Discrete Mathematics

Supervision 8

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11. On surjections and injections

11.1. Basic exercises

1. Give two examples of functions that are surjective, and two examples of functions that are not.
2. Give two examples of functions that are injective, and two examples of functions that are not.

11.2. Core exercises

1. Explain and justify the phrase *injections can be undone*.
2. Show that $f : A \to B$ is a surjection if and only if for all sets C and functions $g, h : B \to C$, $g \circ f = h \circ f$ implies $g = h$.

   What would be an analogous condition for injections?
3. Use the above sufficient condition to show that the identity function is a surjection, and the composition of surjections is a surjection.

12. On images

12.1. Basic exercises

1. Let $R_2 = \{(m, n) \mid m = n^2\} : \mathbb{N} \nrightarrow \mathbb{Z}$ be the integer square-root relation. What is the direct image of $\mathbb{N}$ under R_2? And what is the inverse image of $\mathbb{N}$?
2. For a relation $R : A \nrightarrow B$, show that:

   a) $\overrightarrow{R}(X) = \bigcup_{x \in X} \overrightarrow{R}(\{x\})$ for all $X \subseteq A$

   b) $\overleftarrow{R}(Y) = \left\{ a \in A \,\middle|\, \overrightarrow{R}(\{a\}) \subseteq Y \right\}$ for all $Y \subseteq B$.

12.2. Core exercises

1. For $X \subseteq A$, prove that the direct image $\overrightarrow{f}(X) \subseteq B$ under an injective function $f : A \rightarrowtail B$ is in bijection with X; that is, $X \cong \overrightarrow{f}(X)$.
2. Prove that for a surjective function $f : A \twoheadrightarrow B$, the direct image function $\overrightarrow{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ is surjective.
3. Show that any function $f : A \to B$ can be decomposed into an injection and a surjection: that is, there exists a set X, a surjection $s : A \twoheadrightarrow X$ and an injection $i : X \rightarrowtail B$ such that $f = i \circ s$.
4. For a relation $R : A \nrightarrow B$, prove that

   a) $\overrightarrow{R}\left(\bigcup \mathcal{F}\right) = \bigcup \left\{ \overrightarrow{R}(X) \,\middle|\, X \in \mathcal{F} \right\}$ for all $\mathcal{F} \subseteq \mathcal{P}(A)$

   b) $\overleftarrow{R}\left(\bigcap \mathcal{G}\right) = \bigcap \left\{ \overleftarrow{R}(Y) \,\middle|\, Y \in \mathcal{G} \right\}$ for all $\mathcal{G} \subseteq \mathcal{P}(B)$

5. Show that, by the inverse image, every map $A \to B$ induces a *Boolean algebra map* $\mathcal{P}(B) \to \mathcal{P}(A)$. That is, for every function $f : A \to B$, its inverse image preserves set operations:
   - $\overleftarrow{f}(\emptyset) = \emptyset$
   - $\overleftarrow{f}(B) = A$
   - $\overleftarrow{f}(X \cup Y) = \overleftarrow{f}(X) \cup \overleftarrow{f}(Y)$
   - $\overleftarrow{f}(X \cap Y) = \overleftarrow{f}(X) \cap \overleftarrow{f}(Y)$
   - $\overleftarrow{f}(X^{\mathrm{c}}) = \left(\overleftarrow{f}(X)\right)^{\mathrm{c}}$

13. On countability

13.1. Basic exercises

1. Prove that every finite set is countable.
2. Demonstrate that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable sets.

13.2. Core exercises

1. Let A be an infinite subset of $\mathbb{N}$. Show that $A \cong \mathbb{N}$. *Hint*: Adapt the argument shown in the proof of Proposition 144, showing that the map $\mathbb{N} \to A$ is both injective and surjective.
2. For an infinite set A, prove that the following are equivalent:
   a) There is a bijection $\mathbb{N} \xrightarrow{\cong} A$.
   b) There is a surjection $\mathbb{N} \twoheadrightarrow A$.
   c) There is an injection $A \rightarrowtail \mathbb{N}$.
3. Prove that:
   a) Every subset of a countable set is countable.
   b) The product and disjoint union of countable sets is countable.
4. For a set A, prove that there is no injection $\mathcal{P}(A) \rightarrowtail A$.

13.3. Optional advanced exercise

1. Prove that if A and B are countable sets then so are A^*, $\mathcal{P}_{\mathrm{fin}}(A)$ and $\mathrm{PFun}_{\mathrm{fin}}(A, B)$.