Discrete Mathematics

Supervision 7

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9. On bijections

9.1. Basic exercises

1. a) Define a function that has (i) none, (ii) exactly one, and (iii) more than one retraction.

   b) Define a function that has (i) none, (ii) exactly one, and (iii) more than one section.

2. Let n be an integer.

   a) How many sections are there for the absolute-value map $x \mapsto |x| : [-n..n] \to [0..n]$?

   b) How many retractions are there for the exponential map $x \mapsto 2^x : [0..n] \to [0..2^n]$?

3. Give an example of two sets A and B and a function $f : A \to B$ such that f has a retraction but no section. Explain how you know that f has these properties.

4. Prove that the identity function is a bijection and that the composition of bijections is a bijection.

5. For $f : A \to B$, prove that if there are $g, h : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ then $g = h$. Conclude as a corollary that, whenever it exists, the inverse of a function is unique.

9.2. Core exercises

1. We say that two functions $s : A \to B$ and $r : B \to A$ are a *section-retraction pair* whenever $r \circ s = \mathrm{id}_A$; and that a function $e : B \to B$ is an *idempotent* whenever $e \circ e = e$. This question demonstrates that section-retraction pairs and idempotents are closely connected: any section-retraction pair gives rise to an idempotent function, and any idempotent function can be split into a section-retraction pair.

   a) Let $f : C \to D$ and $g : D \to C$ be functions such that $f \circ g \circ f = f$.

      (i) Can you conclude that $f \circ g$ is idempotent? What about $g \circ f$? Justify your answers.

      (ii) Define a map g' using f and g that satisfies both

      $$f \circ g' \circ f = f \qquad \text{and} \qquad g' \circ f \circ g' = g'$$

   b) Show that if $s : A \to B$ and $r : B \to A$ are a section-retraction pair then the composite $s \circ r : B \to B$ is idempotent.

   c) Show that for every idempotent $e : B \to B$ there exists a set A (called a *retract* of B) and a section-rectraction pair $s : A \to B$ and $r : B \to A$ such that $s \circ r = e$.

10. On equivalence relations

10.1. Basic exercises

1. Prove that the isomorphism relation $\cong$ between sets is an equivalence relation.
2. Prove that the identity relation id_A on a set A is an equivalence relation, and that $A/\mathrm{id}_A \cong A$.
3. Show that, for a positive integer m, the relation $\equiv_m$ on $\mathbb{Z}$ given by

$$x \equiv_m y \iff x \equiv y \pmod{m}$$

   is an equivalence relation. What are the equivalence classes of this relation?
4. Show that the relation $\equiv$ on $\mathbb{Z} \times \mathbb{Z}^+$ given by

$$(a,b) \equiv (x,y) \iff a \cdot y = x \cdot b$$

   is an equivalence relation. What are the equivalence classes of this relation?

10.2. Core exercises

1. Let E_1 and E_2 be two equivalence relations on a set A. Either prove or disprove the following statements.
   a) $E_1 \cup E_2$ is an equivalence relation on A.
   b) $E_1 \cap E_2$ is an equivalence relation on A.
2. For an equivalence relation E on a set A, show that $[a_1]_E = [a_2]_E$ iff $a_1 \mathrel{E} a_2$, where

$$[a]_E = \{\, x \in A \mid x \mathrel{E} a \,\}.$$

3. For a function $f : A \to B$ define a relation $\equiv_f$ on A by the rule: for all $a, a' \in A$,

$$a \equiv_f a' \iff f(a) = f(a')$$

   a) Show that for every function $f : A \to B$, the relation $\equiv_f$ is an equivalence relation on A.
   b) Prove that every equivalence relation E in a set A is equal to $\equiv_q$, where $q : A \twoheadrightarrow A/E$ is the quotient function $q(a) = [a]_E$.
   c) Prove that for every surjection $f : A \twoheadrightarrow B$,

$$B \cong \big(A/\equiv_f\big)$$