# Logic and Proof

*Supervision 2*

## 5. Formal reasoning in first-order logic

1. a) For each of the following FOL formulas, circle the free variables, and underline the bound variables. Connect each bound variable to its binding occurrence (either graphically, by numbering, or whatever works for you).

   i. $\forall x.\, x = x$

   ii. $\exists x.\, P(x, y) \wedge \forall y.\, \neg P(y, x)$

   iii. $(\forall x.\, P(x, y)) \to (\exists y.\, P(x, y))$

   iv. $\exists z.\, P(x, y) \wedge Q(x, z)$

   v. $\forall x.\, (P(x) \to Q(x)) \wedge S(x, y)$

   vi. $P(x, \forall z.\, (\exists x.\, Q(y, x)) \to Q(x, z))$

   b) Apply the substitution $[x \mapsto f(x, y), y \mapsto g(z)]$ to the formulas above.

2. Consider the following proof attempt of the set-theoretic conjecture:

$$\forall A, B.\ \mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$$

*Proof.* Let $A, B$ be sets. We prove the proposition using equational reasoning:

$$\begin{aligned}
\mathcal{P}(A \cup B) &= \{X \mid X \subseteq A \cup B\} && \text{(def. of powerset)}\\
&= \{X \mid \forall x.\, x \in X \to x \in A \cup B\} && \text{(def. of subsets)}\\
&= \{X \mid \forall x.\, x \in X \to (x \in A \vee x \in B)\} && \text{(def. of union)}\\
&= \{X \mid \forall x.\, (x \in X \to x \in A) \vee (x \in X \to x \in B)\} && \text{(lemma of prop. logic)}\\
&= \{X \mid (X \subseteq A) \vee (X \subseteq B)\} && \text{(def. of subsets)}\\
&= \{X \mid X \subseteq A\} \cup \{X \mid X \subseteq B\} && \text{(def. of union)}\\
&= \mathcal{P}(A) \cup \mathcal{P}(B) && \text{(def. of powerset)} \quad \square
\end{aligned}$$

Is this a valid proof? Why or why not?

3. Verify the following equivalences by appealing to the truth definition of FOL.

$$\neg(\exists x.\, P(x)) \simeq \forall x.\, \neg P(x) \qquad (\forall x.\, P(x)) \wedge R \simeq \forall x.\, (P(x) \wedge R)$$
$$(\exists x.\, P(x)) \vee (\exists x.\, Q(x)) \simeq \exists x.\, (P(x) \vee Q(x))$$

4. Prove the equivalence $(\forall x.\, P(x) \vee P(a)) \simeq P(a)$.

5. Prove the following sequents. *Hint*: the last one requires two uses of $(\forall l)$.

$$(\forall x.\, P(x)) \wedge (\forall x.\, Q(x)) \Rightarrow \forall y.\, (P(y) \wedge Q(y))$$
$$\forall x.\, P(x) \wedge Q(x) \Rightarrow (\forall y.\, P(y)) \wedge (\forall y.\, Q(y))$$
$$\forall x.\, P(x) \to P(f(x)), P(a) \Rightarrow P(f(f(a)))$$

6. Prove the following sequents. *Hint*: the last one requires two uses of $(\exists r)$.

$$
\begin{aligned}
P(a) \lor \exists x.\, P(f(x)) &\Rightarrow \exists y.\, P(y) \\
\exists x.\, P(x) \lor Q(x) &\Rightarrow (\exists y.\, P(y)) \lor (\exists y.\, Q(y)) \\
&\Rightarrow \exists z.\, P(z) \to P(a) \land P(b)
\end{aligned}
$$

7. Prove the formula $\neg\forall y.\, (Q(a) \lor Q(b)) \land \neg Q(y)$ using equivalences, and then formally using the sequent calculus.

## 6. Clause methods for propositional logic

1. Outline the steps of the Davis–Putnam–Logeman–Loveland method. Explain the goal of the method, and why the steps of the algorithm are sound. Why does the empty clause represent a contradiction?
2. Apply the DPLL procedure to the clause set:

$$\{P, Q\} \qquad \{\neg P, Q\} \qquad \{P, \neg Q\} \qquad \{\neg P, \neg Q\}$$

3. Explain the resolution algorithm and how it differs from DPLL.
4. Use resolution (showing the steps of converting the formula into clauses) to prove at least three of the following formulas.

$$
\begin{gathered}
(P \to Q \lor R) \to ((P \to Q) \lor (P \to R)) \\
((P \to Q) \to P) \to P \\
(Q \to R) \land (R \to P \land Q) \land (P \to Q \lor R) \to (P \leftrightarrow Q) \\
(P \land Q \to R) \land (P \lor Q \lor R) \to ((P \leftrightarrow Q) \to R) \\
(P \to R) \land (R \land P \to S) \to (P \land Q \to R \land S)
\end{gathered}
$$

5. Convert these axioms to clauses, showing all steps. Then prove $\text{Winterstorm} \to \text{Miserable}$ by resolution.

$$
\begin{gathered}
\text{Wet} \land \text{Cold} \to \text{Miserable} \\
\text{Winterstorm} \to \text{Storm} \land \text{Cold} \\
\text{Storm} \to \text{Rain} \land \text{Windy} \\
\text{Rain} \land (\text{Windy} \lor \neg\text{Umbrella}) \to \text{Wet}
\end{gathered}
$$

## 7. Skolem functions, Herbrand's Theorem and unification

1. a) Explain the process of Skolemisation on a formula of your choice.
   b) The notes state that "[Skolemisation] does not preserve the meaning of a formula. However, it does preserve *inconsistency*, which is the critical property". Justify the two claims in this statement, demonstrating them on your example above.

2. Skolemize the following formulas, dropping all quantifiers.

$$\forall u.\, \exists x, y.\, P(x, y) \qquad \exists x, y.\, \forall z.\, \exists w.\, P(x, y, z, w) \qquad \forall u.\, (\exists x.\, P(x, x)) \land \forall v.\, \exists y.\, Q(u, y)$$

3. Consider a first-order language $A$ with $z$ and $o$ as constant symbols, with $n$ as a 1-place function symbol and $a$ as a 2-place function symbol, and with $C$ as a 2-place predicate symbol.

   a) Describe the Herbrand universe for this language.

   b) This language has an interpretation $\mathcal{I}$ in the domain $D = \mathbb{Z}$ of integers, with $z$ and $o$ interpreted as $0 \in \mathbb{Z}$ and $1 \in \mathbb{Z}$ respectively, $n$ being the negation function $x \mapsto -x$, $a$ being the addition function $x, y \mapsto x + y$, and $C$ being the less-than comparison relation $x, y \mapsto x < y$. What is the Herbrand model of the symbols of the language with respect to this interpretation $\mathcal{I}$?

4. For at least three of the following pairs of terms, give a most general unifier or explain why none exists. Do not rename variables prior to performing the unification.

$$\begin{array}{rl} f(g(x), z) & f(y, h(y)) \\ j(x, y, z) & j(f(y, y), f(z, z), f(a, a)) \\ j(x, z, x) & j(y, f(y), z) \\ j(f(x), y, a) & j(y, z, z) \\ j(g(x), a, y) & j(z, x, f(z, z)) \end{array}$$

5. Which of the following substitutions are most general unifiers for the terms $f(x, y, z)$ and $f(w, w, v)$?

$$[x/y, x/w, v/z] \qquad [y/x, y/w, v/z] \qquad [y/x, v/z]$$
$$[x/y, x/z, x/w, x/v] \qquad [u/x, u/y, u/w, y/z, y/v]$$

# 8. First-order resolution

1. What techniques allow us to convert first-order formulas into "propositional" clauses, and prove them using resolution? How are quantifiers and variables handled?

2. Is the clause $\{P(x, b), P(a, y)\}$ logically equivalent to the unit clause $\{P(a, b)\}$? Is the clause $\{P(y, y), P(y, a)\}$ logically equivalent to $\{P(y, a)\}$? Explain both answers.

3. Show that every set $S$ of definite clauses is consistent. *Hint*: first consider propositional logic, then extend your argument to first order logic.

4. Convert the following formulas into clauses, showing each step: negating the formula, eliminating $\to$ and $\leftrightarrow$, pushing in negations, Skolemising, dropping the universal quantifiers, and converting the resulting formula into CNF. Apply resolution (and possibly factoring) to prove or

disprove the formulas in each case.

$$(\exists x.\ \forall y.\ R(x,y)) \to (\forall y.\ \exists x.\ R(x,y))$$
$$(\forall y.\ \exists x.\ R(x,y)) \to (\exists x.\ \forall y.\ R(x,y))$$
$$\exists x.\ \forall y,z.\ (P(y) \to Q(z)) \to (P(x) \to Q(x))$$
$$\neg(\exists y.\ \forall x.\ R(x,y) \leftrightarrow \neg(\exists z.\ R(x,z) \land R(z,x)))$$

5. Refute the following set of clauses using resolution and factoring.

$$①\ \{P(x,b), P(a,y)\} \qquad ②\ \{\neg P(x,b), \neg P(c,y)\} \qquad ③\ \{\neg P(x,d), \neg P(a,y)\}$$

6. Prove the following formulas by resolution, showing all steps of the conversion into clauses. Note that $P$ is just a predicate symbol, so in particular, $x$ is not free in $P$.

$$(\forall x.\ P \lor Q(x)) \to (P \lor \forall x.\ Q(x)) \qquad \exists x,y.\ (R(x,y) \to \forall z,w.\ R(z,w))$$

## 9. Optional exercises

1. In your own words, explain the motivation behind Herbrand interpretations.

   - How is a Herbrand interpretation constructed from a set of clauses $S$?
   - Why do we need Herbrand interpretations?
   - What is the significance of the Skolem–Gödel–Herbrand Theorem?

   If you wish, consult a pre-2013 version of the course lecture notes, which discuss Herbrand models in more detail.

2. Consider the Prolog program consisting of the definite clauses

$$P(f(x,y)) \leftarrow Q(x), R(y)$$
$$Q(g(z)) \leftarrow R(z)$$
$$R(a) \leftarrow$$

   Describe the Prolog computation starting from the goal clause $\leftarrow P(v)$. Keep track of the substitutions affecting $v$ to determine what answer the Prolog system would return.