Logic and Proof

Supervision 1

1. Introduction

1. Determine whether the truth value of the following sentences is true, false, unknown, or if the sentence is not a proper logical statement. Justify all of your answers, and give satisfying and falsifying valuations of the variables where appropriate.

   a) Broccoli is delicious.

   b) x likes broccoli.

   c) The area of a circle of radius r is πr^2.

   d) Larry Paulson was born in x.

   e) Larry Paulson est né aux États-Unis.

   f) This sentence is false.

   g) Every even number greater than 2 can be written as the sum of two primes.

   h) There exists a unique prime n that satisfies $a^n + b^n = c^n$ for some positive integers a, b and c.

   i) There are two people in Peterborough with the exact same number of hairs on their head.

2. a) Write two closed statements (without variables): one true and one false.

   b) Write three statements with at least two variables: one valid, one satisfiable and one unsatisfiable. Give satisfying and falsifying interpretations where appropriate.

   c) Write a set S of at least three statements containing at least one (common) variable such that every statement is satisfiable, but the set is inconsistent.

2. Propositional logic

1. Verify de Morgan's laws and Peirce's Law using truth tables.

2. Each of the following formulas is satisfiable but not valid. Exhibit an interpretation that makes the formula true and another that makes the formula false.

$$P \to Q \qquad \neg(P \lor Q \lor R) \qquad P \lor Q \to P \land Q \qquad \neg(P \land Q) \land \neg(Q \lor R) \land (P \lor R)$$

3. Associate each of the following terms with one of the propositional equivalences on Slide 205. Give a number-theoretic example for each property.

   unit, distributivity, idempotence, commutativity, inverse, associativity, annihilation

4. Convert each of the following propositional formulas into Conjunctive Normal Form and also into Disjunctive Normal Form. For each formula, state whether it is valid, satisfiable, or unsatisfiable; justify each answer.

$$(P \to Q) \land (Q \to P) \qquad ((P \land Q) \lor R) \land \neg(P \lor R) \qquad \neg(P \lor Q \lor R) \lor ((P \land Q) \lor R)$$

3. Proof systems for propositional logic

1. Briefly compare and contrast the following three formal proof systems:
    - Hilbert-style deductive system
    - Gentzen-style natural deduction
    - Gentzen-style sequent calculus
2. a) Proof systems employ many "if X then Y"-style connectives and assertions on different reasoning levels. Explain, with examples, the differences and similarities between

$$A \wedge B \to C \qquad A, B \vdash C \qquad A, B \vDash C \qquad A, B \Rightarrow C \qquad \frac{A \quad B}{C} \qquad \begin{array}{c}[A \wedge B]\\ \vdots \\ C\end{array}$$

   b) Separate the statements above based on whether they belong to the *syntax* or the *semantics* of propositional logic. Which proof systems are the constructs associated with?
3. Prove the following sequents:

$$\neg\neg A \Rightarrow A \qquad A \wedge B \Rightarrow B \wedge A \qquad (A \vee B) \wedge (A \vee C) \Rightarrow A \vee (B \wedge C)$$
$$\neg(A \vee B) \Rightarrow \neg A \wedge \neg B \qquad \Rightarrow (A \wedge \neg A) \to B \qquad \Rightarrow ((A \to B) \to A) \to A$$

   (You can write the proof trees upside-down if you prefer, so you don't have to reserve space.)
4. Derive the sequent calculus rules for the connectives $\leftrightarrow$ (iff) and $\oplus$ (exclusive or). Note that other connectives must not appear in these rules.

4. First-order logic

1. To test your understanding of quantifiers, consider the following formulas: *everybody loves somebody* vs. *there is somebody that everybody loves*:

$$\forall x.\, \exists y.\, \mathrm{loves}(x, y) \qquad \exists y.\, \forall x.\, \mathrm{loves}(x, y)$$

   Does the first imply the second? Does the second imply the first? Consider both the informal meaning and the formal semantics defined in the course.
2. Let $\approx$ be a 2-place predicate symbol, which we write using infix notation as $x \approx y$ instead of $\approx(x, y)$.
   a) Give three formulas in FOL describing the axioms that make $\approx$ an *equivalence relation*.
   b) Let the universe be the set $\mathbb{N}$ of natural numbers. Which axioms hold if $I[\approx]$ is:
      (i) the empty relation, $\emptyset$?
      (ii) the universal relation, $\{(x, y) \mid x, y \in \mathbb{N}\}$?
      (iii) the equality relation, $\{(x, x) \mid x \in \mathbb{N}\}$?
      (iv) the relation $\{(x, y) \mid x, y \in \mathbb{N} \wedge x + y \text{ is even}\}$?

(v) the relation $\{(x, y) \mid x, y \in \mathbb{N} \land x + y = 100\}$?

(vi) the relation $\{(x, y) \mid x, y \in \mathbb{N} \land x \leq y\}$?

3. Taking R as 2-place relation symbol and $=$ denoting equality, consider the following axioms:

$$\forall x.\, \neg R(x, x) \tag{1}$$
$$\forall x, y.\, \neg(R(x, y) \land R(y, x)) \tag{2}$$
$$\forall x, y, z.\, R(x, y) \land R(y, z) \to R(x, z) \tag{3}$$
$$\forall x, y.\, R(x, y) \lor (x = y) \lor R(y, x) \tag{4}$$
$$\forall x, y.\, R(x, z) \to \exists y.\, R(x, y) \land R(y, z) \tag{5}$$

a) Exhibit two interpretations that satisfy axioms 1–5.

b) Exhibit two interpretations that satisfy axioms 1–4 and falsify axiom 5.

c) Exhibit two interpretations that satisfy axioms 1–3 and falsify axioms 4 and 5.

Hint: Consider simple examples such as $R(x, y) = x < y$ on some appropriate domain.

4. Some textbook and paper authors like to use nonstandard notation, or even define their own syntax for some particular formal system. While often difficult to read and generally not recommended, there is nothing inherently wrong with this practice – as long as the meaning (semantics) is defined appropriately, the syntax can be whatever the author chooses to use.

Suppose that a particularly creative logician decides to reinvent the syntax of first-order logic. Below are three logically valid formulas written in the syntax which should hold for any formula A and predicate P and Q containing a free variable x.

$$\overbrace{\left[P(x), \boxed{P(x)}\right]}^{x} \qquad \boxed{\boxed{A}} \triangleright A \qquad \boxed{\overbrace{\left[P(x), \boxed{Q(x)}\right]}^{x}} \bowtie \underbrace{\left\langle \boxed{P(x)}, Q(x) \right\rangle}_{x}$$

a) Describe the formal syntax of this logic as a grammar, giving the different syntactic forms of a formula A (see e.g. Slide 201). You may assume that the syntax already includes atomic formulas made up of a relation symbol $P, Q, R \ldots$ applied to some number of terms.

b) Consider a formula A in the syntax presented above. Let $\mathcal{I}$ be an interpretation of the symbols and V a valuation for the formula. Give a *truth definition* for the formula A by defining a predicate $\models_{\mathcal{I},V} A$ which holds when A is true in $\mathcal{I}$ under V. Make sure that the three formulas above are true with your definition.

c) The formula below is a statement about arithmetic on natural numbers. Give an interpretation $\mathcal{I} = (D, I)$ of the constant, function, and relation symbols, and a valuation V of the free variables, which satisfy this formula.

$$\left\langle \overbrace{\left[n \sim \bullet, \underbrace{n \sim k^{+}}_{k} \right]}^{n}, \left\langle v \sim \bullet^{+++}, \boxed{\underbrace{\bullet \sim \ell^{+}}_{\ell}} \right\rangle \right\rangle$$

Optional exercise

1. Using OCaml, define datatypes for representing propositions and interpretations. Write a function to test whether or not a proposition holds under an interpretation (both supplied as arguments). Write a function to convert a proposition to Negation Normal Form.