Discrete Mathematics

Supervision 5

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5. On sets

5.1. Basic exercises

1. Prove that $\subseteq$ is a partial order, that is, it is:

   a) reflexive: $\forall$ sets $A.\ A \subseteq A$

   b) transitive: $\forall$ sets $A, B, C.\ (A \subseteq B \ \land\ B \subseteq C) \Longrightarrow A \subseteq C$

   c) antisymmetric: $\forall$ sets $A, B.\ (A \subseteq B \ \land\ B \subseteq A) \iff A = B$

2. Prove the following statements:

   a) $\forall$ sets $A.\ \emptyset \subseteq A$

   b) $\forall$ sets $A.\ (\forall x.\ x \notin A) \iff A = \emptyset$

3. Find the union, and intersection of:

   a) $\{1,2,3,4,5\}$ and $\{-1,1,3,5,7\}$

   b) $\{x \in \mathbb{R} \mid x > 7\}$ and $\{x \in \mathbb{N} \mid x > 5\}$

4. Find the Cartesian product and disjoint union of $\{1,2,3,4,5\}$ and $\{-1,1,3,5,7\}$.

5. Let $I = \{2,3,4,5\}$ and for each $i \in I$, let $A_i = \{i, i+1, i-1, 2 \cdot i\}$.

   a) List the elements of all sets A_i for $i \in I$.

   b) Let $\{A_i \mid i \in I\}$ stand for $\{A_2, A_3, A_4, A_5\}$. Find $\bigcup\{A_i \mid i \in I\}$ and $\bigcap\{A_i \mid i \in I\}$.

6. Let U be a set. For all $A, B \in \mathcal{P}(U)$, prove that:

   a) $A^c = B \iff (A \cup B = U \ \land\ A \cap B = \emptyset)$

   b) Double complement elimination: $(A^c)^c = A$

   c) The de Morgan laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$

5.2. Core exercises

1. Prove that for all for all sets U and subsets $A, B \subseteq U$:

   a) $\forall X.\ A \subseteq X \ \land\ B \subseteq X \iff (A \cup B) \subseteq X$ b) $\forall Y.\ Y \subseteq A \land Y \subseteq B \iff Y \subseteq (A \cap B)$

2. Either prove or disprove that, for all sets A and B,

   a) $A \subseteq B \Longrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

   b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

   c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

   d) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$
   e) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$

3. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent.

   a) $A \cup B = B$ b) $A \subseteq B$ c) $A \cap B = A$ d) $B^c \subseteq A^c$

4. For sets A, B, C, D, prove or disprove at least three of the following statements:
   a) $(A \subseteq C \ \wedge\ B \subseteq D) \Longrightarrow A \times B \subseteq C \times D$
   b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$
   c) $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$
   d) $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
   e) $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D)$
5. For sets A, B, C, D, prove or disprove at least three of the following statements:
   a) $(A \subseteq C \ \wedge\ B \subseteq D) \Longrightarrow A \uplus B \subseteq C \uplus D$
   b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$
   c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$
   d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$
   e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$
6. Prove the following properties of the big unions and intersections of a family of sets $\mathcal{F} \subseteq \mathcal{P}(A)$:

   a) $\forall U \subseteq A.\ (\forall X \in \mathcal{F}.\ X \subseteq U) \iff \bigcup \mathcal{F} \subseteq U$

   b) $\forall L \subseteq A.\ (\forall X \in \mathcal{F}.\ L \subseteq X) \iff L \subseteq \bigcap \mathcal{F}$

7. Let A be a set.
   a) For a family $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} \triangleq \{ U \subseteq A \mid \forall S \in \mathcal{F}.\ S \subseteq U \}$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$.
   b) Analogously, define the family $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.

5.3. Optional advanced exercises

1. Prove that for all families of sets $\mathcal{F}_1$ and $\mathcal{F}_2$,

$$\left(\bigcup \mathcal{F}_1\right) \cup \left(\bigcup \mathcal{F}_2\right) = \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2)$$

   State and prove the analogous property for intersections of non-empty families of sets.

2. For a set U, prove that $(\mathcal{P}(U), \subseteq, \cup, \cap, U, \emptyset, (\cdot)^c)$ is a Boolean algebra.