Discrete Mathematics

Supervision 6

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6. On relations

6.1. Basic exercises

1. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $C = \{x, y, z\}$.
   Let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} : A \nrightarrow B$
   and $S = \{(b, x), (b, y), (c, y), (d, z)\} : B \nrightarrow C$.

   Draw the internal diagrams of the relations. What is the composition $S \circ R : A \nrightarrow C$?

2. Prove that relational composition is associative and has the identity relation as the neutral element.

3. For a relation $R : A \nrightarrow B$, let its *opposite*, or *dual relation*, $R^{\mathrm{op}} : B \nrightarrow A$ be defined by:

   $$b\, R^{\mathrm{op}}\, a \iff a\, R\, b$$

   For $R, S : A \nrightarrow B$ and $T : B \nrightarrow C$, prove that:

   a) $R \subseteq S \implies R^{\mathrm{op}} \subseteq S^{\mathrm{op}}$

   b) $(R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} \cap S^{\mathrm{op}}$

   c) $(R \cup S)^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}}$

   d) $(T \circ S)^{\mathrm{op}} = S^{\mathrm{op}} \circ T^{\mathrm{op}}$

6.2. Core exercises

1. Let $R, R' \subseteq A \times B$ and $S, S' \subseteq B \times C$ be two pairs of relations and assume $R \subseteq R'$ and $S \subseteq S'$. Prove that $S \circ R \subseteq S' \circ R'$.

2. Let $\mathcal{F} \subseteq \mathcal{P}(A \times B)$ and $\mathcal{G} \subseteq \mathcal{P}(B \times C)$ be two collections of relations from A to B and from B to C, respectively. Prove that

   $$\left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right) = \bigcup \{ S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G} \} : A \nrightarrow C$$

   Recall that the notation $\{ S \circ R : A \nrightarrow C \mid R \in \mathcal{F}, S \in \mathcal{G} \}$ is common syntactic sugar for the formal definition $\{ T \in \mathcal{P}(A \times C) \mid \exists R \in \mathcal{F}.\ \exists S \in \mathcal{G}.\ T = S \circ R \}$. Hence,

   $$T \in \{ S \circ R \in A \nrightarrow C \mid R \in \mathcal{F}, S \in \mathcal{G} \} \iff \exists R \in \mathcal{F}.\ \exists S \in \mathcal{G}.\ T = S \circ R$$

   What happens in the case of big intersections?

3. Suppose R is a relation on a set A. Prove that

   a) R is reflexive iff $\mathrm{id}_A \subseteq R$

   b) R is symmetric iff $R = R^{\mathrm{op}}$

c) R is transitive iff $R \circ R \subseteq R$

d) R is antisymmetric iff $R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_A$

4. Let R be an arbitrary relation on a set A, for example, representing an undirected graph. We are interested in constructing the smallest transitive relation (graph) containing R, called the *transitive closure* of R: a relation $\mathrm{Cl}_t[R]$ that satisfies ① $R \subseteq \mathrm{Cl}_t[R]$; ② $\mathrm{Cl}_t[R]$ is transitive; and ③ $\mathrm{Cl}_t[R]$ is the smallest such relation.

   a) We define the family of relations which are transitive supersets of R:

   $$\mathcal{T}_R \triangleq \{ Q : A \nrightarrow A \mid R \subseteq Q \text{ and } Q \text{ is transitive} \}$$

   R is not necessarily going to be an element of this family, as it might not be transitive. However, R is a *lower bound* for $\mathcal{T}_R$, as it is a subset of every element of the family.

   Prove that the set $\bigcap \mathcal{T}_R$ is the transitive closure for R.

   b) $\bigcap \mathcal{T}_R$ is the intersection of an infinite number of relations so it's difficult to compute the transitive closure this way. A better approach is to start with R, and keep adding the missing connections until we get a transitive graph. This can be done by repeatedly composing R with itself: after n compositions, all paths of length n in the graph represented by R will have a transitive connection between their endpoints.

   Prove that the (at least once) iterated composition $R^{\circ+} \triangleq R \circ R^{\circ*}$ is the transitive closure for R, i.e. it coincides with the greatest lower bound of $\mathcal{T}_R$: $R^{\circ+} = \bigcap \mathcal{T}_R$. *Hint*: show that $R^{\circ+}$ is both an element and a lower bound of $\mathcal{T}_R$.

7. On partial functions

7.1. Basic exercises

1. Let $A_2 = \{1,2\}$ and $A_3 = \{a,b,c\}$. List the elements of the sets $\mathrm{PFun}(A_i, A_j)$ for $i, j \in \{2,3\}$. *Hint*: there may be quite a few, so you can think of ways of characterising all of them without giving an explicit listing.

2. Prove that a relation $R : A \nrightarrow B$ is a partial function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$.

3. Prove that the identity relation is a partial function, and that the composition of partial functions is a partial function.

7.2. Core exercises

1. Show that $(\mathrm{PFun}(A,B), \subseteq)$ is a partial order. What is its least element, if it exists?

2. Let $\mathcal{F} \subseteq \mathrm{PFun}(A,B)$ be a non-empty collection of partial functions from A to B.

   a) Show that $\bigcap \mathcal{F}$ is a partial function.

   b) Show that $\bigcup \mathcal{F}$ need not be a partial function by defining two partial functions $f, g : A \rightharpoonup B$ such that $f \cup g : A \nrightarrow B$ is a non-functional relation.

c) Let $h: A \rightharpoonup B$ be a partial function. Show that if every element of $\mathcal{F}$ is below h then $\bigcup \mathcal{F}$ is a partial function.

8. On functions

8.1. Basic exercises

1. Let $A_2 = \{1,2\}$ and $A_3 = \{a,b,c\}$. List the elements of the sets $\mathrm{Fun}(A_i, A_j)$ for $i, j \in \{2,3\}$.
2. Prove that the identity partial function is a function, and the composition of functions yields a function.
3. Prove or disprove that $(\mathrm{Fun}(A,B), \subseteq)$ is a partial order.
4. Find endofunctions $f, g : A \to A$ such that $f \circ g \neq g \circ f$.

8.2. Core exercises

1. A relation $R: A \nrightarrow B$ is said to be *total* if $\forall a \in A.\ \exists b \in B.\ a\,R\,b$. Prove that this is equivalent to $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$. Conclude that a relation $R: A \nrightarrow B$ is a function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$ and $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$.
2. Let $\chi : \mathcal{P}(U) \to (U \Rightarrow [2])$ be the function mapping subsets $S \subseteq U$ to their characteristic functions $\chi_S : U \to [2]$.

   a) Prove that for all $x \in U$,

   - $\chi_{A\cup B}(x) = (\chi_A(x) \vee \chi_B(x)) = \max(\chi_A(x), \chi_B(x))$
   - $\chi_{A\cap B}(x) = (\chi_A(x) \wedge \chi_B(x)) = \min(\chi_A(x), \chi_B(x))$
   - $\chi_{A^c}(x) = \neg(\chi_A(x)) = (1 - \chi_A(x))$

   b) For what construction $A?B$ on sets A and B does it hold that

   $$\chi_{A?B}(x) = (\chi_A(x) \oplus \chi_B(x)) = (\chi_A(x) +_2 \chi_B(x))$$

   for all $x \in U$, where $\oplus$ is the *exclusive or* operator? Prove your claim.

8.3. Optional advanced exercise

Consider a set A together with an element $a \in A$ and an endofunction $f : A \to A$.

Say that a relation $R: \mathbb{N} \nrightarrow A$ is (a, f)-*closed* whenever

$$R(0,a) \quad \text{and} \quad \forall n \in \mathbb{N}, x \in A.\ R(n,x) \Longrightarrow R(n+1, f(x))$$

Define the relation $F : \mathbb{N} \nrightarrow A$ as

$$F \triangleq \bigcap \{ R: \mathbb{N} \nrightarrow A \mid R \text{ is } (a,f)\text{-closed} \}$$

a) Prove that F is (a, f)-closed.

b) Prove that F is total, that is: $\forall n \in \mathbb{N}.\ \exists y \in A.\ F(n, y)$.

c) Prove that F is a function $\mathbb{N} \to A$, that is: $\forall n \in \mathbb{N}.\ \exists! y \in A.\ F(n, y)$.

Hint: Proceed by induction. Observe that, in view of the previous item, to show that $\exists! y \in A.\, F(k, y)$ it suffices to exhibit an (a, f)-closed relation R_k such that $\exists! y \in A.\, R_k(k, y)$. (Why?) For instance, as the relation $R_0 = \{ (m, y) \in \mathbb{N} \times A \mid m = 0 \Longrightarrow y = a \}$ is (a, f)-closed one has that $F(0, y) \Longrightarrow R_0(0, y) \Longrightarrow y = a$.

d) Show that if h is a function $\mathbb{N} \to A$ with $h(0) = a$ and $\forall n \in \mathbb{N}.\, h(n+1) = f(h(n))$ then $h = F$.

Thus, for every set A together with an element $a \in A$ and an endofunction $f : A \to A$ there exists a unique function $F : \mathbb{N} \to A$, typically said to be *inductively defined*, satisfying the recurrence relation

$$F(n) = \begin{cases} a & \text{for } n = 0 \\ f(F(n-1)) & \text{for } n \geq 1 \end{cases}$$