S.\; x\stackrel{s}{-\!\!\!\subset} \;\&\; f(s) = a$$

for all $x \in \mathcal{C}(S)$, $a \in A$.

Say f is *innocent* when it is both +-innocent and −-innocent, *i.e.*

$$s \rightarrowtriangle s' \ \&\ pol(s) = + \ \&\ f(s) \text{ is defined} \implies f(s') \text{ is defined} \ \&\ f(s) \rightarrowtriangle f(s')\,,$$
$$s \rightarrowtriangle s' \ \&\ pol(s') = - \ \&\ f(s') \text{ is defined} \implies f(s) \text{ is defined} \ \&\ f(s) \rightarrowtriangle f(s')\,.$$

Proposition 4.7. *A pre-strategy $\sigma : A \rightarrow\!\!\!\!+ B$ is receptive, respectively +/−-innocent, iff both the partial maps σ_1 and σ_2 of its span are receptive, respectively +/−-innocent.*

Proposition 4.8. *For $\sigma : A \rightarrow\!\!\!\!+ B$ a pre-strategy, σ_1 is receptive, respectively +/−-innocent, iff $(\sigma^\perp)_2$ is receptive, respectively +/−-innocent; σ is receptive and innocent iff $\sigma^\perp$ is receptive and innocent.*

The next lemma will play a major role in importing receptivity and innocence to compositions of pre-strategies.

Lemma 4.9. *For pre-strategies $\sigma : A \rightarrow\!\!\!\!+ B$ and $\tau : B \rightarrow\!\!\!\!+ C$, if σ_1 is receptive, respectively +/−-innocent, then $(\tau \odot \sigma)_1$ is receptive, respectively +/−-innocent.*

Proof. Abbreviate $\tau \odot \sigma$ to υ.
Receptivity: We show the receptivity of υ_1 assuming that σ_1 is receptive. Let $x \in \mathcal{C}(T \odot S)$ such that $\upsilon_1 x \overset{a}{\longrightarrow\!\!\!\subset}$ in $\mathcal{C}(A^\perp)$ with $pol_{A^\perp}(a) = -$. By Proposition 4.2, $\sigma_1 \pi_1 \bigcup x \overset{a}{\longrightarrow\!\!\!\subset}$ with $\pi_1 \bigcup x \in \mathcal{C}(S)$. As σ_1 is receptive there is a unique $s \in S$ such that $\pi_1 \bigcup x \overset{s}{\longrightarrow\!\!\!\subset}$ in S and $\sigma_1(s) = a$. It follows that $\bigcup x \overset{(s,*)}{\longrightarrow\!\!\!\subset} z$, for some z, in $\mathcal{C}(T) \odot \mathcal{C}(S)$. Defining $p =_{\text{def}} [(s,*)]_z$ we obtain $x \overset{p}{\longrightarrow\!\!\!\subset}$ and $\upsilon_1(p) = a$, with p the unique such event.
Innocence: Assume that σ_1 is innocent. To show the +-innocence of υ_1 we first establish a property of the $\rightarrowtriangle$-relation in the event structure $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, the synchronized composition of event structures S and T, before projection to V:

> *If $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$ with $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ defined, then $e' \in V$ and $\upsilon_1(e')$ is defined.*

Assume $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, $e \in V$, $pol(e) = +$ and $\upsilon_1(e)$ is defined. From the definition of $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, the event e is a prime configuration of $\mathcal{C}(T) \odot \mathcal{C}(S)$ where $max(e)$ must have the form $(s,*)$, for some event s of S where $\sigma_1(s)$ is defined. By Lemma 3.16, $max(e')$ has the form $(s',*)$ or (s',t) with $s \rightarrowtriangle s'$ in S. Now, as $s \rightarrowtriangle s'$ and $pol(s) = +$, from the +-innocence of σ_1, we obtain $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp \| A$. Whence $\sigma_1(s')$ is defined ensuring $max(e') = (s',*)$. It follows that $e' \in V$ and $\upsilon_1(e')$ is defined.

Now suppose $e \rightarrowtriangle e'$ in $T \odot S$. Then either
(i) $e \rightarrowtriangle e'$ in $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S))$, or

(ii) $e \rightarrowtriangle e_1 < e'$ in $\Pr(\mathcal{C}(T)\odot\mathcal{C}(S))$ for some 'invisible' event $e_1 \notin V$.

But the above argument shows that case (ii) cannot occur when $pol(e) = +$ and $\upsilon_1(e)$ is defined. It follows that whenever $e \rightarrowtriangle e'$ in $T\odot S$ with $pol(e) = +$ and $\upsilon_1(e)$ defined, then $\upsilon_1(e')$ is defined and $\upsilon_1(e) \rightarrowtriangle \upsilon_1(e')$, as required.

The argument showing $-$-innocence of υ_1 assuming that of σ_1 is similar. □

Corollary 4.10. *For pre-strategies $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow B$ and $\tau : B \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow C$, if τ_2 is receptive, respectively $+/-$-innocent, then $(\tau\odot\sigma)_2$ is receptive, respectively $+/-$-innocent.*

Proof. By duality using Lemma 4.9: if τ_2 is receptive, respectively $+/-$-innocent, then $(\tau^\perp)_1$ is receptive, respectively $+/-$-innocent, and hence $(\sigma^\perp\odot\tau^\perp)_1 = ((\tau\odot\sigma)^\perp)_1 = (\tau\odot\sigma)_2$ is receptive, respectively $+/-$-innocent. □

Lemma 4.11. *For an event structure with polarity A, the pre-strategy copy-cat $\gamma_A : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow A$ is receptive and innocent.*

Proof. Receptive: Suppose $x \in \mathcal{C}(\mathbb{C}_A)$ such that $\gamma_A x \overset{c}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{C}(A^\perp \| A)$ where $pol_{A^\perp\|A}(c) = -$. Now $\gamma_A x = x$ and $x' =_{\mathrm{def}} x \cup \{c\} \in \mathcal{C}(A^\perp\|A)$. Proposition 4.1(ii) characterizes those configurations of $A^\perp\|A$ which are also configurations of $\mathbb{C}_A$: the characterization applies to x and to its extension $x' = x \cup \{c\}$ because of the $-$ve polarity of c. Hence $x' \in \mathcal{C}(\mathbb{C}_A)$ and $x \overset{c}{\longrightarrow\!\!\!\!\subset} x'$ in $\mathcal{C}(\mathbb{C}_A)$, and clearly c is unique so $\gamma_A(c) = c$.
$-$*-Innocent:* Suppose $c \rightarrowtriangle c'$ in $\mathbb{C}_A$ and $pol(c') = -$. By Proposition 4.1(i), $c \rightarrowtriangle c'$ in $A^\perp\|A$. The argument for $+$-innocence is similar. □

Theorem 4.12. *Let $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow B$ be a pre-strategy from A to B. If $\sigma\odot\gamma_A \cong \sigma$ and $\gamma_B\odot\sigma \cong \sigma$, then σ is receptive and innocent.*

Let $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow B$ and $\tau : B \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow C$ be pre-strategies which are both receptive and innocent. Then their composition $\tau\odot\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow C$ is receptive and innocent.

Proof. We know the copy-cat pre-strategies γ_A and γ_B are receptive and innocent—Lemma 4.11. Assume $\sigma\odot\gamma_A \cong \sigma$ and $\gamma_B\odot\sigma \cong \sigma$. By Lemma 4.9, $(\sigma\odot\gamma_A)_1$ is receptive and innocent so σ_1 is receptive and innocent. From its dual, Corollary 4.10, $(\gamma_B\odot\sigma)_2$ so σ_2 is receptive and innocent. Hence σ is receptive and innocent.

Assume that $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow B$ and $\tau : B \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow C$ are receptive and innocent. The fact that σ is receptive and innocent ensures that $(\tau\odot\sigma)_1$ is receptive and innocent, that τ is receptive and innocent that $(\tau\odot\sigma)_2$ is too. Combining, we obtain that $\tau\odot\sigma$ is receptive and innocent. □

In other words, if a pre-strategy is to compose well with copy-cat, in the sense that copy-cat behaves as an identity w.r.t. composition, the pre-strategy must be receptive and innocent. Copy-cat behaving as identity is a hallmark of game-based semantics, so any sensible definition of concurrent strategy will have to ensure receptivity and innocence.

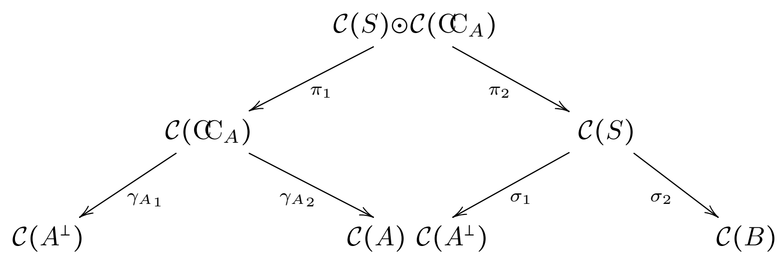
4.4.2 Sufficiency of receptivity and innocence

In fact, as we will now see, not only are the conditions of receptivity and innocence on pre-strategies necessary to ensure that copy-cat acts as identity. They are also sufficient.

Technically, this section establishes that for a pre-strategy $\sigma : A \longrightarrow\!\!\!\!\!\!\!+\; B$ which is receptive and innocent both the compositions $\sigma \odot \gamma_A$ and $\gamma_B \odot \sigma$ are isomorphic to σ. We shall concentrate on the isomorphism from $\sigma \odot \gamma_A$ to σ. The isomorphism from $\gamma_B \odot \sigma$ to σ follows by duality.

Recall, from Section 4.3.2, the construction of the pre-strategy $\sigma \odot \gamma_A$ as a total map $S \odot \mathbb{C}_A \to A^\perp \| B$. The event structure $S \odot \mathbb{C}_A$ is built from the synchronized composition of stable families $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$, a restriction of the product of stable families to events

$$\begin{aligned}&\{(c,*) \mid c \in \mathbb{C}_A \ \&\ \gamma_{A1}(c) \text{ is defined}\} \cup \\ &\{(c,s) \mid c \in \mathbb{C}_A \ \&\ s \in S \ \&\ \gamma_{A2}(c) = \overline{\sigma_1(s)}\} \cup \\ &\{(*,s) \mid s \in S \ \&\ \sigma_2(t) \text{ is defined}\} :\end{aligned}$$



Finally $S \odot \mathbb{C}_A$ is obtained from the prime configurations of $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$ whose maximum events are defined under $\gamma_{A1}\pi_1$ or $\sigma_2 \pi_2$.

We will first present the putative isomorphism from $\sigma \odot \gamma_A$ to σ as a total map of event structures $\theta : S \odot \mathbb{C}_A \to S$. The definition of θ depends crucially on the lemmas below. They involve special configurations of $\mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$, *viz.* those of the form $\bigcup x$, where x is a configuration of $S \odot \mathbb{C}_A$.

Lemma 4.13. *For* $x \in \mathcal{C}(S \odot \mathbb{C}_A)$,

$$(c,s) \in \bigcup x \implies (\overline{c},*) \in \bigcup x\,.$$

Proof. The case when $pol(c) = +$ follows directly because then $\overline{c} \rightarrowtriangle c$ in $\mathbb{C}_A$ so $(\overline{c},*) \rightarrowtriangle_{\bigcup x} (c,s)$.

Suppose the lemma fails in the case when $pol(c) = -$, so there is a $\leq_{\bigcup x}$-maximal $(c,s) \in \bigcup x$ such that

$$pol(c) = - \ \&\ (\overline{c},*) \notin \bigcup x\,. \qquad (\dagger)$$

The event (c,s) cannot be maximal in $\bigcup x$ as its maximal events take the form $(c',*)$ or $(*,s')$. There must be $e \in \bigcup x$ for which

$$(c,s) \rightarrowtriangle_{\bigcup x} e\,.$$

Consider the possible forms of e:
Case $e = (c', s')$*:* Then, by Lemma 3.16, either $c \rightarrowtriangle c'$ in $ℂ_A$ or $s \rightarrowtriangle s'$ in S. However if $s \rightarrowtriangle s'$ then, as $pol(s) = +$ by innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp$, so $\gamma_{A2}(c) \rightarrowtriangle \gamma_{A2}(c')$ in A; but then $c \rightarrowtriangle c'$ in $ℂ_A$. Either way, $c \rightarrowtriangle c'$ in $ℂ_A$.

Suppose $pol(c') = +$. Then,

$$(c, s) \rightarrowtriangle_{\bigcup x} (\overline{c}, *) \rightarrowtriangle_{\bigcup x} (\overline{c'}, *) \rightarrowtriangle_{\bigcup x} (c', s') .$$

But this contradicts $(c, s) \rightarrowtriangle_{\bigcup x} (c', s')$.

Suppose $pol(c') = -$. Because (c, s) is maximal such that (†), $(\overline{c'}, *) \in \bigcup x$. But $(\overline{c}, *) \rightarrowtriangle_{\bigcup x} (\overline{c'}, *)$ whence $(\overline{c}, *) \in \bigcup x$, contradicting (†).
Case $e = (*, s')$*:* Now $(c, s) \rightarrowtriangle_{\bigcup x} (*, s')$. By Lemma 3.16, $s \rightarrowtriangle s'$ in S with $pol(s) = +$. By innocence, $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ and in particular $\sigma_1(s')$ is defined, which forbids $(*, s')$ as an event of $\mathcal{C}(S) \odot \mathcal{C}(ℂ_A)$.
Case $e = (c', *)$*:* Now $(c, s) \rightarrowtriangle_{\bigcup x} (c', *)$. By Lemma 3.16, $c \rightarrowtriangle c'$ in $ℂ_A$. Because (c, s) and $(c', *)$ are events of $\mathcal{C}(S) \odot \mathcal{C}(ℂ_A)$ we must have $\gamma_2(c)$ and $\gamma_1(c')$ are defined—they are in different components of $ℂ_A$. By Proposition 4.1, $c' = \overline{c}$, contradicting (†).

In all cases we obtain a contradiction—hence the lemma. □

Lemma 4.14. *For* $x \in \mathcal{C}(S \odot ℂ_A)$*,*

$$\sigma_1 \pi_2 \bigcup x \subseteq^- \gamma_{A1} \pi_1 \bigcup x .$$

Proof. As a direct corollary of Lemma 4.13, we obtain:

$$\sigma_1 \pi_2 \bigcup x \subseteq \gamma_{A1} \pi_1 \bigcup x .$$

The current lemma will follow provided all events of +ve polarity in $\gamma_{A1}\pi_1 \bigcup x$ are in $\sigma_1 \pi_2 \bigcup x$. However, $(\overline{c}, s) \rightarrowtriangle_{\bigcup x} (c, *)$, for some $s \in S$, when $pol(c) = +$. □

Lemma 4.15. *For* $x \in \mathcal{C}(S \odot ℂ_A)$*,*

$$\sigma \pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A\, x .$$

Proof.

$$\begin{aligned} \sigma \pi_2 \bigcup x &= \{1\} \times \sigma_1 \pi_2 \bigcup x \ \cup\ \{2\} \times \sigma_2 \pi_2 \bigcup x \\ &\subseteq^- \{1\} \times \gamma_{A1} \pi_1 \bigcup x \ \cup\ \{2\} \times \sigma_2 \pi_2 \bigcup x , \quad \text{by Lemma 4.14} \\ &= \sigma \odot \gamma_A\, x , \quad \text{by Proposition 4.2.} \end{aligned}$$

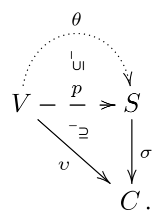
□

Lemma 4.15 is the key to defining a map $\theta : S \odot ℂ_A \to S$ via the following map-lifting property of receptive, −-innocent maps:

Lemma 4.16. *Let* $\sigma : S \to C$ *be a total map of event structures with polarity which is receptive and −-innocent. Let* $p : \mathcal{C}(V) \to \mathcal{C}(S)$ *be a monotonic* *function, i.e. such that $p(x) \subseteq p(y)$ whenever $x \subseteq y$ in $\mathcal{C}(V)$. Let $\upsilon : V \to C$ be a total map of event structures with polarity such that*

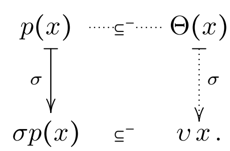
$$\forall x \in \mathcal{C}(V).\ \sigma p(x) \subseteq^- \upsilon\, x\,.$$

Then, there is a unique total map $\theta : V \to S$ of event structures with polarity such that $\forall x \in \mathcal{C}(V).\ p(x) \subseteq^- \theta\, x$ and $\upsilon = \sigma\theta$:



[We use a broken arrow to signify that p is not a map of event structures.]

Proof. Let $x \in \mathcal{C}(V)$. Then $\sigma p(x) \subseteq^- \upsilon\, x$. Define $\Theta(x)$ to be the unique configuration of $\mathcal{C}(S)$, determined by the receptivity of σ—Lemma 4.5, such that



Define θ_x to be the composite bijection

$$\theta_x : x \;\cong\; \upsilon x \;\cong\; \Theta(x)$$

where the bijection $x \cong \upsilon x$ is that determined locally by the total map of event structures υ, and the bijection $\upsilon x \cong \Theta(x)$ is the inverse of the bijection $\sigma \upharpoonright \Theta(x) : \Theta(x) \cong \upsilon\, x$ determined locally by the total map σ.

Now, let $y \in \mathcal{C}(V)$ with $x \subseteq y$. We claim that θ_x is the restriction of θ_y. This will follow once we have shown that $\Theta(x) \subseteq \Theta(y)$. Then, treating the inclusions as inclusion maps, both squares in the diagram below will commute:

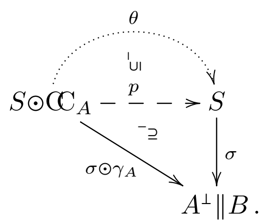
$$\begin{array}{ccccc} \theta_y : y & \cong & \upsilon\, y & \cong & \Theta(y) \\ \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid \\ \theta_x : x & \cong & \upsilon\, x & \cong & \Theta(x) \end{array}$$

This will make the composite rectangle commute, *i.e.* make θ_x the restriction of θ_y.

To show $\Theta(x) \subseteq \Theta(y)$ we suppose otherwise. Then there is an event $s \in \Theta(x)$ of minimum depth w.r.t. $\leq_S$ such that $s \notin \Theta(y)$. Note that $pol(s) = -$, as otherwise $s \in p(x) \subseteq p(y) \subseteq \Theta(y)$. As $\sigma(s) \in \upsilon\, x \subseteq \upsilon\, y$ there is $s' \in \Theta(y)$ such that $\sigma(s') = \sigma(s)$. From the minimality of s, both $[s), [s') \subseteq \Theta(y)$ ensuring the compatibility of $[s)$ and $[s')$. By Lemma 4.4(ii), $s = s'$ and $s \in \Theta(y)$—a contradiction.

By Proposition 2.6, the family θ_x, $x \in \mathcal{C}(V)$, determines the unique total map $\theta : V \to S$ such that $\theta\, x = \Theta(x)$. By construction, $p(x) \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(V)$, and $\upsilon = \sigma\theta$. This property in itself ensures that $\theta\, x = \Theta(x)$ so determines θ uniquely. □

In Lemma 4.16, instantiate $p : \mathcal{C}(S \odot \mathbb{C}\!\mathbb{C}_A) \to \mathcal{C}(S)$ to the function $p(x) = \pi_2 \bigcup x$ for $x \in \mathcal{C}(S \odot \mathbb{C}\!\mathbb{C}_A)$, the map σ to the pre-strategy $\sigma : S \to A^\perp \| B$ and υ to the pre-strategy $\sigma \odot \gamma_A$. By Lemma 4.15, $\sigma \pi_2 \bigcup x \subseteq^- \sigma \odot \gamma_A\, x$, so the conditions of Lemma 4.16 are met and we obtain a total map $\theta : S \odot \mathbb{C}\!\mathbb{C}_A \to S$ such that $\pi_2 \bigcup x \subseteq^- \theta\, x$, for all $x \in \mathcal{C}(S \odot \mathbb{C}\!\mathbb{C}_A)$, and $\sigma\theta = \sigma \odot \gamma_A$:



The next lemma is used in showing θ is an isomorphism.

Lemma 4.17. *(i) Let $z \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}\!\mathbb{C}_A)$. If $e \leq_z e'$ and $\pi_2(e)$ and $\pi_2(e')$ are defined, then $\pi_2(e) \leq_S \pi_2(e')$. (ii) The map π_2 is surjective on configurations.*

Proof. (i) It suffices to show when

$$e \rightarrowtriangle_z e_1 \rightarrowtriangle_z \cdots \rightarrowtriangle_z e_{n-1} \rightarrowtriangle_z e'$$

with $\pi_2(e)$ and $\pi_2(e')$ defined and all $\pi_2(e_i)$, $1 \leq i \leq n-1$, undefined, that $\pi_2(e) \leq_S \pi_2(e')$.
Case $n = 1$, so $e \rightarrowtriangle_z e'$: Use Lemma 3.16. If either e or e' has the form $(*, s)$ then the other event must have the form $(*, s')$ or (c', s') with $s \rightarrowtriangle s'$ in S. In the remaining case $e = (c, s)$ and $e' = (c', s')$ with either (1) $c \rightarrowtriangle c'$ in $\mathbb{C}\!\mathbb{C}_A$, and $\gamma_{A2}(c) \rightarrowtriangle \gamma_{A2}(c')$ in A, or (2) $s \rightarrowtriangle s'$ in S. If (1), $\sigma_1(s) \rightarrowtriangle \sigma_1(s')$ in $A^\perp$ where $s, s' \in \pi_2 z$. By Proposition 3.5, $s \leq_S s'$. In either case (1) or (2), $\pi_2(e) \leq_S \pi_2(e')$.
Case $n > 1$: Each e_i has the form $(c_i, *)$, for $1 \leq i \leq n-1$. By Lemma 3.16, events e and e' must have the form (c, s) and (c', s') with $c \rightarrowtriangle c_1$ and $c_{n-1} \rightarrowtriangle c'$ in $\mathbb{C}\!\mathbb{C}_A$. As $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, $c_1 = \overline{c}$ and similarly $c_{n-1} = \overline{c'}$. Again by Lemma 3.16, $c_i \rightarrowtriangle c_{i+1}$ in $\mathbb{C}\!\mathbb{C}_A$ for $1 \leq i \leq i-2$. Consequently $\gamma_{A2}(c) \leq_A \gamma_{A2}(c')$. Now, $s, s' \in \pi_2 z$ with $\sigma_1(s) \leq_{A^\perp} \sigma_1(s')$. By Proposition 3.5, $s \leq_S s'$, as required.
(ii) Let $y \in \mathcal{C}(S)$. Then $\sigma_1 y \in \mathcal{C}(A^\perp)$ and by the clear surjectivity of γ_{A2} on configurations there exists $w \in \mathcal{C}(\mathbb{C}\!\mathbb{C}_A)$ such that $\gamma_{A2} w = \sigma_1 y$. Now let

$$\begin{aligned} z = &\{(c, *) \mid c \in w \ \&\ \gamma_{A1}(c) \text{ is defined}\} \\ &\cup \{(c, s) \mid c \in w \ \&\ s \in y \ \&\ \gamma_{A2}(c) = \sigma_1(s)\} \\ &\cup \{(*, s) \mid s \in y \ \&\ \sigma_2(s) \text{ is defined}\}\,. \end{aligned}$$

Then, from the definition of the product of stable families—3.3.1, it can be checked that $z \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$. By construction, $\pi_2 z = y$. Hence π_2 is surjective on configurations. □

Theorem 4.18. *$\theta : \sigma \odot \gamma_A \cong \sigma$, an isomorphism of pre-strategies.*

Proof. We show θ is an isomorphism of event structures by showing θ is rigid and both surjective and injective on configurations (Lemma 3.3 of [7]). The rest is routine.
Rigid: It suffices to show $p \rightarrowtriangle p'$ in $S \odot \mathbb{C}_A$ implies $\theta(p) \leq_S \theta(p')$. Suppose $p \rightarrowtriangle p'$ in $S \odot \mathbb{C}_A$ with $max(p) = e$ and $max(p') = e'$. Take $x \in \mathcal{C}(S \odot \mathbb{C}_A)$ containing p' so p too. Then

$$e \rightarrowtriangle_{\bigcup x} e_1 \rightarrowtriangle_{\bigcup x} \cdots \rightarrowtriangle_{\bigcup x} e_{n-1} \rightarrowtriangle_{\bigcup x} e'$$

where $e, e' \in V_0$ and $e_i \notin V_0$ for $1 \leq i \leq n-1$. (V_0 consists of 'visible' events of the form $(c, *)$ with $\gamma_{A1}(c)$ defined, or $(*, s)$, with $\sigma_2(s)$ defined.)
Case $n = 1$, so $e \rightarrowtriangle_{\bigcup x} e'$: By Lemma 3.16, either (i) $e = (*, s)$ and $e' = (*, s')$ with $s \rightarrowtriangle s'$ in S, or (ii) $e = (c, *)$ and $e' = (c', *)$ with $c \rightarrowtriangle c'$ in $\mathbb{C}_A$.
If (i), we observe, via $\sigma\theta = \sigma \odot \gamma_A$, that $s \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s)$, so $\theta(p) = s$ by the local injectivity of σ. Similarly, $\theta(p') = s'$, so $\theta(p) \leq_S \theta(p')$.
If (ii), we obtain $\theta(p), \theta(p') \in \theta x$ with $\sigma_1 \theta(p) = \gamma_{A1}(c)$, $\sigma_1 \theta(p') = \gamma_{A1}(c')$ and $\gamma_{A1}(c) \rightarrowtriangle \gamma_{A1}(c')$ in $A^\perp$. By Proposition 3.5, $\theta(p) \leq_S \theta(p')$.
Case $n > 1$: Note $e_i = (c_i, s_i)$ for $1 \leq i \leq n-1$, and that $s_1 \leq_S s_{n-1}$ by Lemma 4.17(i). Consider the case in which $e = (c, *)$ and $e' = (c', *)$—the other cases are similar. By Lemma 3.16, $c \rightarrowtriangle c_1$ and $c_{n-1} \rightarrowtriangle c'$ in $\mathbb{C}_A$. But $\gamma_{A1}(c)$ and $\gamma_{A2}(c_1)$ are defined, so $c_1 = \overline{c}$, and similarly $c_{n-1} = \overline{c'}$. We remark that $\theta(p) = s_1$, by the local injectivity of σ, as both $s_1 \in \pi_2 \bigcup x \subseteq \theta x$ and $\theta(p) \in \theta x$ with $\sigma(\theta(p)) = \sigma(s_1)$. Similarly $\theta(p') = s_{n-1}$, whence $\theta(p) \leq_S \theta(p')$.
Surjective: Let $y \in \mathcal{C}(S)$. By Lemma 4.17(ii), there is $z \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$ such that $\pi_2 z = y$. Let

$$z' = z \cup \{(c, *) \mid pol(c) = + \ \&\ \exists s \in S.\ (\overline{c}, s) \in z\}\,.$$

It is straightforward to check $z' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$. Now let

$$z'' = z' \setminus \{(c, *) \mid pol(c) = - \ \&\ \forall s \in S.\ (\overline{c}, s) \notin z'\}\,.$$

Then $z'' \in \mathcal{C}(S) \odot \mathcal{C}(\mathbb{C}_A)$ by the following argument. The set z'' is certainly consistent, so it suffices to show

$$pol(c) = - \ \&\ (c, *) \leq_{z'} e \in z'' \implies \exists s \in S.\ (\overline{c}, s) \in z'\,,$$

for all $c \in \mathbb{C}_A$ and $e \in z''$. This we do by induction on the number of events between $(c, *)$ and e. Suppose

$$pol(c) = - \ \&\ (c, *) \rightarrowtriangle_{z'} e_1 \leq_{z'} e \in z'\,.$$

In the case where $e_1 = (c_1, s_1)$, we deduce $c \rightarrowtriangle c_1$ in $ℂ_A$ and as $\gamma_{A1}(c)$ is defined while $\gamma_{A2}(c_1)$ is defined, we must have $c_1 = \overline{c}$, as required. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = -$, by induction, we obtain $(\overline{c_1}, s_1) \in z'$ for some $s_1 \in S$. Also $c \rightarrowtriangle c_1$, so $\overline{c} \rightarrowtriangle \overline{c_1}$ in $ℂ_A$. As z' is a configuration we must have $(\overline{c}, s) \leq_{z'} (\overline{c_1}, s_1)$, for some $s \in S$, so $(\overline{c}, s) \in z'$. In the case where $e_1 = (c_1, *)$ and $pol(c_1) = +$, we have $c \rightarrowtriangle c_1$ in $ℂ_A$. Moreover, $(\overline{c}_1, s) \in z'$, for some $s \in S$, as z' is a configuration and $\overline{c_1} \rightarrowtriangle c_1$ in $ℂ_A$. Again, from the fact that z' is a configuration, there must be $(\overline{c}, s) \in z'$ for some $s \in S$. We have exhausted all cases and conclude $z'' \in \mathcal{C}(S) \odot \mathcal{C}(ℂ_A)$ with $\theta z'' = \pi_2 z = y$, as required to show θ is surjective on configurations.
Injective: Abbreviate $\sigma \odot \gamma_A$ to υ. Assume $\theta x = \theta y$, where $x, y \in \mathcal{C}(S \odot ℂ_A)$. Via the commutativity $\upsilon = \sigma\theta$, we observe

$$\upsilon x = \sigma\theta\, x = \sigma\theta\, y = \upsilon y\,.$$

Recall by Proposition 4.2, that $\upsilon_1 x = \gamma_{A1}\pi_1 \bigcup x = \pi_1 \bigcup x$. It follows that

$$(c, *) \in \bigcup x \iff c \in \upsilon_1 x \iff c \in \upsilon_1 y \iff (c, *) \in \bigcup y\,.$$

Observe

$$(*, s) \in \bigcup x \iff \sigma_2(s) \text{ is defined } \& \ s \in \theta x\ :$$

"⇒" by the local injectivity of σ_2, as $p =_{\text{def}} [(*, s)]_{\bigcup x}$ yields $\theta(p) \in \theta x$ and $s \in \pi_2 \bigcup x \subseteq \theta x$ with $\sigma_2(\theta(p)) = \sigma_2(s)$, so $\theta(p) = s$; "⇐" as $\sigma_2(s)$ defined and $s \in \theta x$ entails $s = \theta(p)$ for some $p \in x$, necessarily with $max(p) = (*, s)$. Hence

$$\begin{aligned}(*, s) \in \bigcup x &\iff \sigma_2(s) \text{ is defined } \& \ s \in \theta x \\ &\iff \sigma_2(s) \text{ is defined } \& \ s \in \theta y \\ &\iff (*, s) \in \bigcup y\,.\end{aligned}$$

Assuming $(c, s) \in \bigcup x$ we now show $(c, s) \in \bigcup y$. (The converse holds by symmetry.) There is $p \in x$, such that $(c, s) \in p$. If $max(p) = (*, s')$ (also in $\bigcup y$ as it is visible) then as π_2 is rigid, $s \leq s'$ and we must have $(c', s) \in \bigcup y$. Otherwise, $max(p) = (d, *)$ and we can suppose (by taking p minimal) that $(c, s) \leq_{\bigcup x} (d', s') \rightarrowtriangle_{\bigcup x} (d, *)$. But then $\theta(p) = s' \in \theta x = \theta y$. Also $s \leq_S s'$, by the rigidity of π_2, and, as we have seen before, $d' = \overline{d}$ with d' −ve. Hence s' is +ve and as θy is a −ve extension of $\pi_2 \bigcup y$ we must have $s' \in \pi_2 \bigcup y$. Hence there is $(*, s')$ or (c'', s') in $\bigcup y$, and as $s \leq_S s'$ there is some $(c', s) \in \bigcup y$. In both cases, $\gamma_{A2}(c') = \overline{\sigma_1(s)} = \gamma_{A2}(c)$, so $c' = c$, and thus $(c, s) \in \bigcup y$.

We conclude $\bigcup x = \bigcup y$, so $x = y$, as required for injectivity. □

4.5 Concurrent strategies

Define a *strategy* to be a pre-strategy which is receptive and innocent. We obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \rightarrow\!\!\!\!\!\rightarrow B$ and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the functoriality of synchronized composition). The isomorphisms expressing associativity and the identity of copy-cat are those of Proposition 4.3 and Theorem 4.18 with its dual.

4.5.1 Alternative characterizations

Via saturation conditions

An alternative description of concurrent strategies exhibits the correspondence between innocence and earlier "saturation conditions," *reflecting* specific independence, in [8, 9, 10]:

Proposition 4.19. *A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that*
(i) $\sigma x \stackrel{a}{\longrightarrow\!\!\!\subset}$ & $pol_A(a) = - \Rightarrow \exists! s \in S.\ x \stackrel{s}{\longrightarrow\!\!\!\subset}$ & $\sigma(s) = a$, *for all* $x \in \mathcal{C}(S)$, $a \in A$.
(ii)(+) If $x \stackrel{e}{\longrightarrow\!\!\!\subset} x_1 \stackrel{e'}{\longrightarrow\!\!\!\subset}$ & $pol_S(e) = +$ *in* $\mathcal{C}(S)$ *and* $\sigma x \stackrel{\sigma(e')}{\longrightarrow\!\!\!\subset}$ *in* $\mathcal{C}(A)$, *then* $x \stackrel{e'}{\longrightarrow\!\!\!\subset}$ *in* $\mathcal{C}(S)$.
(ii)(−) If $x \stackrel{e}{\longrightarrow\!\!\!\subset} x_1 \stackrel{e'}{\longrightarrow\!\!\!\subset}$ & $pol_S(e') = -$ *in* $\mathcal{C}(S)$ *and* $\sigma x \stackrel{\sigma(e')}{\longrightarrow\!\!\!\subset}$ *in* $\mathcal{C}(A)$, *then* $x \stackrel{e'}{\longrightarrow\!\!\!\subset}$ *in* $\mathcal{C}(S)$.

Proof. Note that if $x \stackrel{e}{\longrightarrow\!\!\!\subset} x_1 \stackrel{e'}{\longrightarrow\!\!\!\subset}$ then either $e\ co\ e'$ or $e \rightarrowtriangle e'$. Condition (ii) is a contrapositive reformulation of innocence. □

Via lifting conditions

Let x and x' be configurations of an event structure with polarity. Write $x \subseteq^+ x'$ to mean $x \subseteq x'$ and $pol(x' \setminus x) \subseteq \{+\}$, *i.e.* the configuration x' extends the configuration x solely by events of +ve polarity. With this notation in place we can give an attractive characterization of concurrent strategies:

Proposition 4.20. *A strategy S in a game A comprises a total map of event structures with polarity $\sigma : S \to A$ such that*
(i) whenever $y \subseteq^+ \sigma x$ in $\mathcal{C}(A)$ there is a (necessarily unique) $x' \in \mathcal{C}(S)$ so that $x' \subseteq x$ & $\sigma x' = y$, i.e.

$$\begin{array}{ccc} x' & \subseteq & x \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ y & \subseteq^+ & \sigma x\,, \end{array}$$

and

(ii) whenever $\sigma x \subseteq^- y$ *in* $\mathcal{C}(A)$ *there is a unique* $x' \in \mathcal{C}(S)$ *so that* $x \subseteq x'$ & $\sigma x' = y$ *, i.e.*

$$\begin{array}{ccc} x & \cdots\subseteq\cdots & x' \\ {\scriptstyle\sigma}\downarrow & & \vdots\,{\scriptstyle\sigma} \\ \sigma x & \subseteq^- & y\,. \end{array}$$

Proof. Let $\sigma : S \to A$ be a total map of event structures with polarity. It is claimed that σ is a strategy iff (i) and (ii).

"Only if": Lemma 4.5 directly implies (ii). To establish (i) it suffices to show the seemingly weaker property (ii)$'$ that

$$y \stackrel{a}{\relbar\joinrel\subset} \sigma x \ \&\ pol(a) = + \implies \exists x' \in \mathcal{C}(S).\, x' \relbar\joinrel\subset x \ \&\ \sigma x' = y$$

for $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$. Then (ii), with $y \subseteq^+ \sigma x$, follows by considering a covering chain $y \relbar\joinrel\subset \cdots \relbar\joinrel\subset \sigma x$. (The uniqueness of x is a direct consequence of σ being a total map of event structures.) To show (ii)$'$, suppose $y \stackrel{a}{\relbar\joinrel\subset} \sigma x$ with a +ve. Then $\sigma(s) = a$ for some unique $s \in x$ with s +ve. Supposing s were not $\le$-maximal in x, then $s \rightarrowtriangle s'$ for some $s' \in x$. By +-innocence $a = \sigma(s) \rightarrowtriangle \sigma(s') \in \sigma x$ implying a is not $\le$-maximal in σx. This contradicts $y \stackrel{a}{\relbar\joinrel\subset} \sigma x$. Hence s is $\le$-maximal and $x' =_{\text{def}} x \setminus \{s\} \in \mathcal{C}(S)$ with $x' \relbar\joinrel\subset x$ and $\sigma x' = y$.

"If": Assume σ satisfies (i) and (ii). Clearly σ is receptive by (ii). We establish innocence via Proposition 4.19.

Suppose $x \stackrel{s}{\relbar\joinrel\subset} x_1 \stackrel{s'}{\relbar\joinrel\subset} x'$ and $pol(s) = +$ with $\sigma x \stackrel{\sigma(s')}{\relbar\joinrel\subset} y_2$. Then $y_2 \stackrel{\sigma(s)}{\relbar\joinrel\subset} \sigma x'$ with $pol(\sigma(s)) = +$. From (i) we obtain a unique $x_2 \in \mathcal{C}(S)$ such that $x_2 \subseteq x'$ and $\sigma x_2 = y_2$. As σ is a total map of event structures, we obtain $x_2 \stackrel{s}{\relbar\joinrel\subset} x'$ and subsequently $x \stackrel{s'}{\relbar\joinrel\subset} x_2$, as required by Proposition 4.19(ii)+.

Suppose $x \stackrel{s}{\relbar\joinrel\subset} x_1 \stackrel{s'}{\relbar\joinrel\subset} x'$ and $pol(s') = -$ with $\sigma x \stackrel{\sigma(s')}{\relbar\joinrel\subset} y_2$. The case where $pol(s) = +$ is covered by the previous argument: we obtain $x \stackrel{s'}{\relbar\joinrel\subset} x_2$, as required by Proposition 4.19(ii)−. Suppose $pol(s) = -$. We have

$$\sigma x \stackrel{\sigma(s')}{\relbar\joinrel\subset} y_2 \stackrel{\sigma(s)}{\relbar\joinrel\subset} \sigma x'\,.$$

As σ is already known to be receptive, we obtain

$$x \stackrel{e'}{\relbar\joinrel\subset} x_2 \stackrel{e}{\relbar\joinrel\subset} x'' \ \&\ \sigma x_2 = y_2 \ \&\ \sigma x'' = \sigma x'\,.$$

From the uniqueness part of (ii) we deduce $x'' = x'$. As σ is a total map of event structures, $e = s$ and $e' = s'$ ensuring $x \stackrel{s'}{\relbar\joinrel\subset}$, as required by Proposition 4.19(ii)−.
□

As its proof makes clear, condition (i) in Proposition 4.20 can be replaced by: for all $a \in A, x \in \mathcal{C}(S), y \in \mathcal{C}(A)$,

$$y \stackrel{+}{-\!\!\!\subset} \sigma x \implies \exists x' \in \mathcal{C}(S).\, x' -\!\!\!\subset x \;\&\; \sigma x' = y\,, \quad i.e.$$

$$\begin{array}{ccc} x' & \cdots -\!\!\!\subset \cdots & x \\ {\scriptstyle \sigma}\,\vdots & & \downarrow{\scriptstyle \sigma} \\ y & \stackrel{+}{-\!\!\!\subset} & \sigma x\,, \end{array}$$

where the relation $\stackrel{+}{-\!\!\!\subset}$ signifies the covering relation induced by an event of +ve polarity.

Via +-moves

A strategy is determined by its +-moves. More precisely, a strategy $\sigma : S \to A$ determines a monotone function $d : \mathcal{C}(S^+) \to \mathcal{C}(A)$ given by $d(x) = \sigma[x]_S$ for $x \in \mathcal{C}(S^+)$. The event structure S^+ is the projection of S to its purely +-ve moves. Intuitively, d specifies the position in the game at which Player moves occur. The function d determines the original strategy σ via the universal property described in the proposition below.

Proposition 4.21. *Let $\sigma : S \to A$ be a receptive −-innocent pre-strategy. Define $q : S \to S^+$ be the partial map of event structures with polarity mapping S to its projection S^+ comprising only the +ve events of S, so $q\,y = y^+$ for $y \in \mathcal{C}(S)$. Define the function $d : \mathcal{C}(S^+) \to \mathcal{C}(A)$ to act as $d(x) = \sigma[x]_S$ for $x \in \mathcal{C}(S^+)$. Then, $d(qy) \subseteq^- \sigma y$ for all $y \in \mathcal{C}(S)$, i.e.*

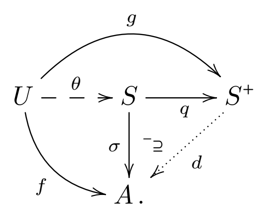
$$\begin{array}{ccc} S & \xrightarrow{q} & S^+ \\ {\scriptstyle \sigma}\downarrow & {}^-\!\supseteq & \swarrow{\scriptstyle d} \\ A\,. & & \end{array} \tag{1}$$

[The dotted line indicates that d is not a map of event structures.]
Suppose $f : U \to A$ is a total map and $g : U \to S^+$ a partial map of event structures with polarity such that $d(gy) \subseteq^- fy$ for all $y \in \mathcal{C}(U)$, i.e.

$$\begin{array}{ccc} U & \xrightarrow{g} & S^+ \\ {\scriptstyle f}\downarrow & {}^-\!\supseteq & \swarrow{\scriptstyle d} \\ A\,. & & \end{array} \tag{2}$$

Then, there is a unique total map of event structures with polarity $\theta : U \to S$

such that $f = \sigma\theta$ *and* $g = q\theta$*,*



(3)

Proof. We first check (1). Letting $y \in \mathcal{C}(S)$,

$$d(q\,y) = d(y^+) = \sigma[y^+]_S \subseteq^- y\,.$$

Suppose (2). Define $p : \mathcal{C}(U) \to \mathcal{C}(S)$ by taking

$$p(z) =_{\rm def} [g\,z]_S\,.$$

Clearly p is monotonic and

$$\sigma p(z) = \sigma[g\,z]_S = d(gz) \subseteq^- f\,z$$

for all $z \in \mathcal{C}(U)$. By Lemma 4.16, there is a unique total map of event structures with polarity $\theta : U \to S$ such that

$$f = \sigma\theta \quad \text{and} \quad \forall z \in \mathcal{C}(U).\ p(z) \subseteq^- \theta\,z\,.$$

From the latter, $[g\,z]_S \subseteq^- \theta z$ from which $g\,z = (g\,z)^+ = (\theta\,z)^+$, so $g\,z = q\theta\,z$, for all $z \in \mathcal{C}(U)$. Hence we have the commuting diagram (3). Noting

$$\forall z \in \mathcal{C}(U).\ \ g\,z = (\theta\,z)^+ \iff [g\,z]_S \subseteq^- \theta z\,,$$

we see that θ is the unique map making (3) commute. □

It follows that a strategy σ is determined up to isomorphism by its 'position function' d specifying at what state of the game Player moves are made. The position functions d which arise from strategies have been characterized by Alex Katovsky and GW [11].

Chapter 5

Deterministic strategies

This chapter concentrates on the important special case of deterministic concurrent strategies and their properties. They are shown to coincide with Melliès and Mimram's *receptive ingenuous strategies*.

5.0.2 Definition

We say an event structure with polarity S is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} S.\ Neg[X] \in \text{Con}_S \implies X \in \text{Con}_S\,,$$

where $Neg[X] =_{\text{def}} \{s' \in S \mid pol(s') = - \ \&\ \exists s \in X.\ s' \leq s\}$. In other words, S is deterministic iff any finite set of moves is consistent when it causally depends only on a consistent set of opponent moves. We say a strategy $\sigma : S \to A$ is deterministic if S is deterministic.

There is a simple, more local, characterisation of what it means to be deterministic.

Lemma 5.1. *An event structure with polarity S is deterministic iff*

$$\forall s, s' \in S, x \in \mathcal{C}(S).\ \ x \overset{s}{\longrightarrow\!\subset} \ \&\ x \overset{s'}{\longrightarrow\!\subset} \ \&\ pol(s) = + \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: Assume S is deterministic, $x \overset{s}{\longrightarrow\!\subset}$, $x \overset{s'}{\longrightarrow\!\subset}$ and $pol(s) = +$. Take $X =_{\text{def}} x \cup \{s, s'\}$. Then $Neg[X] \subseteq x \cup \{s'\}$ so $Neg[X] \in \text{Con}_S$. As S is deterministic, $X \in \text{Con}_S$ and being down-closed $X = x \cup \{s, s'\} \in \mathcal{C}(S)$.
"If": Assume S satisfies the property stated above in the proposition. Let $X \subseteq_{\text{fin}} S$ with $Neg[X] \in \text{Con}_S$. Then the down-closure $[Neg[X]] \in \mathcal{C}(S)$. Clearly $[Neg[X]] \subseteq [X]$ where all events in $[X] \setminus [Neg[X]]$ are necessarily positive. Suppose, to obtain a contradiction, that $X \notin \text{Con}_S$. Then there is a maximal $z \in \mathcal{C}(S)$ such that

$$[Neg[X]] \subseteq z \subseteq [X]$$

and some $e \in [X] \setminus z$, necessarily positive, for which $[e) \subseteq z$. Take a covering chain

$$[e) \stackrel{s_1}{-\!\!\!\subset} z_1 \stackrel{s_2}{-\!\!\!\subset} \cdots \stackrel{s_k}{-\!\!\!\subset} z_k = z\,.$$

As $[e) \stackrel{e}{-\!\!\!\subset} [e]$ with e positive, by repeated use of the property of the lemma—illustrated below—we obtain $z \stackrel{e}{-\!\!\!\subset} z'$ in $\mathcal{C}(S)$ with $[Neg[X]] \subseteq z' \subseteq [X]$, which contradicts the maximality of z.

$$\begin{array}{ccccccccc}
[e] & \stackrel{s_1}{-\!\!\!\subset} & z'_1 & \stackrel{s_2}{-\!\!\!\subset} & \cdots & \stackrel{s_k}{-\!\!\!\subset} & z'_k & = & z' \\
{\scriptstyle e}\Big\uparrow & & {\scriptstyle e}\Big\uparrow & & \cdots & & {\scriptstyle e}\Big\uparrow & & \\
[e) & \stackrel{s_1}{-\!\!\!\subset} & z_1 & \stackrel{s_2}{-\!\!\!\subset} & \cdots & \stackrel{s_k}{-\!\!\!\subset} & z_k & = & z
\end{array}$$

□

So, above, an event structure with polarity can fail to be deterministic in two ways, either with $pol(s) = pol(s') = +$ or with $pol(s) = +$ & $pol(s') = -$. In general for an event structure with polarity A the copy-cat strategy can fail to be deterministic in either way, illustrated in the examples below.

Example 5.2. *(i) Take A to consist of two positive events and one negative event, with any two but not all three events consistent. The construction of $\mathbb{C}_A$ is pictured:*

$$\begin{array}{ccc}
 & \ominus \rightarrow \oplus & \\
A^\perp & \ominus \rightarrow \oplus & A \\
 & \oplus \leftarrow \ominus &
\end{array}$$

Here γ_A is not deterministic: take x to be the set of all three negative events in $\mathbb{C}_A$ and s, s' to be the two positive events in the A component.
(ii) Take A to consist of two events, one positive and one negative event, inconsistent with each other. The construction $\mathbb{C}_A$:

$$\begin{array}{ccc}
A^\perp & \ominus \rightarrow \oplus & A \\
 & \oplus \leftarrow \ominus &
\end{array}$$

To see $\mathbb{C}_A$ is not deterministic, take x to be the singleton set consisting e.g. of the negative event on the left and s, s' to be the positive and negative events on the right.

5.0.3 The bicategory of deterministic strategies

We first characterise those games for which copy-cat is deterministic; they are "race-free" in that they only allow immediate conflict between events of the same polarity.

Lemma 5.3. *Let A be an event structure with polarity. The copy-cat strategy γ_A is deterministic iff A satisfies*

$$\forall x \in \mathcal{C}(A).\ x \stackrel{a}{-\!\!\!\subset}\ \&\ x \stackrel{a'}{-\!\!\!\subset}\ \&\ pol(a) = +\ \&\ pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad \textbf{(Race-free)}$$

Proof. *"Only if":* Suppose $x \in \mathcal{C}(A)$ with $x \stackrel{a}{-\!\!\subset}$ and $x \stackrel{a'}{-\!\!\subset}$ where $pol(a) = +$ and $pol(a') = -$. Construct $y =_{\rm def} \{(1, \overline{b}) \mid b \in x\} \cup \{(1, \overline{a})\} \cup \{(2, b) \mid b \in x\}$. Then $y \in \mathcal{C}(ℂ_A)$ with $y \stackrel{(2,a)}{-\!\!\subset}$ and $y \stackrel{(2,a')}{-\!\!\subset}$, by Proposition 4.1. Assuming $ℂ_A$ is deterministic, we obtain $y \cup \{(2,a),(2,a')\} \in \mathcal{C}(ℂ_A)$, so $y \cup \{(2,a),(2,a')\} \in \mathcal{C}(A^\perp \| A)$. This entails $x \cup \{a, a'\} \in \mathcal{C}(A)$, as required to show (**Race – free**).
"If": Assume A satisfies (**Race – free**). It suffices to show for $X \subseteq_{\rm fin} ℂ_A$, with X down-closed, that $Neg[X] \in {\rm Con}_{ℂ_A}$ implies $X \in {\rm Con}_{ℂ_A}$. Recall $Z \in {\rm Con}_{ℂ_A}$ iff $Z \in {\rm Con}_{A^\perp \| A}$.

Let $X \subseteq_{\rm fin} ℂ_A$ with X down-closed. Assume $Neg[X] \in {\rm Con}_{ℂ_A}$. Observe

(i) $\{c \mid c \in X \;\&\; pol(c) = -\} \subseteq Neg[X]$ and

(ii) $\{\overline{c} \mid c \in X \;\&\; pol(c) = +\} \subseteq Neg[X]$ as by Proposition 4.1, X being down-closed must contain $\overline{c}$ if it contains c with $pol(c) = +$.

Consider $X_2 =_{\rm def} \{a \mid (2,a) \in X\}$. Then X_2 is a finite down-closed subset of A. From (i),

$$X_2^- =_{\rm def} \{a \in X_2 \mid pol(a) = -\} \in {\rm Con}_A .$$

From (ii),

$$X_2^+ =_{\rm def} \{a \in X_2 \mid pol(a) = +\} \in {\rm Con}_A .$$

We show (**Race – free**) implies $X_2 \in {\rm Con}_A$.

Define $z^- =_{\rm def} [X_2^-]$ and $z^+ =_{\rm def} [X_2^+]$. Being down-closures of consistent sets, $z^-, z^+ \in \mathcal{C}(A)$. We show $z^- \uparrow z^+$ in $\mathcal{C}(A)$. First note $z^- \cap z^+ \in \mathcal{C}(A)$. If $a \in z^- \setminus z^- \cap z^+$ then $pol(a) = -$; otherwise, if $pol(a) = +$ then $a \in z^+$ a well as $a \in z^-$ making $a \in z^- \cap z^+$, a contradiction. Similarly, if $a \in z^+ \setminus z^- \cap z^+$ then $pol(a) = +$. We can form covering chains

$$z^- \cap z^+ \stackrel{p_1}{-\!\!\subset} x_1 \stackrel{p_2}{-\!\!\subset} \cdots \stackrel{p_k}{-\!\!\subset} x_k = z^- \quad \text{and} \quad z^- \cap z^+ \stackrel{n_1}{-\!\!\subset} y_1 \stackrel{n_2}{-\!\!\subset} \cdots \stackrel{n_l}{-\!\!\subset} y_l = z^+$$

where each p_i is positive and each n_j is negative.

Consequently, by repeated use of (**Race – free**), we obtain $x_k \cup y_l \in \mathcal{C}(A)$, *i.e.* $z^+ \cup z^- \in \mathcal{C}(A)$, as is illustrated below. But $X_2 \subseteq z^+ \cup z^-$, so $X_2 \in {\rm Con}_A$. A similar argument shows $X_1 =_{\rm def} \{a \in A^\perp \mid (1,a) \in X\} \in {\rm Con}_{A^\perp}$. It follows that $X \in {\rm Con}_{A^\perp \| A}$, so $X \in {\rm Con}_{ℂ_A}$ as required.

$$\begin{array}{ccccccccc}
y_l & \stackrel{p_1}{-\!\!\subset} & x_1 \cup y_l & \stackrel{p_2}{-\!\!\subset} & x_2 \cup y_l & \stackrel{p_3}{-\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\subset} & x_k \cup y_l \\
{\scriptstyle n_l}\Uparrow & & {\scriptstyle n_l}\Uparrow & & {\scriptstyle n_l}\Uparrow & & & & {\scriptstyle n_l}\Uparrow \\
\vdots & & \vdots & & \vdots & \cdots & \cdots & \cdots & \vdots \\
{\scriptstyle n_2}\Uparrow & & {\scriptstyle n_2}\Uparrow & & {\scriptstyle n_2}\Uparrow & & & & {\scriptstyle n_2}\Uparrow \\
y_1 & \stackrel{p_1}{-\!\!\subset} & x_1 \cup y_1 & \stackrel{p_2}{-\!\!\subset} & x_2 \cup y_1 & \stackrel{p_3}{-\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\subset} & x_k \cup y_1 \\
{\scriptstyle n_1}\Uparrow & & {\scriptstyle n_1}\Uparrow & & {\scriptstyle n_1}\Uparrow & & & & {\scriptstyle n_1}\Uparrow \\
z^- \cap z^+ & \stackrel{p_1}{-\!\!\subset} & x_1 & \stackrel{p_2}{-\!\!\subset} & x_2 & \stackrel{p_3}{-\!\!\subset} & \cdots & \stackrel{p_k}{-\!\!\subset} & x_k
\end{array}$$

□

Via the next lemma, when games satisfy $(\mathbf{Race-free})$ we can simplify the condition for a strategy to be deterministic.

Lemma 5.4. *Let $\sigma : S \to A$ be a strategy. Suppose $x \stackrel{s}{\longrightarrow\!\!\!\subset} y$ & $x \stackrel{s'}{\longrightarrow\!\!\!\subset} y'$ & $pol_S(s) = -$. Then, $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A) \implies y \uparrow y'$ in $\mathcal{C}(S)$.*

Proof. Assume $\sigma y \uparrow \sigma y'$ in $\mathcal{C}(A)$, so $\sigma y' \stackrel{\sigma(s)}{\longrightarrow\!\!\!\subset} \sigma y \cup \sigma y'$ in $\mathcal{C}(A)$. As $\sigma(s)$ is −ve, by receptivity, there is a unique $s'' \in S$, necessarily negative, such that $\sigma(s'') = \sigma(s)$ and $y' \stackrel{s''}{\longrightarrow\!\!\!\subset} x \cup \{s', s''\}$ in $\mathcal{C}(S)$. In particular, $x \cup \{s', s''\} \in \mathcal{C}(S)$. By −-innocence, we cannot have $s' \rightarrowtriangle s''$, so $x \cup \{s''\} \in \mathcal{C}(S)$. But now $x \stackrel{s}{\longrightarrow\!\!\!\subset}$ and $x \stackrel{s''}{\longrightarrow\!\!\!\subset}$ with $\sigma(s) = \sigma(s'')$ and both s, s'' negative and hence $s'' = s$ by the uniqueness part of receptivity. We conclude that $x \cup \{s', s\} \in \mathcal{C}(S)$ so $y \uparrow y'$. □

Corollary 5.5. *Assume A satisfies $(\mathbf{Race-free})$ of Lemma 5.3. A strategy $\sigma : S \to A$ is deterministic iff for all positive events $s, s' \in S$ and configurations $x \in \mathcal{C}(S)$,*

$$x \stackrel{s}{\longrightarrow\!\!\!\subset} \;\&\; x \stackrel{s'}{\longrightarrow\!\!\!\subset} \implies x \cup \{s, s'\} \in \mathcal{C}(S)\,.$$

Proof. *"Only if"*: clear. *"If"*: Let $x \stackrel{s}{\longrightarrow\!\!\!\subset}$ and $x \stackrel{s'}{\longrightarrow\!\!\!\subset}$ where $pol_S(s) = +$. For S to be deterministic we require $x \cup \{s, s'\} \in \mathcal{C}(S)$. The above assumption ensures this when $pol_S(s') = +$. Otherwise $pol_S(s') = -$ with $\sigma x \stackrel{\sigma(s)}{\longrightarrow\!\!\!\subset}$ and $\sigma x \stackrel{\sigma(s')}{\longrightarrow\!\!\!\subset}$. As A satisfies $(\mathbf{Race-free})$, $\sigma x \cup \sigma(s), \sigma(s') \in \mathcal{C}(A)$. Now by Lemma 5.4, $x \cup \{s, s'\} \in \mathcal{C}(S)$. □

Lemma 5.6. *The composition $\tau \odot \sigma$ of deterministic strategies σ and τ is deterministic.*

Proof. Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be deterministic strategies. The composition $T \odot S$ is constructed as $\Pr(\mathcal{C}(T) \odot \mathcal{C}(S)) \downarrow V$, a synchronised composition of event structures S and T projected to visible events $e \in V$ where $max(e)$ has the form $(s, *)$ or $(*, t)$.

We first note a fact about the effect of internal, or "invisible," events not in V on configurations of $\mathcal{C}(T) \odot \mathcal{C}(S)$. If

$$z \stackrel{(s,t)}{\longrightarrow\!\!\!\subset} w \;\&\; z \stackrel{(s',t')}{\longrightarrow\!\!\!\subset} w' \;\&\; w \not\uparrow w' \tag{1}$$

within $\mathcal{C}(T) \odot \mathcal{C}(S)$, then either

$$\pi_1 z \stackrel{s}{\longrightarrow\!\!\!\subset} \pi_1 w \;\&\; \pi_1 z \stackrel{s'}{\longrightarrow\!\!\!\subset} \pi_1 w' \;\&\; \pi_1 w \not\uparrow \pi_1 w'\,, \tag{2}$$

within $\mathcal{C}(S)$, or

$$\pi_2 z \stackrel{t}{\longrightarrow\!\!\!\subset} \pi_2 w \;\&\; \pi_2 z \stackrel{t'}{\longrightarrow\!\!\!\subset} \pi_2 w' \;\&\; \pi_2 w \not\uparrow \pi_2 w'\,, \tag{3}$$

within $\mathcal{C}(T)$. Assume (1). If $t = t'$ then $\sigma(s) = \overline{\tau(t)} = \overline{\tau(t')} = \sigma(s')$ and we obtain (2) as σ is a map of event structures. Similarly if $s = s'$ then (3). Supposing

$s \neq s'$ and $t \neq t'$ then if both (2) and (3) failed we could construct a configuration $z' =_{\text{def}} z \cup \{(s,t),(s',t)\}$ of $\mathcal{C}(T) \odot \mathcal{C}(S)$, contradicting (1); it is easy to check that z' is a configuration of the product $\mathcal{C}(S) \times \mathcal{C}(T)$ and its events are clearly within the restriction used in defining the synchronised composition.

We now show the impossibility of (2) and (3), and so (1). Assume (2) (case (3) is similar). One of s or s' being positive would contradict S being deterministic. Suppose otherwise, that both s and s' are negative. Then, because σ is a strategy, by Lemma 5.4, we have

$$\sigma_2 \pi_1 w \not\uparrow \sigma_2 \pi_1 w'$$

in $\mathcal{C}(B)$. Also, then both t and t' are positive ensuring $\pi_2 w \uparrow \pi_2 w'$ in $\mathcal{C}(T)$, as T is deterministic. This entails

$$\tau_1 \pi_2 w \uparrow \tau_1 \pi_2 w'$$

in $\mathcal{C}(B^\perp)$. But $\sigma_2\pi_1 w$ and $\tau_1\pi_2 w$, respectively $\sigma_2\pi_1 w'$ and $\tau_1 \pi_2 w'$, are the same configurations on the common event structure underlying B and $B^\perp$, of which we have obtained contradictory statements of compatibility.

As (1) is impossible, it follows that

$$z \xrightarrow{(s,t)}\!\!\subset w \ \& \ z \xrightarrow{(s',t')}\!\!\subset w' \implies w \uparrow w' \tag{4}$$

within $\mathcal{C}(T) \odot \mathcal{C}(S)$.

Finally, we can show that $\tau \odot \sigma$ is deterministic. Suppose $x \xrightarrow{p}\!\!\subset y$ and $x \xrightarrow{p'}\!\!\subset y'$ in $\mathcal{C}(T \odot S)$ with $pol(p) = +$. Then,

$$\bigcup x \xrightarrow{e_1}\!\!\subset z_1 \xrightarrow{e_2}\!\!\subset \cdots \xrightarrow{e_k}\!\!\subset z_k = \bigcup y \quad \text{and} \quad \bigcup x \xrightarrow{e_1'}\!\!\subset z_1' \xrightarrow{e_2'}\!\!\subset \cdots \xrightarrow{e_l'}\!\!\subset z_l' = \bigcup y'$$

in $\mathcal{C}(T) \odot \ \mathcal{C}(S)$, where $e_k = max(p)$ and $e_l' = max(p')$, and the events e_i and e_j' otherwise have the form $e_i = (s_i, t_i)$, when $1 \le i < k$, and $e_j' = (s_j', t_j')$, when $1 \le j < l$. By repeated use of (4) we obtain $z_{k-1} \uparrow z_{l-1}'$. (The argument is like that ending the proof of Lemma 5.3, though with the minor difference that now we may have $e_i = e_j'$.) We obtain $w =_{\text{def}} z_{k-1} \cup z_{l-1}' \in \mathcal{C}(T) \odot \mathcal{C}(S)$ with $w \xrightarrow{e_k}\!\!\subset$ and $w \xrightarrow{e_l'}\!\!\subset$ and $pol(e_k) = +$.

Now, $w \cup \{e_k, e_l'\} \in \mathcal{C}(T) \odot \mathcal{C}(S)$ provided $w \cup \{e_k, e_l'\} \in \mathcal{C}(S) \times \mathcal{C}(T)$. Inspect the definition of configurations of the product of stable families in Section 3.3.1. If e_k and e_l' have the form $(s, *)$ and $(s', *)$ respectively, then determinacy of S ensures that the projection $\pi_1 w \cup \{s, s'\} \in \mathcal{C}(S)$ whence $w \cup \{e_k, e_l'\}$ meets the conditions needed to be in $\mathcal{C}(S) \times \mathcal{C}(T)$. Similarly, $w \cup \{e_k, e_l'\} \in \mathcal{C}(S) \times \mathcal{C}(T)$ if e_k and e_l' have the form $(*, t)$ and $(*, t')$. Otherwise one of e_k and e_l' has the form $(s, *)$ and the other $(*, t)$. In this case again an inspection of the definition of configurations of the product yields $w \cup \{e_k, e_l'\} \in \mathcal{C}(S) \times \mathcal{C}(T)$. Forming the set of primes of $w \cup \{e_k, e_l'\}$ in V we obtain $x \cup \{p, p'\} \in \mathcal{C}(T \odot S)$.

This establishes that $T \odot S$ is deterministic. □

We thus obtain a sub-bicategory **DGames** of **Games**; its objects are race-free games—satisfying (**Race−free**) of Lemma 5.3—and its maps are deterministic strategies. The original duality of **Games**, where is $\sigma : A \rightarrow\!\!\!\!+ B$ corresponds to a dual strategy $\sigma^\perp : B^\perp \rightarrow\!\!\!\!+ A^\perp$, is maintained in **DGames**.

5.0.4 A category of deterministic strategies

In fact, **DGames** is equivalent to an order-enriched category via the following lemma. It says deterministic strategies in a game A are essentially certain subfamilies of configurations $\mathcal{C}(A)$, for which we give a characterisation.

Lemma 5.7. *A deterministic strategy is injective on configurations (i.e. is mono as a map of event structures).*

Proof. Let $\sigma : S \to A$ be a deterministic strategy. We show

$$x \supseteq z \longrightarrow\!\!\subset y \ \&\ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, by induction on $|x \smallsetminus z|$.

Suppose $x \supseteq z \stackrel{e}{\longrightarrow\!\!\subset} y$ and $\sigma y \subseteq \sigma x$. There are x_1 and event $e_1 \in S$ such that $z \stackrel{e_1}{\longrightarrow\!\!\subset} x_1 \subseteq x$. If $\sigma(e_1) = \sigma(e)$ then e_1, e have the same polarity; if negative, $e_1 = e$, by receptivity; if positive, $e_1 = e$, by determinacy with the local injectivity of σ. Either way $y \subseteq x$. Suppose $\sigma(e_1) \neq \sigma(e)$. We show in all cases $y \cup \{e_1\} \subseteq x$, so $y \subseteq x$.
Case pol$(e_1) = +$ *or pol*$(e) = +$: As σ is deterministic, e_1 and e are concurrent giving $x_1 \stackrel{e}{\longrightarrow\!\!\subset} y \cup \{e_1\}$. By induction we obtain $y \cup \{e_1\} \subseteq x$.
Case pol$(e_1) =$ *pol*$(e) = -$: From Lemma 5.4, we deduce that e_1 and e are concurrent yielding $x_1 \stackrel{e}{\longrightarrow\!\!\subset} y \cup \{e_1\}$, and by induction $y \cup \{e_1\} \subseteq x$.

Another, simpler induction on $|y \smallsetminus z|$ now yields

$$x \supseteq z \subseteq y \ \&\ \sigma y \subseteq \sigma x \implies y \subseteq x\,,$$

for $x, y, z \in \mathcal{C}(S)$, from which the results follows (taking z to be, for instance, $\varnothing$ or $x \cap y$). Injectivity of σ as a function on configurations is now obvious. □

We can provide an alternative description of deterministic strategies in a game A as certain subfamilies of $\mathcal{C}(A)$. A deterministic strategy $\sigma : S \to A$ determines, as the image of the configurations $\mathcal{C}(S)$, a subfamily $F =_{\mathrm{def}} \sigma\mathcal{C}(S)$ of configurations of $\mathcal{C}(A)$, which satisfies:

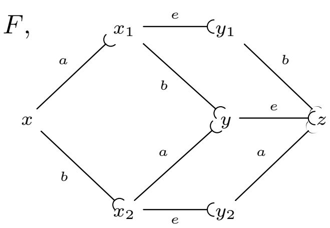
reachability: $\varnothing \in F$ and if $x \in F$ there is a covering chain $\varnothing \stackrel{a_1}{\longrightarrow\!\!\subset} x_1 \stackrel{a_2}{\longrightarrow\!\!\subset} \cdots \stackrel{a_k}{\longrightarrow\!\!\subset} x_k = x$ within F;

determinacy: If $x \stackrel{a}{\longrightarrow\!\!\subset}$ and $x \stackrel{a'}{\longrightarrow\!\!\subset}$ in F with $pol_A(a) = +$, then $x \cup \{a, a'\} \in F$;

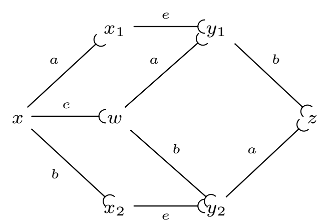
receptivity: If $x \in F$ and $x \stackrel{a}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$ and $pol_A(a) = -$, then $x \cup \{a\} \in F$;

+-innocence: If $x \stackrel{a}{\longrightarrow\!\!\subset} x_1 \stackrel{a'}{\longrightarrow\!\!\subset}$ & $pol_A(a) = +$ in F and $x \stackrel{a'}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$, then $x \stackrel{a'}{\longrightarrow\!\!\subset}$ in F (here receptivity implies −-innocence);

cube: In F,



implies



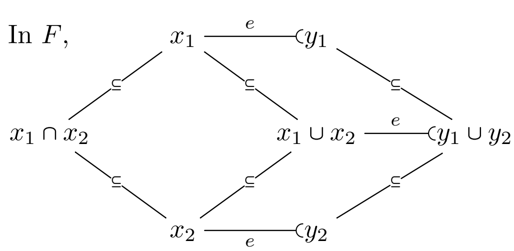
Theorem 5.8. *A subfamily $F \subseteq \mathcal{C}(A)$ satisfies the axioms above iff there is a deterministic strategy $\sigma : S \to A$ s.t. $F = \sigma\mathcal{C}(S)$, the image of $\mathcal{C}(S)$ under σ.*

Proof. (Sketch) It is routine to check that F, the image $\sigma\mathcal{C}(S)$ of a deterministic strategy, satisfies the axioms. Conversely, suppose a subfamily $F \subseteq \mathcal{C}(A)$ satisfies the axioms. We show F is a stable family. First note that from the axioms of determinacy and receptivity we can deduce:

$$\text{if } x \stackrel{a}{\longrightarrow\!\!\subset} \text{ and } x \stackrel{a'}{\longrightarrow\!\!\subset} \text{ in } F \text{ with } x \cup \{a, a'\} \in \mathcal{C}(A), \text{ then } x \cup \{a, a'\} \in F\,.$$

By repeated use of this property, using their reachability, if $x, y \in F$ and $x \uparrow y$ in $\mathcal{C}(A)$ then $x \cup y \in F$; the proof also yields a covering chain from x to $x \cup y$ and from y to $x \cup y$. (In particular, if $x \subseteq y$ in F, then there is a covering chain from x to y —a fact we shall use shortly.) Thus, if $x \uparrow y$ in F then $x \cup y \in F$. As also $\varnothing \in F$, we obtain Completeness, required of a stable family. Coincidence-freeness is a direct consequence of reachability. Repeated use of the cube axiom yields

Cube: In F,



implies $x_1 \cap x_2 \stackrel{e}{\longrightarrow\!\!\subset}$.

We use *Cube* to show stability. Assume $v \uparrow w$ in F. Let $z \in F$ be maximal s.t. $z \subseteq v, w$. We show $z = v \cap w$. Suppose not. Then, forming covering chains in F,

$$z \stackrel{c_1}{\longrightarrow\!\!\subset} v_1 \stackrel{c_2}{\longrightarrow\!\!\subset} \cdots \stackrel{c_k}{\longrightarrow\!\!\subset} v_k = v \quad \text{and} \quad z \stackrel{d_1}{\longrightarrow\!\!\subset} w_1 \stackrel{d_2}{\longrightarrow\!\!\subset} \cdots \stackrel{d_l}{\longrightarrow\!\!\subset} w_l = w\,,$$

there are c_i and d_j s.t. $c_i = d_j$, where we may assume c_i is the earliest event to be repeated as some d_j. Write $e =_{\text{def}} c_i = d_j$. Now, $v_{i-1} \cap w_{j-1} = z$. Also, being bounded above $v_{i-1} \cup w_{j-1} \in F$ and $v_i \cup w_j \in F$. We have an instance of *Cube*: take $x_1 = v_{i-1}$, $x_2 = w_{j-1}$, $y_1 = v_i$ and $y_2 = w_j$. Hence $z \stackrel{e}{\longrightarrow\!\!\subset}$ and $z \cup \{e\} \subseteq x, y$—contradicting the maximality of z. Therefore $z = v \cap w$, as required for stability.

Now we can form an event structure $S =_{\text{def}} \Pr(F)$. The inclusion $F \subseteq \mathcal{C}(A)$ induces a total map $\sigma : S \to A$ for which $F = \sigma\mathcal{C}(S)$. Note that $-$-innocence (*viz.* if $x \stackrel{a}{\longrightarrow\!\!\subset} x_1 \stackrel{a'}{\longrightarrow\!\!\subset}$ & $pol_A(a') = -$ in F and $x \stackrel{a'}{\longrightarrow\!\!\subset}$ in $\mathcal{C}(A)$, then $x \stackrel{a'}{\longrightarrow\!\!\subset}$ in F) is a direct consequence of receptivity. That S is deterministic follows from determinacy, that σ is a strategy from the axioms of receptivity and $+$-innocence. □

We can thus identify deterministic strategies from A to B with subfamilies of $\mathcal{C}(A^{\perp} \| B)$ satisfying the axioms above. Through this identification we obtain an order-enriched category of deterministic strategies (presented as subfamilies) equivalent to **DGames**; the order-enrichment is via the inclusion of subfamilies. As the proof of Theorem 5.8 above makes clear, in the characterization of those subfamilies F corresponding to deterministic families, the cube axiom can be replaced by *stability*: if $v \uparrow w$ in F, then $v \cap w \in F$.

Chapter 6

Games people play

We briefly and incompletely examine special cases of nondeterministic concurrent games in the literature.

6.1 Categories for games

We remark that event structures with polarity appear to provide a rich environment in which to explore structural properties of games and strategies. There are adjunctions

$$\begin{array}{ccccccc}
\mathcal{PA}_r & \overset{\longleftarrow}{\underset{\hookrightarrow}{\top}} & \mathcal{PF}_r & \overset{\longleftarrow}{\underset{\hookrightarrow}{\top}} & \mathcal{PE}_r & \overset{\longleftarrow}{\underset{\hookrightarrow}{\top}} & \mathcal{PE}_t \\
\downarrow \vdash \uparrow & & \downarrow \vdash \uparrow & & & & \\
\mathcal{PA}_r^{\#} & \overset{\longleftarrow}{\underset{\hookrightarrow}{\top}} & \mathcal{PF}_r^{\#} & & & &
\end{array}$$

relating $\mathcal{PE}_t$, the category of event structures with polarity with total maps, to subcategories $\mathcal{PE}_r$, with rigid maps, $\mathcal{PF}_r$ of forest-like (or filiform) event structures with rigid maps, and $\mathcal{PA}_r$, its full subcategory where polarities alternate along a branch; in $\mathcal{PF}_r^{\#}$ and $\mathcal{PA}_r^{\#}$ distinct branches are inconsistent. We shall mainly be considering games in $\mathcal{PE}_t$. Lamarche games and those of sequential algorithms belong to $\mathcal{PA}_r$ [12]. Conway games inhabit $\mathcal{PF}_r^{\#}$, in fact a coreflective subcategory of $\mathcal{PE}_t$ as the inclusion is now full; Conway's 'sum' is obtained by applying the right adjoint to the $\|$-composition of Conway games in $\mathcal{PE}_t$. Further refinements are possible. The 'simple games' of [13, 14] belong to $\mathcal{PA}_r^{-\#}$, the coreflective subcategory of $\mathcal{PA}_r^{\#}$ comprising "polarized" games, starting with moves of Opponent. The 'tensor' of simple games is recovered by applying the right adjoint of $\mathcal{PA}_r^{-\#} \hookrightarrow \mathcal{PE}_t$ to their $\|$-composition in $\mathcal{PE}_t$. Generally, the right adjoints, got by composition, from $\mathcal{PE}_t$ to the other categories fail to conserve immediate causal dependency. Such facts led Melliès *et al.* to the insight that uses of pointers in game semantics can be an artifact of working with models of games which do not take account of the independence of moves [15, 10].

6.2 Related work—early results

6.2.1 Stable spans, profunctors and stable functions

The sub-bicategory of **Games** where the events of games are purely +ve is equivalent to the bicategory of stable spans [7]. In this case, strategies correspond to *stable spans*:

$$\begin{array}{ccccccc} & & S & \longleftrightarrow & & S^+ & \\ & \swarrow^{\sigma_1} & & \searrow^{\sigma_2} & \swarrow^{\sigma_1^-} & & \searrow^{\sigma_2^+} \\ A^{\perp} & & & B \quad A & & & B\,, \end{array}$$

where S^+ is the projection of S to its +ve events; σ_2^+ is the restriction of σ_2 to S^+, necessarily a rigid map by innocence; σ_2^- is a *demand map* taking $x \in \mathcal{C}(S^+)$ to $\sigma_1^-(x) = \sigma_1[x]$; here $[x]$ is the down-closure of x in S. Composition of stable spans coincides with composition of their associated profunctors—see [3]. If we further restrict strategies to be deterministic (and, strictly, event structures to be countable) we obtain a bicategory equivalent to Berry's *dI-domains and stable functions*.

6.2.2 Ingenuous strategies

Via Theorem 5.8, deterministic concurrent strategies coincide with the *receptive ingenuous strategies* of Melliès and Mimram [10].

6.2.3 Closure operators

In [16], deterministic strategies are presented as closure operators. A deterministic strategy $\sigma : S \to A$ determines a closure operator φ on possibly infinite configurations $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x\}\,.$$

Clearly φ preserves intersections of configurations and is continuous. The closure operator φ on $\mathcal{C}^\infty(S)$ induces a *partial* closure operator φ_p on $\mathcal{C}^\infty(A)$. This in turn determines a closure operator $\varphi_p^\top$ on $\mathcal{C}^\infty(A)^\top$, where configurations are extended with a top $\top$, *cf.* [16]: take $y \in \mathcal{C}^\infty(A)^\top$ to the least, fixed point of φ_p above y, if such exists, and $\top$ otherwise.

6.2.4 Simple games

"Simple games" [13, 14] arise when we restrict **Games** to objects and deterministic strategies in $\mathcal{PA}_r^{-\#}$, described in Section 6.1.

6.2.5 Extensions

Games, such as those of [17, 18], allowing copying are being systematized through the use of monads and comonads [14], work now feasible on event structures with symmetry [7]. Nondeterministic strategies can potentially support probability as probabilistic or stochastic event structures [19] to become probabilistic or stochastic strategies.

Chapter 7

Winning ways

What does it mean to win a nondeterministic concurrent game and what is a winning strategy? This chapter extends the work on games and strategies to games with winning conditions and winning strategies.

7.1 Winning strategies

A *game with winning conditions* comprises $G = (A, W)$ where A is an event structure with polarity and $W \subseteq \mathcal{C}^\infty(A)$ consists of the *winning configurations* for Player. We define the *losing conditions* to be $L =_{\text{def}} \mathcal{C}^\infty(A) \setminus W$. Clearly a game with winning conditions is determined once we specify either its winning or losing conditions, and we can define such a game by specifying its losing conditions.

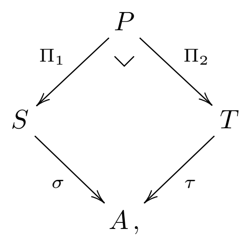
A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes Player moves to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$—a configuration x is +-maximal if whenever $x \stackrel{s}{\longrightarrow\!\!\subset}$ then the event s has −ve polarity. Any achievable position $z \in \mathcal{C}^\infty(S)$ of the game can be extended to a +-maximal, so winning, configuration (via Zorn's Lemma). So a strategy prescribes Player moves to reach a winning configuration whatever state of play is achieved following the strategy. Note that for a game A, if winning conditions $W = \mathcal{C}^\infty(A)$, *i.e.* every configuration is winning, then any strategy in A is a winning strategy.

In the special case of a deterministic strategy $\sigma : S \to A$ in G it is winning iff $\sigma\varphi(x) \in W$ for all $x \in \mathcal{C}^\infty(S)$, where φ is the closure operator $\varphi : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(S)$ determined by σ or, equivalently, the images under σ of fixed points of φ lie outside L. Recall from Section 6.2.3 that a deterministic strategy $\sigma : S \to A$ determines a closure operator φ on $\mathcal{C}^\infty(S)$: for $x \in \mathcal{C}^\infty(S)$,

$$\varphi(x) = x \cup \{s \in S \mid pol(s) = + \ \&\ Neg[\{s\}] \subseteq x\}\,.$$

Clearly, we can equivalently say a strategy $\sigma : S \to A$ in G is winning if it always prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent; a strategy $\sigma : S \to A$ in G is winning if $\sigma x \notin L$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$

Informally, we can also understand a strategy as winning for Player if when played against any counter-strategy of Opponent, the final result is a win for Player. Suppose $\sigma : S \to A$ is a strategy in a game (A, W). A counter-strategy is strategy of Opponent, so a strategy $\tau : T \to A^\perp$ in the dual game. We can view σ as a strategy $\sigma : \varnothing \twoheadrightarrow\!\!\!\!\!\!\!+\; A$ and τ as a strategy $\tau : A \twoheadrightarrow\!\!\!\!\!\!\!+\; \varnothing$. Their composition $\tau \odot \sigma : \varnothing \twoheadrightarrow\!\!\!\!\!\!\!+\; \varnothing$ is not in itself so informative. Rather it is the status of the configurations in $\mathcal{C}^\infty(A)$ their full interaction induces which decides which of Player or Opponent wins. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,



to obtain the event structure P resulting from the interaction of σ and τ. (Note $P \cong \mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))$, in the terms of Chapter 4, by the remarks of Section 4.3.3.) Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^\infty(P)$. A maximal configuration z in $\mathcal{C}^\infty(P)$ images to a configuration $\sigma \Pi_1 z = \tau \Pi_2 z$ in $\mathcal{C}^\infty(A)$. Define the set of *results* of the interaction of σ and τ to be

$$\langle \sigma, \tau \rangle =_{\mathrm{def}} \{ \sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P) \} .$$

We shall show the strategy σ is a winning for Player iff all the results of the interaction $\langle \sigma, \tau \rangle$ lie within the winning configurations W, for any counter-strategy $\tau : T \to A^\perp$ of Opponent.

It will be convenient later to have proved facts about +-maximality in the broader context of the composition of arbitrary strategies.

Convention 7.1. Refer to the construction of the composition of pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : B^\perp \| C$ in Chapter 4 We shall say a configuration x of either $\mathcal{C}^\infty(S), \mathcal{C}^\infty(T)$ or $(\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ is +-maximal if whenever $x \overset{e}{\longrightarrow\!\!\subset}$ then the event e has $-$ve polarity. In the case of $(\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ an event of $-$ve polarity is deemed to be one of the form $(s, *)$, with s $-$ve in S, or $(*, t)$, with t $-$ve in T. We shall say a configuration z of $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)))$ is +-maximal if whenever $z \overset{p}{\longrightarrow\!\!\subset}$ then $max(p)$ has $-$ve polarity.

Lemma 7.2. *Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be receptive pre-strategies. Then,*

$$z \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty \text{ is } +\text{-maximal iff}$$
$$\pi_1 z \in \mathcal{C}^\infty(S) \text{ is } +\text{-maximal } \& \ \pi_2 z \in \mathcal{C}^\infty(T) \text{ is } +\text{-maximal.}$$

Proof. Let $z \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$. *"Only if":* Assume z is +-maximal. Suppose, for instance, $\pi_1 z$ is not +-maximal. Then, $\pi_1 z \xrightarrow{s} \subset$ for some +ve event $s \in S$. Consider the two cases. *Case $\sigma_1(s)$ is defined:* Form the configuration $z \cup \{(s, *)\} \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$, to contradict the +-maximality of z. *Case $\sigma_2(s)$ is defined:* As s is +-ve by the receptivity of τ there is $t \in T$ such that $\tau_1(t) = \overline{\sigma_2(s)}$. Form the configuration $z \cup \{(s, t)\} \in (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$, to contradict the +-maximality of z. The argument showing $\pi_2 z$ is +-maximal is similar.

"If": Assume both $\pi_1 z$ and $\pi_2 z$ are +-maximal. Suppose z were not +-maximal. Then, either

- $z \xrightarrow{(s,*)} \subset$ or $z \xrightarrow{(s,t)} \subset$ with s is a +ve event of S, or
- $z \xrightarrow{(*,t)} \subset$ or $z \xrightarrow{(s,t)} \subset$ with t a +ve event of T.

But then either $\pi_1 z \xrightarrow{s} \subset$, contradicting the +-maximality of $\pi_1 z$, or $\pi_2 z \xrightarrow{t} \subset$, contradicting contradict the +-maximality of $\pi_2 z$. □

Corollary 7.3. *, Let $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$ be receptive pre-strategies. Then,*

$$x \in \mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))) \text{ is } +\text{-maximal iff}$$
$$\Pi_1 x \in \mathcal{C}^\infty(S) \text{ is } +\text{-maximal } \& \ \Pi_2 x \in \mathcal{C}^\infty(T) \text{ is } +\text{-maximal.}$$

Proof. From Lemma 7.2, noting the order isomorphism $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))) \cong (\mathcal{C}(T) \odot \mathcal{C}(S))^\infty$ given by $x \mapsto \bigcup x$ and that $\Pi_1 x = \pi_1 \bigcup x$, $\Pi_2 x = \pi_2 \bigcup x$. □

Lemma 7.4. *Let $\sigma : S \to A$ be a strategy in a game (A, W). The strategy σ is winning for Player iff $\langle \sigma, \tau \rangle \subseteq W$ for all (deterministic) strategies $\tau : T \to A^\perp$.*

Proof. "Only if": Suppose σ is winning, *i.e.* $\sigma x \in W$ for all +-maximal $x \in \mathcal{C}^\infty(S)$. Let $\tau : T \to A^\perp$ be a strategy. By Corollary 7.3,

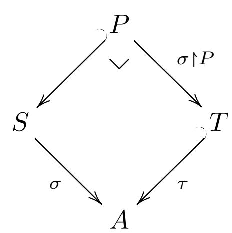
$$x \in \mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S))) \text{ is } +\text{-maximal}$$
$$\text{iff}$$
$$\Pi_1 x \in \mathcal{C}^\infty(S) \text{ is } +\text{-maximal } \& \ \Pi_2 x \in \mathcal{C}^\infty(T) \text{ is } +\text{-maximal.}$$

Letting x be maximal in $\mathcal{C}^\infty(\mathrm{Pr}(\mathcal{C}(T) \odot \mathcal{C}(S)))$ it is certainly +-maximal, whence $\Pi_1 x$ is +-maximal in $\mathcal{C}^\infty(S)$. It follows that $\sigma \Pi_1 x \in W$ as σ is winning. Hence $\langle \sigma, \tau \rangle \subseteq W$.

"If": Assume $\langle\sigma,\tau\rangle \subseteq W$ for all strategies $\tau : T \to A^\perp$. Suppose x is +-maximal in $\mathcal{C}^\infty(S)$. Define T to be the event structure given as the restriction

$$T =_{\text{def}} A^\perp \upharpoonright \sigma x \cup \{a \in A^\perp \mid pol_{A^\perp} = -\}\,.$$

Let $\tau : T \to A^\perp$ be the inclusion map $T \hookrightarrow A^\perp$. The pre-strategy τ can be checked to be receptive and innocent, so a strategy. (In fact, τ is a *deterministic* strategy as all its +ve events lie within the configuration σx.) One way to describe a pullback of τ along σ is as the "inverse image" $P =_{\text{def}} S \upharpoonright \{s \in S \mid \sigma(s) \in T\}$:



From the definition of T and P we see $x \in \mathcal{C}^\infty(P)$; and moreover that x is maximal in $\mathcal{C}^\infty(P)$ as x is +-maximal in $\mathcal{C}^\infty(S)$. Hence $\sigma x \in \langle\sigma,\tau\rangle$ ensuring $\sigma x \in W$, as required.

The proof is unaffected if we restrict to *deterministic* counter-strategies $\tau : T \to A^\perp$. □

Corollary 7.5. *There are the following four equivalent ways to say that a strategy $\sigma : S \to A$ is winning in (A, W)—we write L for the losing configurations $\mathcal{C}^\infty(A) \setminus W$:*

1. *$\sigma x \in W$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to reach a winning configuration, no matter what the activity or inactivity of Opponent;*

2. *$\sigma x \notin L$ for all +-maximal configurations $x \in \mathcal{C}^\infty(S)$, i.e. the strategy prescribes Player moves to avoid ending up in a losing configuration, no matter what the activity or inactivity of Opponent;*

3. *$\langle\sigma,\tau\rangle \subseteq W$ for all strategies $\tau : T \to A^\perp$, i.e. all plays against counter-strategies of the Opponent result in a win for Player;*

4. *$\langle\sigma,\tau\rangle \subseteq W$ for all deterministic strategies $\tau : T \to A^\perp$, i.e. all plays against deterministic counter-strategies of the Opponent result in a win for Player.*

Not all games with winning conditions have winning strategies. Consider the game A consisting of one player move $\oplus$ and one opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as its winning conditions. This game has no winning strategy; any strategy $\sigma : S \to A$, being receptive, will have an event $s \in S$ with $\sigma(s) = \ominus$, and so the losing $\{s\}$ as a +-maximal configuration.

7.2 Operations

7.2.1 Dual

There is an obvious dual of a game with winning conditions $G = (A, W_G)$:

$$G^\perp = (A^\perp, W_{G^\perp})$$

where, for $x \in \mathcal{C}^\infty(A)$,

$$x \in W_{G^\perp} \text{ iff } \overline{x} \notin W_G .$$

We are using the notation $a \leftrightarrow \overline{a}$, giving the correspondence between events of A and $A^\perp$, extended to their configurations: $\overline{x} =_{\text{def}} \{\overline{a} \mid a \in x\}$, for $x \in \mathcal{C}^\infty(A)$. As usual the dual reverses the roles of Player and Opponent and correspondingly the roles of winning and losing conditions.

7.2.2 Parallel composition

The parallel composition of two games with winning conditions $G = (A, W_G)$, $H = (B, W_H)$ is

$$G \| H =_{\text{def}} (A \| B,\ W_G \| \mathcal{C}^\infty(B) \cup \mathcal{C}^\infty(A) \| W_H)$$

where $X \| Y = \{\{1\} \times x \cup \{2\} \times y \mid x \in X \ \&\ y \in Y\}$ when X and Y are subsets of configurations. In other words, for $x \in \mathcal{C}^\infty(A \| B)$,

$$x \in W_{G \| H} \text{ iff } x_1 \in W_G \text{ or } x_2 \in W_H ,$$

where $x_1 = \{a \mid (1, a) \in x\}$ and $x_2 = \{b \mid (2, b) \in x\}$. To win in $G \| H$ is to win in either game. Its losing conditions are $L_A \| L_B$—to lose is to lose in both games G and H.[1] The unit of $\|$ is $(\varnothing, \varnothing)$. In order to disambiguate the various forms of parallel composition, we shall sometimes use the linear-logic notation $G \parr H$ for the parallel composition $G \| H$ of games with winning strategies.

7.2.3 Tensor

Defining $G \otimes H =_{\text{def}} (G^\perp \| H^\perp)^\perp$ we obtain a game where to win is to win in both games G and H—so to lose is to lose in either game. More explicitly,

$$(A, W_A) \otimes (B, W_B) =_{\text{def}} (A \| B,\ W_A \| W_B) .$$

The unit of $\otimes$ is $(\varnothing, \{\varnothing\})$.

[1] I'm grateful to Nathan Bowler, Pierre Clairambault and Julian Gutierrez for guidance in the definition of parallel composition of games with winning conditions.

7.2.4 Function space

With $G \multimap H =_{\text{def}} G^{\perp} \| H$ a win in $G \multimap H$ is a win in H conditional on a win in G.

Proposition 7.6. *Let $G = (A, W_G)$ and $H = (B, W_H)$ be games with winning conditions. Write $W_{G \multimap H}$ for the winning conditions of $G \multimap H$, so $G \multimap H = (A^{\perp} \| B, W_{G \multimap H})$. For $x \in \mathcal{C}^{\infty}(A^{\perp} \| B)$,*

$$x \in W_{G \multimap H} \quad \textit{iff} \quad \overline{x_1} \in W_G \implies x_2 \in W_H \,.$$

Proof. Letting $x \in \mathcal{C}^{\infty}(A^{\perp} \| B)$,

$$\begin{aligned} x \in W_{G \multimap H} \ &\text{iff} \ \ x \in W_{G^{\perp} \| H} \\ &\text{iff} \ \ x_1 \in W_{G^{\perp}} \text{ or } x_2 \in W_H \\ &\text{iff} \ \ \overline{x_1} \notin W_G \text{ or } x_2 \in W_H \\ &\text{iff} \ \ \overline{x_1} \in W_G \implies x_2 \in W_H \,. \end{aligned}$$

□

7.3 The bicategory of winning strategies

We can again follow Joyal and define strategies between games now with winning conditions: a (winning) strategy from G, a game with winning conditions, to another H is a (winning) strategy in $G \multimap H = G^{\perp} \| H$. We compose strategies as before. We first show that the composition of winning strategies is winning.

Lemma 7.7. *Let σ be a winning strategy in $G^{\perp} \| H$ and τ be a winning strategy in $H^{\perp} \| K$. Their composition $\tau \odot \sigma$ is a winning strategy in $G^{\perp} \| K$.*

Proof. Let $G = (A, W_G)$, $H = (B, W_H)$ and $K = (C, W_K)$.

Suppose $x \in \mathcal{C}^{\infty}(T \odot S)$ is $+$-maximal. Then $\bigcup x \in (\mathcal{C}(T) \odot \mathcal{C}(S))^{\infty}$. By Zorn's Lemma we can extend $\bigcup x$ to a maximal configuration $z \supseteq \bigcup x$ in $(\mathcal{C}(T) \odot \mathcal{C}(S))^{\infty}$ with the property that all events of $z \setminus \bigcup x$ are synchronizations of the form (s, t) for $s \in S$ and $t \in T$. Then, z will be $+$-maximal in $(\mathcal{C}(T) \odot \mathcal{C}(S))^{\infty}$ with

$$\sigma_1 \pi_1 z = \sigma_1 \pi_1 \bigcup x \quad \& \quad \tau_2 \pi_2 z = \tau_2 \pi_2 \bigcup x \,. \tag{1}$$

By Lemma 7.2,

$$\pi_1 z \text{ is } +\text{-maximal in } S \quad \& \quad \pi_2 z \text{ is } +\text{-maximal in } T \,.$$

As σ and τ are winning,

$$\sigma \pi_1 z \in W_{G^{\perp} \| H} \quad \& \quad \tau \pi_2 z \in W_{H^{\perp} \| K} \,.$$

Now $\sigma \pi_1 z \in W_{G^{\perp} \| H}$ expreses that

$$\overline{\sigma_1 \pi_1 z} \in W_G \implies \sigma_2 \pi_1 z \in W_H \tag{2}$$

and $\tau\pi_2 z \in W_{H^\perp\|K}$ that

$$\overline{\tau_1\pi_2 z} \in W_H \implies \tau_2\pi_2 z \in W_K\,, \tag{3}$$

by Proposition 7.6. But $\sigma_2\pi_1 z = \overline{\tau_1\pi_2 z}$, so (2) and (3) yield

$$\overline{\sigma_1\pi_1 z} \in W_G \implies \tau_2\pi_2 z \in W_K\,.$$

By (1)

$$\overline{\sigma_1\pi_1 \bigcup x} \in W_G \implies \tau_2\pi_2 \bigcup x \in W_K\,,$$

*i.e.*by Proposition 4.2,

$$\overline{\upsilon_1 x} \in W_G \implies \upsilon_2 x \in W_K$$

in the span of the composition $\tau\odot\sigma$. Hence $x \in W_{G^\perp\|K}$, as required. □

For a general game with winning conditions (A, W) the copy-cat strategy need not be winning, as shown in the following example.

Example 7.8. *Let A consist of two events, one +ve event $\oplus$ and one −ve event $\ominus$, inconsistent with each other. Take as winning conditions the set $\{\{\oplus\}\}$. The event structure $\mathbb{C}\mathbb{C}_A$:*

$$\begin{array}{cccc} A^\perp & \ominus \rightarrowtriangle \oplus & A \\ & \oplus \leftarrowtriangle \ominus & \end{array}$$

To see $\mathbb{C}\mathbb{C}_A$ is not winning consider the configuration x consisting of the two −ve events in $\mathbb{C}\mathbb{C}_A$. Then x is +-maximal as any +ve event is inconsistent with x. However, $\overline{x}_1 \in W$ while $x_2 \notin W$, failing the winning condition of $(A, W) \multimap (A, W)$.

Each event structure with polarity$A$ possesses a Scott order on its configurations $\mathcal{C}^\infty(A)$:

$$x' \sqsubseteq x \;\text{ iff }\; x' \supseteq^- x \cap x' \subseteq^+ x\,.$$

Exercise 7.9. Prove that the Scott order is indeed a partial order. □

A necessary and sufficient for copy-cat to be winning w.r.t. a game (A, W):

$$\forall x, x' \in \mathcal{C}^\infty(A).\;\; \text{if } x' \sqsubseteq x \;\&\; x' \text{ is +-maximal} \;\&\; x \text{ is −-maximal}, \quad \text{then } \; x \in W \implies x' \in W\,. \tag{Cwins}$$

Lemma 7.10. *Let (A, W) be a game with winning conditions. The copy-cat strategy $\gamma_A : \mathbb{C}\mathbb{C}_A \to A^\perp\|A$ is winning iff (A, W) satisfies* **(Cwins)**.

Proof. By Lemma **??**,

$$z \in \mathcal{C}^\infty(\mathbb{C}\mathbb{C}_A) \;\text{ iff }\; z = \{1\} \times \overline{x} \cup \{2\} \times x' \;\text{ with }\; x' \sqsubseteq_A x\,,$$

for $x, x' \in \mathcal{C}^\infty(A)$. In this situation z is +-maximal iff both x is −-maximal and x' is +-maximal. Thus **(Cwins)** expresses precisely that copy-cat is winning. □

A robust sufficient condition on an event structure with polarity A which ensures that copy-cat is a winning strategy for all choices of winning conditions is that A is race-free

$$\forall x \in \mathcal{C}(A).\ x \overset{a}{\longrightarrow\!\!\!\!\subset} \ \&\ x \overset{a'}{\longrightarrow\!\!\!\!\subset} \ \&\ pol(a) = + \ \&\ pol(a') = - \implies x \cup \{a, a'\} \in \mathcal{C}(A)\,. \qquad (\ddagger)$$

This property, which says immediate conflict respects polarity, is seen earlier in Lemma 5.3 (characteriziing those A for which copy-cat is deterministic).

Proposition 7.11. *Let A be an event structure with polarity. Copy-cat is a winning strategy for all games (A, W) with winning conditions W iff A is race-free.*

Proof. *"If":* Assume ($\ddagger$). Let $W \subseteq \mathcal{C}^\infty(A)$. We show **(Cwins)** holds for the game with winning conditions (A, W). For $x, x' \in \mathcal{C}^\infty(A)$, assume

$$x' \sqsubseteq x \ \&\ x' \text{ is } +\text{-maximal} \ \&\ x \text{ is } -\text{-maximal.}$$

Then, as $x' \supseteq^- x \cap x' \subseteq^+ x$, there are covering chains associated with purely +ve and −ve events from $x \cap x'$ to x and x', respectively:

$$\begin{array}{l} x \cap x' \overset{+}{\longrightarrow\!\!\!\!\subset} \cdots \overset{+}{\longrightarrow\!\!\!\!\subset} x\,, \\ x \cap x' \overset{-}{\longrightarrow\!\!\!\!\subset} \cdots \overset{-}{\longrightarrow\!\!\!\!\subset} x'\,. \end{array}$$

If one of the covering chains is of zero length then so must the other be—otherwise we contradict one or other of the maximality assumptions. On the other hand, if both are nonempty, by repeated use of ($\ddagger$) we again contradict a maximality assumption, *e.g.*

$$\begin{array}{ccccccc} y_1 & \overset{+}{\longrightarrow\!\!\!\!\subset} & x_1 \cup x_1' & \overset{+}{\longrightarrow\!\!\!\!\subset} & \cdots & \overset{+}{\longrightarrow\!\!\!\!\subset} & x \cup x_1' \\ {\scriptstyle -}\Big\uparrow & & {\scriptstyle -}\Big\uparrow & & & & {\scriptstyle -}\Big\uparrow \\ x \cap x' & \overset{+}{\longrightarrow\!\!\!\!\subset} & x_1 & \overset{+}{\longrightarrow\!\!\!\!\subset} & \cdots & \overset{+}{\longrightarrow\!\!\!\!\subset} & x \end{array}$$

shows how a repeated use of ($\ddagger$) contradicts the −-maximality of x. We conclude $x = x \cap x' = x'$ so certainly $x \in W \implies x' \in W$, as required to fulfil **(Cwins)**.

"Only if": Suppose A failed ($\ddagger$), *i.e.* $x \overset{a}{\longrightarrow\!\!\!\!\subset} x_1$ & $x \overset{a'}{\longrightarrow\!\!\!\!\subset} x_2$ with $x_1 \not\uparrow x_2$ and $pol_A(a) = +$ and $pol(a') = -$ within the finite configurations of A. The set $\{1\} \times \overline{x}_1 \cup \{2\} \times x_2$ is certainly a finite configuration of $A^\perp \| A$ and is easily checked to also be a configuration of $\mathbb{C}_A$. Define winning conditions by

$$W = \{x \in \mathcal{C}^\infty(A) \mid a \in x\}\,.$$

Let $z \in \mathcal{C}^\infty(\mathbb{C}_A)$ be a +-maximal extension of $\{1\} \times \overline{x}_1 \cup \{2\} \times x_2$ (the maximal extension exists by Zorn's Lemma). Take $z_1 = \{a \mid (1, a) \in z\}$ and $z_2 = \{a \mid (2, a) \in z\}$. Then $\overline{z}_1 \supseteq x_1$ and $z_2 \supseteq x_2$. As $a \in \overline{z}_1$ we obtain $\overline{z}_1 \in W$, whereas $z_2 \notin W$ because z_2 extends y which is inconsistent with a. Hence copy-cat is not winning in $(A, W)^\perp \| (A, W)$. $\square$

We can now refine the bicategory of strategies **Games** to the bicategory **WGames** with objects games with winning conditions $G, H, \cdots$ satisfying (**Cwins**) and arrows winning strategies $G \twoheadrightarrow H$; 2-cells, their vertical and horizontal composition is as before. Its restriction to deterministic strategies yields a bicategory **WDGames** equivalent to a simpler order-enriched category.

7.4 Total strategies

As an application of winning conditions we apply them to pick out a subcategory of "total strategies," informally strategies in which Player can always answer a move of Opponent.[2]

We restrict attention to 'simple games' (games and strategies are alternating and begin with opponent moves—see Section 6.2.4). Here a strategy is *total* if all its finite maximal sequences are even, so ending in a +ve move, *i.e.* a move of Player. In general, the composition of total strategies need not be total—see the Exercise below. However, as we will see, we can pick out a subcategory of 'simple games' with suitable winning conditions. Within this full subcategory of games with winning conditions winning strategies will be total and moreover compose.

Exercise 7.12. Exhibit two total strategies whose composition is not total. □

As objects of the subcategory we choose simple games with winning strategies,

$$(A, W_A)$$

where A is a simple game and W_A is a subset of possibly infinite sequences $s_1 s_2 \cdots$ satisfying

$$W_A \cap \text{Finite}(A) = \text{Even}(A) \qquad \textbf{(Tot)}$$

i.e. the finite sequences in W_A are precisely those of even length. Note that winning strategies in such a game will be total. (Below we use 'sequence' to mean allowable finite or infinite sequences of the appropriate simple game.)

The function space $(A, W_A) \multimap (B, W_B)$, given as $(A, W_A)^\perp \| (B, W_B)$, has winning conditions W such that

$$s \in W \text{ iff } s_A \in W_A \implies s_B \in W_B \, .$$

We use s_A for the projection of the sequence s to its subsequence in A and, similarly, s_B for its projection to B.

Lemma 7.13. *For s a finite sequence of $A^\perp \| B$, s is even iff s_A is odd or s_B is even.*

Proof. By parity, considering the final move of the sequence.
"Only if": Assume s is even, *i.e.* its final event is +ve. If s ends in B, s_B ends in + so is even. If s ends in A, s_A ends in − so is odd.

[2] This section is inspired by [23, 13].

"If": Assume s_A is odd or s_B is even. Suppose, to obtain a contradiction, that s is not even, *i.e.* s is odd so ends in $-$. If s ends in B, s_B ends in $-$ so is odd and consequently s_A even (as the length of s is the sum of the lengths of s_A and s_B). Similarly, if s ends in A, s_A ends in $+$ so s_A is even and s_B is odd. Either case contradicts the initial assumption. Hence s is even. □

It follows that W, the winning conditions of the function space, satisfies (**Tot**): Let s be a finite sequence of a strategy in $A^\perp \| B$. Then, for a finite sequence s,

$$\begin{aligned} s \in W &\text{ iff } s_A \in W_A \implies s_B \in W_B \\ &\text{ iff } s_A \notin W_A \text{ or } s_B \in W_B \\ &\text{ iff } s_A \text{ is odd or } s_B \text{ is even} \\ &\text{ iff } s \text{ is even.} \end{aligned}$$

All maps in the subcategory (which are winning strategies in its function spaces $(A, W_A) \multimap (B, W_B)$) compose (because winning strategies do) and are total (because winning conditions of its function spaces satisfy (**Tot**)).

7.5 On determined games

A game with winning conditions G is said to be *determined* when either Player or Opponent has a winning strategy, *i.e.* either there is a winning strategy in G or in $G^\perp$.[3] Not all games are determined. Neither the game G consisting of one player move $\oplus$ and one opponent move $\ominus$ inconsistent with each other, with $\{\{\oplus\}\}$ as winning conditions, nor the game $G^\perp$ have a winning strategy.

Notation 7.14. *Let $\sigma : S \to A$ be a strategy. We say $y \in \mathcal{C}^\infty(A)$ is* σ-reachable *iff $y = \sigma x$ for some $x \in \mathcal{C}^\infty(S)$. Let $y' \subseteq y$ in $\mathcal{C}^\infty(A)$. Say y' is* $-$-maximal in y *iff $y \overset{-}{\longrightarrow\!\!\subset} y''$ implies $y'' \notin y$. Similarly, say y' is* $+$-maximal in y *iff $y \overset{+}{\longrightarrow\!\!\subset} y''$ implies $y'' \notin y$.*

Lemma 7.15. *Let (A, W) be a game with winning conditions. Let $y \in \mathcal{C}(A)$. Suppose*

$$\begin{aligned} &\forall y' \in \mathcal{C}(A). \\ &y' \subseteq y \ \&\ y' \textit{ is } -\textit{-maximal in } y \ \&\ \textit{not } +\textit{-maximal in } y \\ &\implies \\ &\{y'' \in \mathcal{C}(A) \mid y' \subseteq^+ y'' \ \&\ (y'' \setminus y') \cap y = \varnothing\} \cap W = \varnothing\,. \end{aligned}$$

Then y is σ-reachable in all winning strategies σ.

Proof. Assume the property above of $y \in \mathcal{C}(A)$. Suppose, to obtain a contradiction, that y is not σ-reachable in a winning strategy $\sigma : S \to A$.

As y is finite, there is a maximal $y' \in \mathcal{C}(A)$ with $y' \subsetneq y$ and y' σ-reachable.

[3] This section is based on work with Julian Gutierrez.

By receptivity of σ, the configuration y' is $-$-maximal in y. As $y' \subsetneq y$, we must therefore have $y' \stackrel{+}{\longrightarrow\!\!\!\!\subset} y_0 \subseteq y$ in $\mathcal{C}(A)$, *i.e.* y' is not $+$-maximal in y. From the property assumed of y we deduce

$$y' \notin W \quad \& \quad \forall y'' \in W.\ y' \subseteq^+ y'' \implies (y'' \setminus y') \cap y \neq \varnothing\,.$$

As y' is σ-reachable, $y' = \sigma x'$ for some $x' \in \mathcal{C}(S)$. As σ is winning, there is $+$-maximal extension $x' \subseteq^+ x''$ in $\mathcal{C}(S)$ such that $\sigma x'' \in W$. Hence

$$(\sigma x'' \setminus y') \cap y \neq \varnothing\,.$$

Taking a $\leq_A$-minimal event a_1 in the above set we obtain

$$y' \stackrel{a_1}{\longrightarrow\!\!\!\!\subset} y_1 \subseteq^+ \sigma x''\,.$$

By Proposition 4.20, $y_1 = \sigma x_1$ for some $x_1 \in \mathcal{C}(S)$ with $x' \stackrel{+}{\longrightarrow\!\!\!\!\subset} x_1 \subseteq x''$. But this contradicts the choice of y' as the maximal reachable sub-configuration of y. Hence the original assumption that y is not σ-reachable must be false. □

Recall the property of race-freedomof an event structure with polarity A, first seen in Lemma 5.3, though here rephrased a little:

$$\forall y, y_1, y_2 \in \mathcal{C}(A).\ y \stackrel{-}{\longrightarrow\!\!\!\!\subset} y_1 \ \&\ y \stackrel{+}{\longrightarrow\!\!\!\!\subset} y_2 \implies y_1 \uparrow y_2\,. \qquad (\ddagger)$$

Corollary 7.16. *If A, an event structure with polarity, fails to be race-free, then there are winning conditions W, for which the game (A, W) is not determined.*

Proof. Suppose ($\ddagger$) failed, that $y \stackrel{-}{\longrightarrow\!\!\!\!\subset} y_1$ and $y \stackrel{+}{\longrightarrow\!\!\!\!\subset} y_2$ and $y_1 \not\uparrow y_2$ in $\mathcal{C}(A)$. Assign configurations $\mathcal{C}^\infty(A)$ to winning conditions W or its complement as follows:

(i) for y'' with $y_1 \subseteq^+ y''$, assign $y'' \notin W$;

(ii) for y'' with $y_2 \subseteq^- y''$, assign $y'' \in W$;

(iii) for y'' with $y' \subseteq^+ y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' $-$-maximal and not $+$-maximal in y, assign $y'' \notin W$;

(iv) for y'' with $y' \subseteq^- y''$ and $(y'' \setminus y') \cap y = \varnothing$, for some sub-configuration y' of y with y' $+$-maximal and not $-$-maximal in y, assign $y'' \in W$;

(v) assign arbitrarily in all other cases.

We should check the assignment is well-defined, that we do not assign a configuration both to W and its complement.

Clearly the first two cases (i) and (ii) are disjoint as $y_1 \not\uparrow y_2$.

The two cases (iii) and (iv) are also disjoint. Suppose otherwise, that both (iii) and (iv) hold for y'', *viz.*

$$\begin{aligned} &y_1' \subseteq^+ y'' \ \&\ (y'' \setminus y_1') \cap y = \varnothing\ \& \\ &\qquad y_1' \text{ is } -\text{-maximal} \ \&\ \text{not } +\text{-maximal in } y\,, \text{ and} \\ &y_2' \subseteq^- y'' \ \&\ (y'' \setminus y_2') \cap y = \varnothing\ \& \\ &\qquad y_2' \text{ is } +\text{-maximal} \ \&\ \text{not } -\text{-maximal in } y\,. \end{aligned}$$

As

$$y'_1 \subseteq^+ y'' \supseteq^- y'_2$$

we deduce ${y'_2}^- \subseteq y'_1$, *i.e.* all the −ve events of y'_2 are in y'_1. Now let $a \in {y'_2}^+$. Then $a \in y$ as $y'_2 \subseteq y$. Therefore $a \notin y'' \setminus y'_1$, by assumption. But $a \in y''$ as $y'_2 \subseteq^- y''$, so $a \in y'_1$. We conclude $y'_2 \subseteq y'_1$. A similar dual argument shows $y'_1 \subseteq y'_2$. Thus $y'_1 = y'_2$. But this implies that y'_1 is both −-maximal and not −maximal in y —a contradiction.

Suppose both the conditions (i) and (iv) are met by y''. From (vi), as y' is +-maximal & not −-maximal in y,

$$y' \overset{a}{\longrightarrow\!\subset} y_0 \subseteq y\,,$$

for some event a with $pol_A(a) = -$ and $y_0 \in \mathcal{C}^\infty(A)$. From (i), $y \subseteq y''$, so

$$y' \overset{a}{\longrightarrow\!\subset} y_0 \subseteq y''\,.$$

Therefore

$$a \in y'' \setminus y' \;\&\; a \in y\,,$$

which contradicts (iv). Similarly the cases (ii) and (iii) are disjoint.

We conclude that the assignment of winning conditions is well-defined.

Then y is reachable for both winning strategies in (A, W) and winning strategies in $(A, W)^\perp$. Suppose σ is a winning strategy σ in (A, W). By (iii) and Lemma 7.15, y is σ-reachable. From receptivity y_1 is σ-reachable, say $y_1 = \sigma x_1$ for some $x_1 \in \mathcal{C}(S)$. There is a +-maximal extension x'_1 of x_1 in $\mathcal{C}^\infty(S)$. By (i), $\sigma x'_1$ cannot be a winning configuration. Hence there can be no winning strategy in (A, W). In a dual fashion, there can be no winning strategy in $(A, W)^\perp$. □

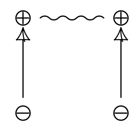
It is tempting to believe that a nondeterministic winning strategy always has a winning (weakly-)deterministic sub-strategy. However, this is not so, as the following examples show.

Example 7.17. *A winning strategy need not have a winning deterministic sub-strategy. Consider the game (A, W) where A consists of two inconsistent events $\ominus$ and $\oplus$, of the indicated polarity, and $W = \{\{\ominus\}, \{\oplus\}\}$. Consider the strategy σ in A given by the identity map $\mathrm{id}_A : a \to A$. Then σ is a nondeterministic winning strategy—all +-maximal configurations in A are winning. However any sub-strategy must include $\ominus$ by receptivity and cannot include $\oplus$ if it is to be deterministic, wherepon it has $\varnothing$ as a +-maximal configuration which is not winning.*

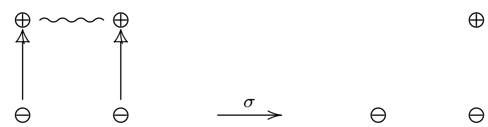
Example 7.18. *Observe that the strategy σ of Example 7.17 is already weakly-deterministic—cf. Corollary* **??**. *A winning strategy need not have a winning* weakly-*deterministic sub-strategy. Consider the game (A, W) where A consists of two −ve events $1, 2$ and one +ve event 3 all consistent with each other and*

$$W = \{\varnothing, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Let S be the event structure



and $\sigma : S \to A$ the only possible total map of event structures with polarity:



Then σ is a winning strategy for which there is no weakly-deterministic substrategy.

7.6 Determinacy for well-founded games

Definition 7.19. A game A is *well-founded* if every configuration in $\mathcal{C}^\infty(A)$ is finite.

It is shown that any well-founded, race-free concurrent game is determined.

7.6.1 Preliminaries

Proposition 7.20. *Let $\mathcal{Q}$ be a family of finite partial orders closed under rigid inclusions, i.e. if $q \in \mathcal{Q}$ and $q' \hookrightarrow q$ is a rigid inclusion (regarded as a map of event structures) then $q' \in \mathcal{Q}$. The family $\mathcal{Q}$ determines an event structure $(P, \leq, \mathrm{Con})$ as follows:*

- *the events P are the prime partial orders in $\mathcal{Q}$, i.e. those finite partial orders in $\mathcal{Q}$ with a top element;*
- *the causal dependency relation $p' \leq p$ holds precisely when there is a rigid inclusion from $p' \hookrightarrow p$;*
- *a finite subset $X \subseteq P$ is consistent, $X \in \mathrm{Con}$, iff there is $q \in \mathcal{Q}$ and rigid inclusions $p \hookrightarrow q$ for all $p \in X$. If $x \in \mathcal{C}(P)$ then $\bigcup x$, the union of the partial orders in x, is in $\mathcal{Q}$. The function $x \mapsto \bigcup x$ is an order-isomorphism from $\mathcal{C}(P)$, ordered by inclusion, to $\mathcal{Q}$, ordered by rigid inclusions.*

Call a family of finite partial orders closed under rigid inclusions a *rigid family*. Observe:

Proposition 7.21. *Any stable family $\mathcal{F}$ determines a rigid family: its configurations x possess a partial order $\leq_x$ such that whenever $x \subseteq y$ in $\mathcal{F}$ there is a rigid inclusion $(x, \leq_x) \hookrightarrow (y, \leq_y)$ between the corresponding partial orders.*

Notation 7.22. *We shall use $\mathrm{Pr}(\mathcal{Q})$ for the construction described in Proposition 7.20. The construction extends that on stable families with the same name.*

Lemma 7.23. *Let $\sigma : S \to A$ be a strategy. Letting $x, y \in \mathcal{C}(S)$,*

$$x^+ \subseteq y^+ \ \&\ \sigma x \subseteq \sigma y \implies x \subseteq y\,.$$

Proof. The proof relies on Proposition 4.20, characterising strategies. We first prove two special cases of the lemma.
Special case $\sigma x \subseteq^- \sigma y$. By assumption $x^+ \subseteq y^+$. Supposing $s \in y^+ \setminus x^+$, via the injectivity of σ on y, we obtain $\sigma y \setminus \sigma x$ contains $\sigma(s)$ a +ve event—a contradiction. Hence $x^+ = y^+$.

From Proposition 4.20(ii), as $\sigma x \subseteq^- \sigma y$, we obtain (a unique) $x' \in \mathcal{C}(S)$ such that $x \subseteq x'$ and $\sigma x' = \sigma y$:

$$\begin{array}{ccc} x & \subseteq & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma x & \subseteq^- & \sigma y\,. \end{array}$$

Now $[x^+] \subseteq^- x$, from which

$$\begin{array}{ccc} [x^+] & \subseteq & x \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma[x^+] & \subseteq^- & \sigma x\,. \end{array}$$

Combining the two diagrams:

$$\begin{array}{ccc} [x^+] & \subseteq & x' \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma[x^+] & \subseteq^- & \sigma y\,. \end{array}$$

As $[y^+] \subseteq^- y$,

$$\begin{array}{ccc} [y^+] & \subseteq & y \\ \sigma \downarrow & & \downarrow \sigma \\ \sigma[y^+] & \subseteq^- & \sigma y\,. \end{array}$$

where, by Proposition 4.20(ii), y is the unique such configuration of S. But $y^+ = x^+$ so this same property is shared by x'. Hence $x' = y$ and $x \subseteq y$.

Thus

$$x^+ \subseteq y^+ \ \&\ \sigma x \subseteq^- \sigma y \implies x \subseteq y\,. \tag{1}$$

Note that, in particular,

$$x^+ = y^+ \ \&\ \sigma x = \sigma y \implies x = y\,. \tag{2}$$

Special case $\sigma x \subseteq^{+} \sigma y$. By Proposition 4.20(i), there is (a unique) $y_1 \in \mathcal{C}(S)$ with $y_1 \subseteq y$ such that $\sigma y_1 = \sigma x$:

$$\begin{array}{ccc} y_1 & \overset{\subseteq}{\cdots\cdots} & y \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma} \\ \sigma x & \subseteq^{+} & \sigma y\,, \end{array}$$

Now $x^{+}, y_1^{+} \subseteq y$ and $\sigma\, x^{+} = (\sigma x)^{+} = \sigma\, y_1^{+}$. So by the local injectivity of σ we obtain $x^{+} = y_1^{+}$. By (2) above, $x = y_1$, whence $x \subseteq y$. Thus

$$x^{+} \subseteq y^{+} \;\&\; \sigma x \subseteq^{+} \sigma y \implies x \subseteq y\,. \tag{3}$$

Any inclusion $\sigma x \subseteq \sigma y$ can be built as a composition of inclusions $\subseteq^{-}$ and $\subseteq^{+}$, so the lemma follows from the special cases (1) and (3). □

Lemma 7.24. *Let $\sigma : S \to A$ be a strategy for which no +ve event of S appears as a −ve event in A. Defining*

$$\mathcal{F}_\sigma \;=_{\mathrm{def}}\; \{x^{+} \cup (\sigma x)^{-} \mid x \in \mathcal{C}(S)\}$$

yields a stable family for which

$$\alpha_\sigma(s) = \begin{cases} s & \text{if } s \text{ is +ve,} \\ \sigma(s) & \text{if } s \text{ is −ve.} \end{cases}$$

is a map of stable families $\alpha_\sigma : \mathcal{C}(S) \to \mathcal{F}_\sigma$ which induces an order-isomorphism

$$(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$$

taking $x \in \mathcal{C}(S)$ to $\alpha_\sigma\, x = x^{+} \cup (\sigma x)^{-}$. Defining

$$f_\sigma(e) = \begin{cases} \sigma(e) & \text{if } e \text{ is +ve,} \\ e & \text{if } e \text{ is −ve} \end{cases}$$

on events e of $\mathcal{F}_\sigma$ yields a map of stable families $f_\sigma : \mathcal{F}_y \to \mathcal{C}(A)$ such that

$$\begin{array}{ccc} \mathcal{C}(S) & \xrightarrow{\alpha_\sigma} & \mathcal{F}_\sigma \\ & {\scriptstyle\sigma}\searrow & \big\downarrow{\scriptstyle f_\sigma} \\ & & \mathcal{C}(A) \end{array}$$

commutes.

Proof. A configuration $x \in \mathcal{C}(S)$ has direct image

$$\alpha_\sigma x = x^{+} \cup (\sigma x)^{-}$$

under the function α_σ. Direct image under α_σ is clearly surjective and preserves inclusions, and by Lemma 7.23 yields an order-isomorphism $(\mathcal{C}(S), \subseteq) \cong (\mathcal{F}_\sigma, \subseteq)$: if $\alpha_\sigma x \subseteq \alpha_\sigma y$, for $x, y \in \mathcal{C}(S)$, then $x^+ \subseteq y^+$ and $(\sigma x)^- \subseteq (\sigma y)^-$ by the disjointness of S^+ and A, whence $\sigma x \subseteq \sigma y$ so $x \subseteq y$.

It is now routine to check that $\mathcal{F}_\sigma$ is a stable family and α_σ is a map of stable families. For instance to show the stability property required of $\mathcal{F}_\sigma$, assume $\alpha_\sigma x, \alpha_\sigma y \subseteq \alpha_\sigma z$. Then $x, y \subseteq z$ so $\sigma\, x \cap y = (\sigma x) \cap (\sigma y)$ as σ is a map of event structures, and consequently $(\sigma\, x \cap y)^- = (\sigma x)^- \cap (\sigma y)^-$. Now reason

$$\begin{aligned}
(\alpha_\sigma x) \cap (\alpha_\sigma y) =& (x^+ \cup (\sigma x)^-) \cap (y^+ \cup (\sigma y)^-) \\
=& (x^+ \cap y^+) \cup ((\sigma x)^- \cap (\sigma y)^-) \\
& \text{—by distributivity with the disjointness of } S^+ \text{ and } A\,, \\
=& (x \cap y)^+ \cup (\sigma\, x \cap y)^- \\
=& (\alpha_\sigma\, x \cap y) \in \mathcal{F}_\sigma\,.
\end{aligned}$$

From the definitions of α_σ and f_σ it is clear that $f_\sigma \alpha_\sigma(s) = \sigma(s)$ for all events of S. Any configuration of $\mathcal{F}_\sigma$ is sent under f_σ to a configuration in $\mathcal{C}(A)$ in a locally injective fashion, making f_σ a map of stable families; this follows from the matching properties of σ. □

When we "glue" strategies together it can be helpful to assume that all the initial −ve moves of the strategies are exactly the same:

Lemma 7.25. *Let $\sigma : S \to A$ be a strategy. Then $\sigma \cong \sigma'$, a strategy $\sigma' : S' \to A$ for which*

$$\forall s' \in S'.\ pol_{S'}[s']_S = \{-\} \implies s' = [\sigma(s')]_A\,.$$

Proof. Without loss of generality we may assume no +ve event of S appears as a −ve event in A. Take $f_\sigma : \mathcal{F}_\sigma \to \mathcal{C}(A)$ given by Lemma 7.25 and construct σ' as the composite map

$$\mathrm{Pr}(\mathcal{F}_\sigma) \xrightarrow{\mathrm{Pr}(\sigma)} \mathrm{Pr}(\mathcal{C}(A)) \overset{max}{\cong} A$$

—recall *max* takes a prime $[a]_A$ to a, where $a \in A$. □

7.7 Determinacy proof

Definition 7.26. *Let A be an event structure with polarity. Let $W \subseteq \mathcal{C}^\infty(A)$. Let $y \in \mathcal{C}^\infty(A)$. Define A/y to be the event structure with polarity comprising events*

$$\{a \in A \setminus y \mid y \cup [a]_A \in \mathcal{C}^\infty(A)\}\,,$$

also called A/y, with consistency relation

$$X \in \mathrm{Con}_{A/y} \text{ iff } X \subseteq_{\mathrm{fin}} A/y \ \&\ y \cup [X]_A \in \mathcal{C}^\infty(A)\,,$$

and causal dependency the restriction of that on A. Define $W/y \subseteq \mathcal{C}^\infty(A/y)$ *by*

$$z \in W/y \text{ iff } z \in \mathcal{C}^\infty(A/y) \ \&\ y \cup z \in W\,.$$

Finally, define $(A,W)/y =_{\rm def} (A/y, W/y)$.

Proposition 7.27. *Let A be an event structure with polarity and* $y \in \mathcal{C}^\infty(A)$. *Then,*

$$z \in \mathcal{C}^\infty(A/y) \text{ iff } z \subseteq A/y \ \&\ y \cup z \in \mathcal{C}^\infty(A)\,.$$

Assume A is a *well-founded* event structure with polarity with winning conditions $W \subseteq \mathcal{C}(A)$. Assume the property (‡) of A, *i.e.* that A is race-free:

$$\forall y, y_1, y_2 \in \mathcal{C}(A).\ y \overset{-}{\longrightarrow\!\!\subset} y_1 \ \&\ y \overset{+}{\longrightarrow\!\!\subset} y_2 \implies y_1 \uparrow y_2\,. \qquad (‡)$$

Observe that by repeated use of (‡), if $x, y \in \mathcal{C}(A)$ with $x \cap y \subseteq^+ x$ and $x \cap y \subseteq^- y$, then $x \cup y \in \mathcal{C}(A)$.

We show that the game (A,W) is determined. Assuming Player has no winning strategy we build a winning (counter) strategy for Opponent based on the following lemma.

Lemma 7.28. *Assume game A is well-founded and race-free. Let* $W \subseteq \mathcal{C}(A)$. *Assume* (A,W) *has no winning strategy (for Player). Then,*

$$\begin{array}{l} \forall x \in \mathcal{C}(A).\ \varnothing \subseteq^+ x \ \&\ x \in W \\ \implies \\ \exists y \in \mathcal{C}(A).\ x \subseteq^- y \ \&\ y \notin W \ \&\ (A,W)/y \textit{ has no winning strategy.} \end{array}$$

Proof. Suppose otherwise, that under the assumption that (A,W) has no winning strategy, there is some $x \in \mathcal{C}(A)$ such that

$$\begin{array}{l} \varnothing \subseteq^+ x \ \&\ x \in W \\ \& \\ \forall y \in \mathcal{C}(A).\ x \subseteq^- y \ \&\ y \notin W \implies (A,W)/y \text{ has a winning strategy.} \end{array}$$

We shall establish a contradiction by constructing a winning strategy for Player.

For each $y \in \mathcal{C}(A)$ with $x \subseteq^- y$ and $y \notin W$, choose a winning strategy

$$\sigma_y : S_y \to A/y\,.$$

By Lemma 7.25, we can replace σ_y by a stable family $\mathcal{F}_y$ with all −ve events in A and a map of stable families $f_y : \mathcal{F}_y \to \mathcal{C}(A)$. It is easy to arrange that, within the collection of all such stable families, $\mathcal{F}_{y_1}$ and $\mathcal{F}_{y_2}$ are disjoint on +ve events whenever y_1 and y_2 are distinct. We build a putative stable family as

$$\begin{aligned} \mathcal{F} =_{\rm def} & \{y \in \mathcal{C}(A) \mid pol_A(y \setminus x) \subseteq \{-\}\} \cup \\ & \{y \cup v \mid y \in \mathcal{C}(A) \ \&\ pol_A(y \setminus x) \subseteq \{-\} \ \&\ x \cup y \notin W\ \& \\ & \qquad v \in \mathcal{F}_{x \cup y} \ \&\ + \in pol\, v \ \&\ y \cup f_{x \cup y} v \in \mathcal{C}(A)\}\,. \end{aligned}$$

[Note, in the second set-component, that $x \cup y$ is a configuration by (‡).]
We assign events of $\mathcal{F}$ the same polarities they have in A and the families $\mathcal{F}_y$.

We check that $\mathcal{F}$ is indeed a stable family.

Clearly $\varnothing \in \mathcal{F}$. Assuming $z_1, z_2 \subseteq z$ in $\mathcal{F}$, we require $z_1 \cup z_2$, $z_1 \cap z_2 \in \mathcal{F}$.

It is easily seen that if both z_1 and z_2 belong to the first set-component, so do their union and intersection. Suppose otherwise, without loss of generality, that z_2 belongs to the second set-component. Then, necessarily, z is in the second set-component of $\mathcal{F}$ and has the form $z = y \cup v$ described there.

Consider the case where $z_1 = y_1 \cup v_1$ and $z_2 = y_2 \cup v_2$, both belonging to the second set-component of $\mathcal{F}$. Then

$$x \cup y_1 = x \cup y_2 = x \cup y\,,$$

from the assumption that families $\mathcal{F}_y$ are disjoint on +ve events for distinct y, and

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x \cup y}\,.$$

It follows that $x \cup (y_1 \cup y_2) = x \cup y \notin W$ and $v_1 \cup v_2 \in \mathcal{F}_{x \cup y} = \mathcal{F}_{x \cup (y_1 \cup y_2)}$. As $z_1, z_z \subseteq z$,

$$(y_1 \cup f_{x \cup y} v_1),\ (y_2 \cup f_{x \cup y} v_2) \subseteq (y \cup f_{x \cup y} v)$$

so

$$(y_1 \cup y_2) \cup f_{x \cup y}(v_1 \cup v_2) = (y_1 \cup f_{x \cup y} v_1) \cup (y_2 \cup f_{x \cup y} v_2) \in \mathcal{C}(A)\,.$$

This ensures $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$. Similarly, $x \cup (y_1 \cap y_2) = (x \cup y_1) \cap (x \cup y_2) = x \cup y \notin W$ and $v_1 \cap v_2 \in \mathcal{F}_{x \cup y} = \mathcal{F}_{x \cup (y_1 \cap y_2)}$. Checking

$$(y_1 \cap y_2) \cup f_{x \cup y}(v_1 \cap v_2) = (y_1 \cup f_{x \cup y} v_1) \cap (y_2 \cup f_{x \cup y} v_2) \in \mathcal{C}(A)$$

ensures $z_1 \cap z_2 = (y_1 \cap y_2) \cup (v_1 \cap v_2) \in \mathcal{F}$.

Consider the case where $z_1 \in \mathcal{C}(A)$ belongs to the first and $z_2 = y_2 \cup v_2$ to the second set-component of $\mathcal{F}$. As $z_1 \subseteq y \cup v$ it has the form $z_1 = y_1 \cup v_1$ where $y_1 \in \mathcal{C}(A)$ with $y_1 \subseteq y$ and $v_1 \in \mathcal{F}_{x \cup y}$ with $v_1 \subseteq v$; all the events of $v_1 = z_1 \setminus (x \cup y)$ have −ve polarity which ensures $v_1 \in \mathcal{F}_{x \cup y}$ by the receptivity of σ_y. Because v_2 and v have +ve events in common,

$$x \cup y_2 = x \cup y\,,$$

while clearly

$$v_1, v_2 \subseteq v \text{ in } \mathcal{F}_{x \cup y}\,.$$

We deduce $x \cup (y_1 \cup y_2) = x \cup y \notin W$ and $v_1 \cup v_2 \in \mathcal{F}_{x \cup y} = \mathcal{F}_{x \cup (y_1 \cup y_2)}$ whence $z_1 \cup z_2 = (y_1 \cup y_2) \cup (v_1 \cup v_2) \in \mathcal{F}$ after an easy check that $(y_1 \cup y_2) \cup f_{x \cup y}(v_1 \cup v_2) \in \mathcal{C}(A)$. We have $y_2 \cup f_{x \cup y} v_2 \in \mathcal{C}(A)$. But $f_{x \cup y}$ is constant on −ve events so

$$z_1 \cap z_2 = z_1 \cap (y_2 \cup v_2) = z_1 \cap (y_2 \cup f_{x \cup y} v_2) \in \mathcal{C}(A)\,,$$

and $z_1 \cap z_2$ belongs to the first set-component of $\mathcal{F}$.

A routine check establishes that $\mathcal{F}$ is coincidence-free, and uses that each family $\mathcal{F}_y$ is coincidence-free when considering configurations of the second set-component.

Having established that $\mathcal{F}$ is a stable family, we define a total map of stable families

$$f : \mathcal{F} \to \mathcal{C}(A)$$

by taking

$$f(e) = \begin{cases} e & \text{if } e \in x \text{ or } e \text{ is } -\text{ve}, \\ f_y(e) & \text{if } e \text{ is a } +\text{ve event of } \mathcal{F}_y. \end{cases}$$

Defining σ to be the composite map of stable families

$$\mathcal{C}(\Pr(\mathcal{F})) \xrightarrow{max} \mathcal{F} \xrightarrow{f} \mathcal{C}(A)$$

we also obtain a map of event structures

$$\sigma : \Pr(\mathcal{F}) \to A$$

as the embedding of event structures in stable families is full and faithful. Ascribe to events p of $\Pr(\mathcal{F})$ the same polarities as events $max(p)$ of $\mathcal{F}$. Clearly σ preserves polarities as f does, so σ is a total map of event structures with polarity. In fact, σ is a winning strategy for (A, W).

To show receptivity of σ it suffices to show for all $z \in \mathcal{F}$ that $fz \overset{-}{\longrightarrow\!\!\subset} y'$ in $\mathcal{C}(A)$ implies $z \overset{z}{\longrightarrow\!\!\subset} {}'$ with $\sigma z' = z$ for some unique $z' \in \mathcal{F}$. If z belongs to the first set-component of $\mathcal{F}$ this is obvious—take $z' = y'$. Otherwise z belongs to the second set-component, and takes the form $y \cup v$, when receptivity follows from the receptivity of $\sigma_{x \cup y}$. No extra causal dependencies, over those of A, are introduced into y in the first set-component of $\mathcal{F}$. Considering $y \cup v$ in the second set-component of $\mathcal{F}$, the only extra causal dependencies introduced in $y \cup v$, above those inherited from its image $y \cup f_{x \cup y} v$ in A, are from v in $\mathcal{F}_{x \cup y}$ and those making a +ve event of v in $y \cup v$ depend on −ve events $y \setminus x$. For these reasons σ is also innocent, and a strategy in A.

To show σ is a winning strategy for (A, W) it suffices to show that $fz \in W$ for every +-maximal configuration $z \in \mathcal{F}$. Let z be a +-maximal configuration of $\mathcal{F}$.

Suppose that z belongs to the first set-component of $\mathcal{F}$ and, to obtain a contradiction, that $fz \notin W$. Then $z = fz \in \mathcal{C}(A)$ and $pol\, z \setminus x \subseteq \{-\}$. By race-freedom, $x \uparrow y$, so $x \subseteq z$ from the +-maximality of z. As $x \subseteq^- z$ and $z \notin W$ the strategy σ_z is winning in $(A, W)/z$. Because z is +-maximal in $\mathcal{F}$ we must have $\varnothing$ is +-maximal in $\mathcal{F}_z$. It follows that $\varnothing \in W/z$, *i.e.* $z \in W$—a contradiction.

Suppose that z belongs to the second set-component of $\mathcal{F}$, so that z has the form $y \cup v$ with $y \in \mathcal{C}(A)$ and $v \in \mathcal{F}_{x \cup y}$. By (‡), $x \subseteq y$, as z is +-maximal in $\mathcal{F}$. Hence $v \in \mathcal{F}_y$ and is necessarily +-maximal in $\mathcal{F}_y$, again from the +-maximality of z. As σ_y is winning, $f_y v \in W/y$. Therefore $fz = y \cup f_y v \in W$.

Finally, we have constructed a winning strategy σ in (A, W)—the contradiction required to establish the lemma. □

Remark. In the proof above we could instead build the strategy for Player, on which the proof by contradiction depends, out of a rigid family of finite partial orders. Recall that stable families, including configurations of event structures, are rigid families w.r.t. the order induced on configurations; finite configurations x determine finite partial orders $(x, \leq_x)$, which we call $q(x)$ in the construction below. Define

$$\begin{aligned}\mathcal{Q} =_{\text{def}} & \{q(y) \mid y \in \mathcal{C}(A) \ \&\ pol_A(y \setminus x) \subseteq \{-\}\} \cup \\ & \{q(y); q(v) \mid y \in \mathcal{C}(A) \ \&\ pol_A(y \setminus x) \subseteq \{-\} \ \&\ x \cup y \notin W \ \&\ \\ & \qquad v \in \mathcal{F}_{x\cup y} \ \&\ + \in pol\, v \ \&\ y \cup f_{x\cup y} v \in \mathcal{C}(A)\}\end{aligned}$$

where above $q(y); q(v)$ is the least partial order on $y \cup v$ in which events inherit causal dependencies from $q(v)$, from their images in $q(y \cup f_{x\cup y} v)$ and in addition have the causal dependencies $y^- \times v^+$. The family $\mathcal{Q}$ can be shown to be closed under rigid inclusions, and so a rigid family. □

Theorem 7.29. *Assume game A is well-founded, race-free and has winning conditions $W \subseteq \mathcal{C}(A)$. If (A, W) has no winning strategy for Player, then there is a winning (counter) strategy for Opponent.*

Proof. Assume (A, W) has no winning strategy for Player.

We build a winning counter-strategy for Opponent out of a rigid family of partial orders, themselves constructed from 'alternating sequences' of configurations of A.

Define an *alternating sequence* to be a sequence

$$x_1, y_1, x_2, y_2, \cdots, x_i, y_i, \cdots, x_k, y_k, x_{k+1}$$

of length $k + 1 \geq 1$ of configurations of A such that

$$\varnothing \subseteq^+ x_1 \subseteq^- y_1 \subseteq^+ x_2 \subseteq^- y_2 \subseteq^- \cdots \subseteq^+ x_i \subseteq^- y_i \subseteq^+ \cdots \subseteq^+ x_k \subseteq^- y_k \subseteq^+ x_{k+1}$$

with

$$x_i \in W \ \&\ y_i \notin W \ \&\ (A, W)/y_i \text{ has no winning strategy,}$$

when $1 \leq i \leq k$. It is important that x_{k+1}, which may be $\varnothing$, need not be in W. In particular, we allow the alternating singleton sequence x_1 comprising a single configuration of A with $\varnothing \subseteq^+ x_1$ without necessarily having $x_1 \in W$.

For each alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$ define the partial order $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ to comprise the partial order on x_{k+1} inherited from A together with additional causal dependencies given by the pairs in

$$x_i^+ \times (y_i \setminus x_i), \text{ where } 1 \leq i \leq k.$$

We define $\mathcal{Q}$ to be the rigid family comprising the set of all partial orders got from alternating sequences, closed under rigid inclusions.

Form the event structure $\Pr(\mathcal{Q})$ as described in Proposition 7.20. Assign the same polarity to an event in $\Pr(\mathcal{Q})$ as its top event in A. Recall from

Proposition 7.20 the order-isomorphism $\mathcal{C}(\mathrm{Pr}(\mathcal{Q})) \cong \mathcal{Q}$ given by $x \mapsto \bigcup x$ for $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{Q}))$. The map

$$\tau : \mathrm{Pr}(\mathcal{Q}) \to A$$

taking $p \in \mathrm{Pr}(\mathcal{Q})$ to its top event is a total map of event structures with polarity. Writing $T : \mathcal{Q} \to \mathcal{C}(A)$ for the function taking $q \in \mathcal{Q}$ to its set of underlying events, $\tau x = T(\bigcup x)$ for all $x \in \mathcal{C}(\mathrm{Pr}(\mathcal{Q}))$, *i.e.* the diagram

$$\begin{array}{ccc} \mathcal{C}(\mathrm{Pr}(\mathcal{Q})) & \cong & \mathcal{Q} \\ & \searrow^{\tau} & \downarrow T \\ & & \mathcal{C}(A) \end{array}$$

commutes. We shall reason about order-properties of τ via the function T.

We claim that τ is a winning counter-strategy, in other words a winning strategy for Opponent, in which the roles of + and − are reversed.

Because the construction of the partial orders in $\mathcal{Q}$ only introduces extra causal dependencies of −ve events on +ve events, τ is innocent (remember the reversal of polarities). To check receptivity of τ it suffices to show that for $q \in \mathcal{Q}$ assuming $T(q) \stackrel{a}{-\!\!\!\subset} z'$ in $\mathcal{C}(A)$, where $pol_A(a) = +$, there is a unique $q' \in \mathcal{Q}$ such that $q -\!\!\!\subset q'$ and $T(q') = z'$. Any such extension q' must comprise the partial order q extended by the event a. As a is +ve the events on which it immediately depends in q' will coincide with those on which a immediately depends in z', guaranteeing the uniqueness of q'. It remains to show the existence of q'.

By assumption, q rigidly embeds in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ for some alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$. In the case where q consists of purely +ve events, take $q' =_{\mathrm{def}} Q(z')$. Otherwise, consider the largest i for which $T(q) \cap (y_i \smallsetminus x_i) \neq \emptyset$. Then,

$$pol_A\, T(q) \smallsetminus y_i \subseteq \{+\}\,. \tag{1}$$

From the construction of $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ and the rigidity of the inclusion of q in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ we obtain

$$x_i^+ \subseteq T(q)\,. \tag{2}$$

From (2), $T(q) \subseteq^- T(q) \cup y_i$ and, by assumption, $T(q) \stackrel{a}{-\!\!\!\subset} z'$ with $pol_A(a) = +$. Using (‡), their union remains in $\mathcal{C}(A)$, and we can define

$$x' =_{\mathrm{def}} T(q) \cup y_i \cup \{a\} \in \mathcal{C}(A)\,.$$

Note that

$$x_1, y_1, \cdots, x_i, y_i, x'$$

is an alternating sequence because $y_i \subseteq^+ x'$ by (1) and it is built from an alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$. Restricting $Q(x_1, y_1, \cdots, x_i, y_i, x')$ to events z we obtain a partial order q' for which $q -\!\!\!\subset q'$ in $\mathcal{Q}$ and $T(q') = z$.

We now show that τ is winning for Opponent. For this it suffices to show that if $q \in \mathcal{Q}$ is $-$-maximal then $T(q) \notin W$. Assume $q \in \mathcal{Q}$ is $-$-maximal in $\mathcal{Q}$. Necessarily q embeds rigidly in $Q(x_1, y_1, \cdots, x_k, y_k, x_{k+1})$ for some alternating sequence $x_1, y_1, \cdots, x_k, y_k, x_{k+1}$.

In the case where q consists of purely +ve events

$$\varnothing \subseteq^+ T(q) \text{ in } \mathcal{C}(A)\,.$$

Suppose $T(q) \in W$. By Lemma 7.28, for some $y \in \mathcal{C}(A)$,

$$T(q) \subseteq^- y \;\&\; y \notin W\,.$$

But then there is a strict extension $q \hookrightarrow Q(T(q), y, \varnothing)$ of q by $-$ve events in $\mathcal{Q}$, and q is not $-$-maximal—a contradiction.

In the case where q has $-$ve events, we may take the largest i for which $T(q) \cap (y_i \smallsetminus x_i) \neq \varnothing$. As earlier,

$$(1)\ pol_A\, T(q) \smallsetminus y_i \subseteq \{+\} \quad \& \quad (2)\ x_i^+ \subseteq T(q)\,.$$

As q is $-$-maximal, $y_i \subseteq T(q)$, whence by (1),

$$y_i \subseteq^+ T(q)\,.$$

Suppose, to obtain a contradiction, that $T(q) \in W$. The game $(A, W)/y_i$ has no winning strategy. By Lemma 7.28, given

$$\varnothing \subseteq^+ x =_{\text{def}} T(q) \smallsetminus y_i$$

in $\mathcal{C}((A, W)/y_i)$ there is $y \in \mathcal{C}((A, W)/y_i)$ with

$$x \subseteq^- y \quad \& \quad y \notin W/y_i\,.$$

Let $x'_{i+1} =_{\text{def}} T(q)$ and $y'_{i+1} =_{\text{def}} y_i \cup y \notin W$. Then,

$$x_1, y_1, \cdots, x_i, y_i, x'_{i+1}, y'_{i+1}, \varnothing$$

is an alternating sequence which strictly extends q by $-$ve events, contradicting its $-$-maximality.

We conclude that τ is a winning strategy for Opponent. □

Corollary 7.30. *If a well-founded game A satisfies (‡) then (A, W) is determined for any winning conditions W.*

7.8 Satisfaction in the predicate calculus

The syntax for predicate calculus: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists x.\ \phi \mid \forall x.\ \phi$$

where R ranges over basic relation symbols of a fixed arity and $x, x_1, x_2, \cdots, x_k$ over variables.

A model M for the predicate calculus comprises a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M. Following Tarski we can then define by structural induction the truth of a formula of predicate logic w.r.t. an assignment of values in V_M to the variables of the formula. We write

$$\rho \vDash_M \phi$$

iff formula ϕ is true in M w.r.t. environment ρ; we take an environment to be a function from variables to values.

W.r.t. a model M and an environment ρ, we can denote a formula ϕ by $[\![\phi]\!]_M\rho$, a concurrent game with winning conditions, so that $\rho \vDash_M \phi$ iff the game $[\![\phi]\!]_M\rho$ has a winning strategy.

The denotation as a game is defined by structural induction:

$$[\![R(x_1, \cdots, x_k)]\!]_M\rho = \begin{cases} (\varnothing, \{\varnothing\}) & \text{if } \rho \vDash_M R(x_1, \cdots, x_k)\,, \\ (\varnothing, \varnothing) & \text{otherwise.} \end{cases}$$

$$[\![\phi \wedge \psi]\!]_M\rho = [\![\phi]\!]_M\rho \otimes [\![\psi]\!]_M\rho$$

$$[\![\phi \vee \psi]\!]_M\rho = [\![\phi]\!]_M\rho \parr [\![\psi]\!]_M\rho$$

$$[\![\neg\phi]\!]_M\rho = ([\![\phi]\!]_M\rho)^\perp$$

$$[\![\exists x.\ \phi]\!]_M\rho = \bigoplus_{v \in V_M} [\![\phi]\!]_M\rho[v/x]$$

$$[\![\forall x.\ \phi]\!]_M\rho = \bigominus_{v \in V_M} [\![\phi]\!]_M\rho[v/x]\,.$$

We use $\rho[v/x]$ to mean the environment ρ updated to assign value v to variable x. The game $(\varnothing, \{\varnothing\})$ the unit w.r.t. $\otimes$ is the game used to denote true and the game $(\varnothing, \{\varnothing\})$ the unit w.r.t. $\parr$ to denote false. Denotations of conjunctions and disjunctions are denoted by the operations of $\otimes$ and $\parr$ on games, while negations denote dual games. Universal and existential quantifiers denote *prefixed sums* of games, operations which we now describe.

The prefixed game $\oplus.(A, W)$ comprises the event structure with polarity $\oplus.A$ in which all the events of A are made to causally depend on a fresh +ve event $\oplus$. Its winning conditions are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. The game $\bigoplus_{v \in V}(A_v, W_v)$ has underlying event structure with polarity the sum (=coproduct) $\sum_{v \in V} \oplus.A_v$ with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. Note in particular that the empty configuration of $\bigoplus_{v \in V} G_v$ is not winning—Player must make a move in order to win. The game $\bigominus_{v \in V} G_v$ is defined dually, as $(\bigoplus_{v \in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move. More explicitly, the prefixed game $\ominus.(A, W)$ comprises the event structure with polarity $\ominus.A$ in which all the events of A are made to causally depend on the previous occurrence of an opponent event $\ominus$, with winning configurations either the empty configuration or of the form $\{\ominus\} \cup y$ where $y \in W$. Writing $G_v = (A_v, W_v)$, the underlying event structure of $\ominus_{v\in V} G_v$ is the sum $\sum_{v\in V} \ominus.A_v$ with a configuration winning iff it is empty or the image under injection of a winning configuration in a prefixed component.

It is easy to check by structural induction that:

Proposition 7.31. *For any formula ϕ the game $[\![\phi]\!]_M \rho$ is well-founded and race-free, so a determined game by the result of the last section.*

The following facts are useful for building strategies.

Proposition 7.32.

(i) If $\sigma : S \to A$ is a strategy in A and $\tau : T \to B$ is a strategy in B, then $\sigma\|\tau : S\|T \to A\|B$ is a strategy in $A\|B$.

(ii) If $\sigma : S \to T$ is a strategy in T and $\tau : T \to B$ is a strategy in B, then their composition as maps of event structures with polarity $\tau\sigma : S \to B$ is a strategy in B.

Proof. It is easy to check that the properties of receptivity and innocence are preserved by parallel composition and composition of maps. □

There are 'projection' strategies from a tensor product of games to its components:

Proposition 7.33. *Let $G = (A, W_G)$ and $H = (B, W_H)$ be race-free games with winning conditions. The map of event structures with polarity*

$$\mathrm{id}_{A^\perp}\|\gamma_B : A^\perp\|\mathbb{C}_B \to A^\perp\|B^\perp\|B$$

is a winning strategy $p_H : G\otimes H \multimap\!\!\!\!\to H$. The map of event structures with polarity

$$\mathrm{id}_{B^\perp}\|\gamma_A : B^\perp\|\mathbb{C}_A \to B^\perp\|A^\perp\|A \cong A^\perp\|B^\perp\|A$$

is a winning strategy $p_G : G \otimes H \multimap\!\!\!\!\to G$.

Proof. By Proposition 7.32, as $\mathrm{id}_{A^\perp}$ is a strategy in $A^\perp$ and γ_B is a strategy in $B^\perp\|B$ the map $p_H = \mathrm{id}_{A^\perp}\|\gamma_B$ is certainly a strategy in $A^\perp\|B^\perp\|B$.

We need to check that p_H is a winning strategy in $G \otimes H \multimap H$. Consider x, a +-maximal configuration of $A^\perp\|\mathbb{C}_B$. As B is race-free, the copy-cat strategy γ_B is winning in $H \multimap H$. Consequently if x images to a winning configuration in $G\otimes H$ on the left of $G\otimes H \multimap H$ it will image to a winning configuration in H on the right of $G \otimes H \multimap H$. (Recall a winning configuration of $G \otimes H$ is essentially the union of a winning configuration in G together with a winning configuration in H.) Consequently, x images to a winning configuration in $G \otimes H \multimap H$, as is required for p_H to be a winning strategy.

The strategy p_G is defined analogously but for the isomorphism $B^\perp\|A^\perp\|A \cong A^\perp\|B^\perp\|A$ which does not disturb its winning nature. □

The following lemma is used to build and deconstruct strategies in prefixed sums of games. The lemma concerns the more basic prefixed sums of event structures. These are built as coproducts $\sum_{i\in I} \bullet.B_i$ of event structures $\bullet.B_i$ in which an event $\bullet$ is prefixed to B_i, making all the events in B_i causally depend on $\bullet$.

Lemma 7.34. *Suppose $f : A \to \sum_{i\in I} \bullet.B_i$ is a total map of event structures, with codomain a prefixed sum. Then, A is isomorphic to an prefixed sum, $A \cong \sum_{j\in J} \bullet.A_j$, and there is a function $r : J \to I$ and total maps of event structures $f_j : A_j \to B_{r(j)}$ for which*

$$\begin{array}{ccc} \sum_{j\in J} \bullet.A_j & \cong & A \\ {\scriptstyle [\bullet.f_j]_{j\in J}} \downarrow & \swarrow {\scriptstyle f} & \\ \sum_{i\in I} \bullet.B_i & & \end{array}$$

commutes.

Proof. Let J be the subset of events of A whose images are prefix events $\bullet$ in $\sum_{i\in I} \bullet.B_i$. As f is a map of event structures any distinct pairs of events in J are inconsistent. Moreover, every event of A is $\leq_A$-above a necessarily unique event in J. It follows that the events of J are $\leq_A$-minimal with $A \cong \sum_{j\in J} \bullet.A_j$; the event structure A_j is $A/\{j\}$, that part of the event structure strictly above the event j. Each event $j \in J$ is sent to a unique prefix event $f(j)$ in $\sum_{i\in I} \bullet.B_i$. Thus f determines a function $r : J \to I$ and maps $f_j : A_j \to B_{r(i)}$ for all $j \in J$. By construction the map f is reassembled, up to isomorphism, as the unique mediating map $[\bullet.f_j]_{j\in J}$ for which

$$\begin{array}{ccccc} \bullet.A_j & \xrightarrow{in_j^A} & \sum_{j\in J} \bullet.A_j & \cong & A \\ {\scriptstyle \bullet.f_j} \downarrow & & {\scriptstyle [\bullet.f_j]_{j\in J}} \downarrow & \swarrow {\scriptstyle f} & \\ \bullet.B_{r(j)} & \xrightarrow[in_{r(j)}^B]{} & \sum_{i\in I} \bullet.B_i & & \end{array}$$

commutes for all $j \in J$. □

Lemma 7.35. *Let G, H, G_v, where $v \in V$, be race-free games with winning conditions. Then,*

(i) $G \otimes H$ has a winning strategy iff G has a winning strategy and H has a winning strategy.

(ii) $\bigoplus_{v\in V} G_v$ has a winning strategy iff G_v has a winning strategy for some $v \in V$.

(iii) $\ominus_{v\in V} G_v$ has a winning strategy iff G_v has a winning strategy for all $v \in V$.

If in addition G and H are determined, *(iv) $G \parr H$ has a winning strategy iff G has a winning strategy or H has a winning strategy.*

Proof. Throughout write $G_v = (A_v, W_v)$, where $v \in V$.

(i) *'Only if':* If $G \otimes H$ has a winning strategy $\sigma : (\varnothing, \{\varnothing\}) \twoheadrightarrow G \otimes H$, then the compositions $p_G \odot \sigma$ and $p_H \odot \sigma$ provide winning strategies in G and H, respectively. *'If':* If $G = (A, W_G)$ and $H = (B, W_H)$ have winning strategies given as maps of event structures with polarity $\sigma : S \to A$ and $\tau : T \to B$ then the map $\sigma \| \tau : S \| T \to A \| B$ is a winning strategy in $G \otimes H$.

(ii) *'Only if':* Suppose $\sigma : S \to \sum_{v \in V} \oplus.A_v$ is a winning strategy in $\bigoplus_{v \in V} G_v$. As $\varnothing$ is not winning in the game, S must be nonempty. By Lemma 7.34, S decomposes into a prefixed sum necessarily nonempty and of the form $\sum_{j \in J} \oplus.S_j$ with maps, now necessarily total maps of event structures with polarity, $\sigma_j : S_j \to A_{v(j)}$. Because σ is winning any such map will be a winning strategy in $G_{v(j)}$. *'If':* Suppose $\sigma_v : S_v \to A_v$ is a winning strategy in G_v. Prefixing we obtain $\oplus.\sigma_v : \oplus.S_v \to \oplus.A_v$, a winning strategy in $\oplus.G_v$. Composing with the winning 'injection' strategy $In_v : \oplus.G_v \twoheadrightarrow \sum_{v \in V} \oplus.G_v$ defined below we obtain a winning strategy in $\bigoplus_{v \in V} G_v$. The injection strategy is built from the injection map of event structures with polarity

$$in_v : \oplus.A_v \to \sum_{v \in V} \oplus.A_v \,.$$

as the composite map

$$In_v : \mathbb{C}_{\oplus.A_v} \xrightarrow{\gamma_{\oplus.A_v}} (\oplus.A_v)^\perp \| \oplus .A_v \xrightarrow{\mathrm{id}_{(\oplus.A_v)^\perp} \| in_v} (\oplus.A_v)^\perp \| \textstyle\sum_{v \in V} \oplus.A_v \,.$$

Proposition 7.32 is used to show In_v is a strategy. It can be seen that in_v is both receptive and innocent so a strategy in $\sum_{v \in V} \oplus.A_v$. The map $\mathrm{id}_{(\oplus.A_v)^\perp}$ is a strategy. Hence $\mathrm{id}_{(\oplus.A_v)^\perp} \| in_v$ is a strategy. As the composition of two strategy maps, In_v is a strategy in $(\oplus.A_v)^\perp \| \sum_{v \in V} \oplus.A_v$. It is a winning strategy because, as is easily seen from the explicit composite form of In_v, the image under In_v of a $+$-maximal configuration in $\mathbb{C}_{\oplus.A_v}$ is winning.

(iii) *'Only if':* Defining $P_v =_{\mathrm{def}} In_v^\perp$, where $In_v : \oplus.G_v^\perp \twoheadrightarrow \bigoplus_{v \in V} G_v^\perp$ is an instance of an injection strategy defined above, we obtain by duality a winning strategy

$$P_v : \bigominus_{v \in V} G_v \twoheadrightarrow \ominus .G_v \,,$$

for any $v \in V$. Let $v \in V$. By composition with P_v a winning strategy in $\bigominus_{v \in V} G_v$ yields a winning strategy in the component $\ominus.G_v$. By Lemma 7.34 in a strategy $\sigma : S \to \ominus.A_v$ the event structure S decomposes into a prefixed sum, where the prefixing events are necessarily all $-$ve. As σ is receptive the sum must be a unary prefixed sum of the form $\ominus.S'$. Lemma 7.34 provides a map $\sigma' : S' \to A_v$. From σ being winning the map σ' will be a winning strategy in

G_v. *'If':* Suppose $\sigma_v : S_v \to A_v$ is a winning strategy in G_v, for all $v \in V$. Prefixing we obtain winning strategies $\ominus.\sigma_v : \ominus.S_v \to \ominus.A_v$ in $\ominus.G_v$. Forming the sum $\sum_{v\in V} \ominus.\sigma_v : \sum_{v\in V} \ominus.S_v \to \ominus.\sigma_v : \sum_{v\in V} \ominus.A_v$ we obtain a strategy winning in $\bigominus_{v\in V} G_v$.

(iv) Now suppose G and H are determined. *'If':* The dual winning strategies $p^{\perp}_{G^{\perp}} : G \rightarrow\!\!\!\!+\!\!> G \parr H$ and $p^{\perp}_{H^{\perp}} : H \rightarrow\!\!\!\!+\!\!> G \parr H$ compose with a winning strategy $(\varnothing, \{\varnothing\}) \rightarrow\!\!\!\!+\!\!> G$, or respectively a winning strategy $(\varnothing, \{\varnothing\}) \rightarrow\!\!\!\!+\!\!> H$, to yield a winning strategy $(\varnothing, \{\varnothing\}) \rightarrow\!\!\!\!+\!\!> G \parr H$. *'Only if':* Suppose $G \parr H$ has a winning strategy. Then $G^{\perp} \otimes H^{\perp} = (G \parr H)^{\perp}$ has no winning strategy. Hence by (i), $G^{\perp}$ has no winning strategy or $H^{\perp}$ has no winning strategy. From determinacy, G has a winning strategy or H has a winning strategy. □

Theorem 7.36. *For all predicate-calculus formulae ϕ and environments ρ, $\rho \models_M \phi$ iff the game $[\![\phi]\!]_M \rho$ has a winning strategy.*

Proof. By Proposition 7.31 the games $[\![\phi]\!]_M \rho$ obtained from formulae ϕ are race-free and determined. The proof is by structural induction on ϕ.

The base case where ϕ is $R(x_1, \cdots, x_k)$ is obvious; the game $(\varnothing, \{\varnothing\})$ has as (unique) winning strategy the map $\varnothing \to \varnothing$, while $(\varnothing, \varnothing)$ has no winning strategy.

For the case $\phi \wedge \psi$, reason

$$\begin{aligned}
\rho \models_M \phi \wedge \psi &\iff \rho \models_M \phi \ \&\ \rho \models_M \psi \\
&\iff [\![\phi]\!]_M \rho \text{ has a winning strategy } \&\ [\![\psi]\!]_M \rho \text{ has a winning strategy, by induction,} \\
&\iff [\![\phi]\!]_M \rho \otimes [\![\psi]\!]_M \rho \text{ has a winning strategy, by Lemma 7.35(i),} \\
&\iff [\![\phi \wedge \psi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\phi \vee \psi$,

$$\begin{aligned}
\rho \models_M \phi \vee \psi &\iff \rho \models_M \phi \text{ or } \rho \models_M \psi \\
&\iff [\![\phi]\!]_M \rho \text{ has a winning strategy or } [\![\psi]\!]_M \rho \text{ has a winning strategy, by induction,} \\
&\iff [\![\phi]\!]_M \rho \parr [\![\psi]\!]_M \rho \text{ has a winning strategy, by Lemma 7.35(iv),} \\
&\iff [\![\phi \wedge \psi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\neg\phi$,

$$\begin{aligned}
\rho \models_M \neg\phi &\iff \rho \not\models_M \phi \\
&\iff [\![\phi]\!]_M \rho \text{ has no winning strategy, by induction,} \\
&\iff ([\![\phi]\!]_M \rho)^{\perp} \text{ has a winning strategy, by determinacy.}
\end{aligned}$$

In the case $\exists x.\ \phi$,

$$\begin{aligned}
\rho \models_M \exists x.\phi &\iff \rho[v/x] \models_M \phi \text{ for some } v \in V \\
&\iff [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, for some } v \in V \text{, by induction,} \\
&\iff \bigoplus_{v\in V} [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, by Lemma 7.35(ii),} \\
&\iff [\![\exists x.\phi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}$$

In the case $\forall x.\ \phi$,

$$
\begin{aligned}
\rho \models_M \forall x.\phi &\iff \rho[v/x] \models_M \phi \text{ for all } v \in V \\
&\iff [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, for all } v \in V\text{, by induction,} \\
&\iff \bigominus_{v \in V} [\![\phi]\!]_M \rho[v/x] \text{ has a winning strategy, by Lemma 7.35(iii),} \\
&\iff [\![\forall x.\phi]\!]_M \rho \text{ has a winning strategy.}
\end{aligned}
$$

□

Chapter 8

Games with imperfect information

8.1 Motivation

Consider the game "rock, scissors, paper" in which the two participants Player and Opponent independently sign one of r ("rock"), s ("scissors") or p ("paper"). The participant with the dominant sign w.r.t. the relation

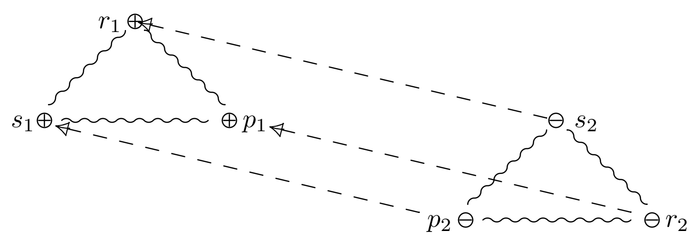
$$r \textit{ beats } s,\ s \textit{ beats } p \text{ and } p \textit{ beats } r$$

wins. It seems sensible to represent this game by RSP, the event structure with polarity

comprising the three mutually inconsistent possible signings of Player in parallel with the three mutually inconsistent signings of Opponent. In the absence of neutral configurations, a reasonable choice is to take the *losing* configurations (for Player) to be

$$\{s_1, r_2\},\ \{p_1, s_2\},\ \{r_1, p_2\}$$

and all other configurations as winning for Player. In this case there is a winning strategy for Player, *viz.* await the move of Opponent and then beat it with a dominant move. Explicitly, the winning strategy $\sigma : S \to RSP$ is given as the obvious map from S, the following event structure with polarity:



But this strategy cheats. In "rock, scissors, paper" participants are intended to make their moves *independently.* The problem with the game RSP as it stands is that it is a game of *perfect information* in the sense that all moves are visible to both participants. This permits the winning strategy above with its unwanted dependencies on moves which should be unseen by Player. To adequately model "rock, scissors, paper" requires a game of *imperfect information* where some moves are masked, or inaccessible, and strategies with dependencies on unseen moves are ruled out.

8.2 Games with imperfect information

We extend concurrent games to games with imperfect information. To do so in way that respects the operations of the bicategory of games we suppose a fixed preorder of *levels* $(\Lambda, \leq)$. The levels are to be thought of as levels of access, or permission. Moves in games and strategies are to respect levels: moves will be assigned levels in such a way that a move is only permitted to causally depend on moves at equal or lower levels; it is as if from a level only moves of equal or lower level can be seen.

An Λ-game (G, l) comprises a game $G = (A, W, L)$ with winning/losing conditions together with a *level function* $l : A \to \Lambda$ such that

$$a \leq_A a' \implies l(a) \leq l(a')$$

for all $a, a' \in A$. A Λ-strategy in the Λ-game (G, l) is a strategy $\sigma : S \to A$ for which

$$s \leq_S s' \implies l\sigma(s) \leq l\sigma(s')$$

for all $s, s' \in S$.

For example, for "rock, scissors, paper" we can take Λ to be the discrete preorder consisting of levels 1 and 2 unrelated to each other under $\leq$. To make RSP into a suitable Λ-game the level function l takes +ve events in RSP to level 1 and −ve events to level 2. The strategy above, where Player awaits the move of Opponent then beats it with a dominant move, is now disallowed because it is not a Λ-strategy—it introduces causal dependencies which do not respect levels. If instead we took Λ to be the unique preorder on a single level the Λ-strategies would coincide with all the strategies.

8.2.1 The bicategory of Λ-games

The introduction of levels meshes smoothly with the bicategorical structure on games.

For a Λ-game (G, l_G), define its dual $(G, l_G)^\perp$ to be $(G^\perp, l_{G^\perp})$ where $l_{G^\perp}(\overline{a}) = l_G(a)$, for a an event of G.

For Λ-games (G, l_G) and (H, l_H), define their parallel composition $(G, l_G)\|(H, l_H)$ to be $(G\|H, l_{G\|H})$ where $l_{G\|H}((1,a)) = l_G(a)$, for a an event of G, and $l_{G\|H}((2,b)) = l_H(b)$, for b an event of H.

A strategy between Λ-games from (G, l_G) to (H, l_H) is a strategy in $(G, l_G)^\perp\|(H, l_H)$.

Proposition 8.1.
(i) Let (G, l_G) be a Λ-game where G satisfies **(Cwins)**. *The copy-cat strategy on G is a Λ-strategy.*
(ii) The composition of Λ-strategies is a Λ-strategy.

Proof. (i) The additional causal links introduced in the construction of the copy-cat strategy are between complementary events in $G^\perp$ and G, at the same level in Λ, and so respect ≤.

(ii) Let (G, l_G), (H, l_H) and (K, l_K) be Λ-games. Let $\sigma : G \longrightarrow\!\!\!\!\!\!+\, H$ and $\tau : H \longrightarrow\!\!\!\!\!\!+\, K$ be Λ-strategies. We show their composition $\tau\odot\sigma$ is a Λ-strategy.

It suffices to show $p \rightarrowtriangle p'$ in $T\odot S$ implies $l_{G^\perp\|K}\tau\odot\sigma(p) \leq l_{G^\perp\|K}\tau\odot\sigma(p')$. Suppose $p \rightarrowtriangle p'$ in $T\odot S$ with $max(p) = e$ and $max(p') = e'$. Take $x \in \mathcal{C}(T\odot S)$ containing p' so p too. Then,

$$e \rightarrowtriangle_{\bigcup x} e_1 \rightarrowtriangle_{\bigcup x} \cdots \rightarrowtriangle_{\bigcup x} e_{n-1} \rightarrowtriangle_{\bigcup x} e'$$

where $e, e' \in V_0$ and $e_i \notin V_0$ for $1 \leq i \leq n-1$. (V_0 consists of 'visible' events of the stable family, those of the form $(s, *)$ with $\sigma_1(s)$ defined, or $(*, t)$, with $\tau_2(t)$ defined.) The events e_i have the form (s_i, t_i) where $\sigma_2(s_i) = \tau_1(t_i)$, for $1 \leq i \leq n-1$.

Any individual link in the chain above has one of the forms:

$$(s,t) \rightarrowtriangle_{\bigcup x} (s',t')\,,\ (s,*) \rightarrowtriangle_{\bigcup x} (s',t')\,,$$
$$(*,t) \rightarrowtriangle_{\bigcup x} (s',t')\,,\ (s,t) \rightarrowtriangle_{\bigcup x} (s',*)\,,\ \text{ or }\ (s,t) \rightarrowtriangle_{\bigcup x} (*,t')\,.$$

By Lemma 3.16, for any link either $s \rightarrowtriangle_S s'$ or $t \rightarrowtriangle_T t'$. As σ and τ are Λ-strategies, this entails

$$l_{G^\perp\|H}\sigma(s) \leq l_{G^\perp\|H}\sigma(s') \ \text{ or }\ l_{H^\perp\|K}\tau(t) \leq l_{H^\perp\|K}\tau(t')$$

for any link. Consequently ≤ is respected across the chain and $l_{G^\perp\|K}\tau\odot\sigma(p) \leq l_{G^\perp\|K}\tau\odot\sigma(p')$, as required. □

W.r.t. a particular choice of access levels $(\Lambda, \leq)$ we obtain a bicategory $\mathbf{WGames}_\Lambda$. Its objects are Λ-games (G, l) where G satisfies **(Cwins)** with arrows the Λ-strategies and 2-cells maps of spans. It restricts to a sub-bicategory of deterministic Λ-strategies, which as before is equivalent to an order-enriched category.

8.3 Hintikka's IF logic

We present a variant of Hintikka's Independence-Friendly (IF) logic and propose a semantics in terms of concurrent games with imperfect information. Assume a preorder $(\Lambda, \leq)$. The syntax for IF logic is essentially that of the predicate calculus, but with levels in Λ associated with quantifiers: formulae are given by

$$\phi, \psi, \cdots ::= R(x_1, \cdots, x_k) \mid \phi \wedge \psi \mid \phi \vee \psi \mid \neg\phi \mid \exists^\lambda x.\ \phi \mid \forall^\lambda x.\ \phi$$

where $\lambda \in \Lambda$, R ranges over basic relation symbols of a fixed arity and $x, x_1, x_2, \cdots$ over variables.

Assume M, a non-empty universe of values V_M and an interpretation for each of the relation symbols as a relation of appropriate arity on V_M; so M is a model for the predicate calculus in which the quantifier levels are stripped away. Again, an environment ρ is a function from variables to values; again, $\rho[v/x]$ means the environment ρ updated to value v at variable x. W.r.t. a model M and an environment ρ, we denote each closed formula ϕ of IF logic by a Λ-game, following very closely the definitions in Section 7.8. The differences are the assignment of levels to events and that the order on Λ has to be respected by the (modified) prefixed sums which quantified formulae denote.

The prefixed game $\oplus^\lambda.(A, W, l)$ comprises the event structure with polarity $\oplus.A$ in which all the events of $a \in A$ where $\lambda \leq l(a)$ are made to causally depend on a fresh +ve event $\oplus$, itself assigned level λ. Its winning conditions are those configurations $x \in \mathcal{C}^\infty(\oplus.A)$ of the form $\{\oplus\} \cup y$ for some $y \in W$. The game $\bigoplus^\lambda_{v \in V}(A_v, W_v, l_v)$ has underlying event structure with polarity the sum $\sum_{v \in V} \oplus^\lambda.A_v$, maintains the same levels as its components, with a configuration winning iff it is the image of a winning configuration in a component under the injection to the sum. The game $\bigominus^\lambda_{v \in V} G_v$ is defined dually, as $(\bigoplus^\lambda_{v \in V} G_v^\perp)^\perp$. In this game the empty configuration is winning but Opponent gets to make the first move.

True denotes the Λ-game the unit w.r.t. $\otimes$ and false denotes he unit w.r.t. $\parr$. Denotations of conjunctions and disjunctions are given by the operations of $\otimes$ and $\parr$ on Λ-games, while negations denote dual games. W.r.t. an environment ρ, universal and existential quantifiers denote the *prefixed sums* of games:

$$[\![\exists^\lambda x.\ \phi]\!]^\Lambda_M \rho = \bigoplus^{\lambda}_{v \in V_M} [\![\phi]\!]^\Lambda_M \rho[v/x]$$

$$[\![\forall^\lambda x.\ \phi]\!]^\Lambda_M \rho = \bigominus^{\lambda}_{v \in V_M} [\![\phi]\!]^\Lambda_M \rho[v/x]\,.$$

As a definition, an IF formula ϕ is satisfied w.r.t. an environment ρ, written

$$\rho \vDash^\Lambda_M \phi\,,$$

iff the Λ-game $[\![\phi]\!]^\Lambda_M \rho$ has a winning strategy.

Chapter 9

Extensions

These notes are incomplete in several ways. They don't account for games with back-tracking, games where play can revisit previous positions. While a little odd from the point of view of everyday games, this feature is very important in game semantics, for instance in order to re-evaluate the argument to a function. The theory has recently been extended to allow back-tracking and copying via event structures with symmetry, which support a rich variety of pseudo (co)monads to achieve this. The determinacy result has been extended to concurrent games with Borel winning conditions (provided the games are race-free and bounded-concurrent). Concurrent strategies have recently been extended to probabilistic and quantum concurrent strategies. The relevant papers can be found at `www.cl.cam.ac.uk/~gw104`.

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Bibliography

[1] Nielsen, M., Plotkin, G., Winskel, G.: Petri nets, event structures and domains. Theoretical Computer Science **13** (1981) 85–108

[2] Winskel, G., Nielsen, M.: Models for concurrency. In Abramsky, S., Gabbay, D., eds.: Semantics and Logics of Computation. OUP (1995)

[3] Saunders-Evans, L., Winskel, G.: Event structure spans for nondeterministic dataflow. Electr. Notes Theor. Comput. Sci. 175(3): 109-129 (2007)

[4] Winskel, G.: Event structure semantics for CCS and related languages. In: ICALP'82 (A report with full proofs is available from www.cl.cam.ac.uk/~gw104). Volume 140 of LNCS., Springer (1982)

[5] Rideau, S., Winskel, G.: Concurrent strategies. In: LICS 2011, IEEE Computer Society (2011)

[6] Joyal, A.: Remarques sur la théorie des jeux à deux personnes. Gazette des sciences mathématiques du Québec, 1(4) (1997)

[7] Winskel, G.: Event structures with symmetry. Electr. Notes Theor. Comput. Sci. 172: 611-652 (2007)

[8] Laird, J.: A games semantics of idealized CSP. Vol 45 of Electronic Books in Theor. Comput. Sci. (2001)

[9] Ghica, D.R., Murawski, A.S.: Angelic semantics of fine-grained concurrency. In: FOSSACS'04, LNCS 2987, Springer (2004)

[10] Melliès, P.A., Mimram, S.: Asynchronous games : innocence without alternation. In: CONCUR '07. Volume 4703 of LNCS., Springer (2007)

[11] Katovsky, A.: Concurrent games. First-year report for PhD study, Computer Lab, Cambridge (2011)

[12] Curien, P.L.: On the symmetry of sequentiality. In: MFPS. Number 802 in LNCS, Springer (1994) 29–71

[13] Hyland, M.: Game semantics. In Pitts, A., Dybjer, P., eds.: Semantics and Logics of Computation. Publications of the Newton Institute (1997)

[14] Harmer, R., Hyland, M., Melliès, P.A.: Categorical combinatorics for innocent strategies. In: LICS '07, IEEE Computer Society (2007)

[15] Melliès, P.A.: Asynchronous games 2: The true concurrency of innocence. Theor. Comput. Sci. 358(2-3): 200-228 (2006)

[16] Abramsky, S., Melliès, P.A.: Concurrent games and full completeness. In: LICS '99, IEEE Computer Society (1999)

[17] Hyland, J.M.E., Ong, C.H.L.: On full abstraction for PCF: I, II, and III. Inf. Comput. 163(2): 285-408 (2000)

[18] Abramsky, S., Jagadeesan, R., Malacaria, P.: Full abstraction for PCF. Inf. Comput. 163(2): 409-470 (2000)

[19] Varacca, D., Völzer, H., Winskel, G.: Probabilistic event structures and domains. Theor. Comput. Sci. 358(2-3): 173-199 (2006)

[20] Hyland, M.: Some reasons for generalising domain theory. Mathematical Structures in Computer Science **20**(2) (2010) 239–265

[21] Cattani, G.L., Winskel, G.: Profunctors, open maps and bisimulation. Mathematical Structures in Computer Science **15**(3) (2005) 553–614

[22] Curien, P.L., Plotkin, G.D., Winskel, G.: Bistructures, bidomains, and linear logic. In: Proof, Language, and Interaction, essays in honour of Robin Milner, MIT Press (2000) 21–54

[23] Abramsky, S.: Semantics of interaction. In Pitts, A., Dybjer, P., eds.: Semantics and Logics of Computation. Publications of the Newton Institute (1997)

Appendix A

Exercises

On event structures and stable families

Exercise A.1. Let $(A, \leq_A, \mathrm{Con}_A), (B, \leq_B, \mathrm{Con}_B)$ be event structures. Let $f : A \rightharpoonup B$. Show f is a map of event structures, $f : (A, \leq_A, \mathrm{Con}_A) \to (B, \leq_B, \mathrm{Con}_B)$, iff

(i) $\forall a \in A, b \in B.\ b \leq_B f(a) \implies \exists a' \in A.\ a' \leq_A a \ \&\ f(a') = b$, and

(ii) $\forall X \in \mathrm{Con}_A.\ fX \in \mathrm{Con}_B \ \&\ \forall a_1, a_2 \in X.\ f(a_1) = f(a_2) \implies a_1 = a_2$.

□

Exercise A.2. Show a map $f : A \to B$ of event structures is mono iff the function $\mathcal{C}(A) \to \mathcal{C}(B)$ taking configuration x to its direct image fx is injective. [Recall a map $f : A \to B$ is mono iff for all maps $g, h : C \to A$ if $fg = fh$ then $g = h$.] Taking B to be the event structure comprising two concurrent events, can you find an event structure A and an example of a total map $f : A \to B$ of event structures which is both mono and where f is not injective as a function on events? □

Exercise A.3. Verify that the finite configurations of an event structure form a stable family. □

Exercise A.4. Say an event structure A is *tree-like* when its concurrency relation is empty (so two events are either causally related or inconsistent). Suppose B is tree-like and $f : A \to B$ is a total map of event structures. Show A must also be tree-like, and moreover that the map f is rigid, *i.e.* preserves causal dependency.

Exercise A.5. Let $\mathcal{F}$ be a nonempty family of finite sets satisfying the Completeness axiom in the definition of stable families. Show $\mathcal{F}$ is coincidence-free iff

$$\forall x, y \in \mathcal{F}.\ x \subsetneq y \implies \exists x_1, e_1.\ x \overset{e_1}{\longrightarrow\!\!\subset} x_1 \subseteq y\,.$$

[Hint: For 'only if' use induction on the size of $y \setminus x$.] □

Exercise A.6. Prove Proposition 3.5: Let $f : \mathcal{F} \to \mathcal{G}$ be a map of stable families. Let $e, e' \in x$, a configuration of $\mathcal{F}$. Show if $f(e) \leq_{fx} f(e')$ (with both $f(e)$ and $f(e')$ defined) then $e \leq_x e'$.

Exercise A.7. Prove the two propositions 3.8 and 3.9. □

Exercise A.8. (From Section 3.2) For an event structure E, show $\mathcal{C}^\infty(E) = \mathcal{C}(E)^\infty$. □

Exercise A.9. (From Section 3.2) Let $\mathcal{F}$ be a stable family. Show $\mathcal{F}^\infty$ satisfies:

Completeness: $\forall Z \subseteq \mathcal{F}^\infty.\ Z\uparrow \implies \bigcup Z \in \mathcal{F}^\infty$;
Stability: $\forall Z \subseteq \mathcal{F}^\infty.\ Z \neq \emptyset\ \&\ Z\uparrow \implies \bigcap Z \in \mathcal{F}^\infty$;
Coincidence-freeness: For all $x \in \mathcal{F}^\infty$, $e, e' \in x$ with $e \neq e'$,

$$\exists y \in \mathcal{F}^\infty.\ y \subseteq x\ \&\ (e \in y \iff e' \notin y)\,;$$

Finiteness: For all $x \in \mathcal{F}^\infty$,

$$\forall e \in x \exists y \in \mathcal{F}.\ e \in y\ \&\ y \subseteq x\ \&\ y \text{ is finite}\ .$$

Show that $\mathcal{F}$ consists of precisely the finite sets in $\mathcal{F}^\infty$. □

Exercise A.10. Let A be the event structure consisting of two distinct events $a_1 \leq a_2$ and B the event structure with a single event b. Following the method of Section 3.3.1 describe the product of event structures $A \times B$. □

Exercise A.11. Let $f : A \to B$ be a total map of event structures. Show

$$a \rightarrowtriangle_A a' \implies f(a) \rightarrowtriangle_B f(a') \text{ or } f(a)\ co_B\ f(a')\,.$$

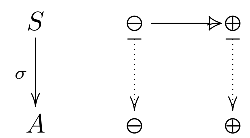
□

On strategies

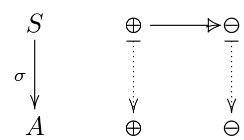
Exercise A.12. Consider the empty map of event structures with polarity $\varnothing \to A$. Is it a strategy? Is it a deterministic strategy? Consider now the identity map $\mathrm{id}_A : A \to A$ on an event structure with polarity A. Is it a strategy? Is it a deterministic strategy? □

Exercise A.13. For each instance of total map σ of event structures with polarity below say whether σ is a strategy and whether it is deterministic. In each case give a short justification for your answer. (Immediate causal dependency within the event structures is represented by an arrow ⇾ and inconsistency, or conflict, by a wiggly line ~~~~ .)

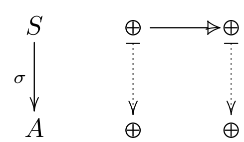
(*i*)



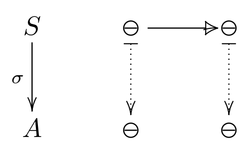
(*ii*)



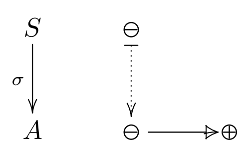
(*iii*)



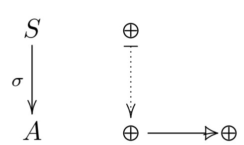
(*iv*)



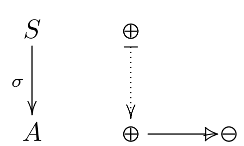
(*v*)

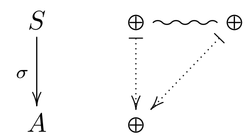


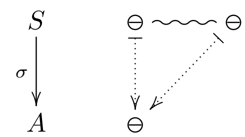
(*vi*)

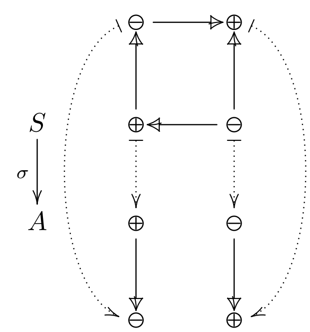


(*vi*)



(vii)


(viii)


(ix)


□

Exercise A.14. Let $\mathrm{id}_A : A \to A$ be the identity map of event structures, sending an event to itself. Show the identity map forms a strategy in the game A. Is it deterministic in general? □

Exercise A.15. Show any strategy $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!> B$ has a dual strategy $\sigma^\perp : B^\perp \rightarrow\!\!\!\!\!+\!\!\!> A^\perp$. In more detail, supposing $\sigma : S \to A^\perp \| B$ is a strategy show $\sigma^\perp : S \to (B^\perp)^\perp \| A^\perp$ is a strategy where

$$\sigma^\perp(s) = \begin{cases} (1, b) & \text{if } \sigma(s) = (2, b) \\ (2, a) & \text{if } \sigma(s) = (1, a)\,. \end{cases}$$

□

Exercise A.16. Say an event structure is *set-like* if its causal dependency relation is the identity relation and all pairs of distinct events are inconsistent. Let A and B be set-like event structures. In this case, can you see a simpler way to describe strategies and deterministic strategies $A \rightarrow\!\!\!\!\!+\!\!\!> B$, and what composition of strategies corresponds to? □

Exercise A.17. By considering the game A comprising two concurrent events, one +ve and one −ve, show there is a nondeterministic pre-strategy $\sigma : S \to A$ such that $s \rightarrowtriangle s'$ in S without $\sigma(s) \rightarrowtriangle \sigma(s')$. Could you find such a counterexample were σ deterministic? Explain. □

Exercise A.18. Let $G =_{\rm def} (A, W)$ be a game with winning conditions. Say a pre-strategy $\sigma : S \to A$ is winning iff $\sigma x \in W$ for all $+$-maximal configurations $x \in \mathcal{C}^\infty(S)$. Show that if G has a winning *receptive* pre-strategy, then the dual game $G^\perp$ has no winning strategy (use Corollary 7.3.) Show that G may have a winning pre-strategy (necessarily not receptive) while $G^\perp$ has a winning strategy. □

Exercise A.19. (Section 7.3.) Each event structure with polarity A possesses a "Scott order" on its configurations $\mathcal{C}^\infty(A)$:

$$x' \sqsubseteq x \ \text{ iff } \ x' \supseteq^- x \cap x' \subseteq^+ x \,.$$

Prove that the Scott order is indeed a partial order. □