Higher-order algebraic theories

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TallCat seminar
June 2020

First-order theories have operators $\mathsf{op} : s^k \Rightarrow s$

$$\frac{\Gamma \vdash t_1 \qquad \cdots \qquad \Gamma \vdash t_k}{\Gamma \vdash \mathsf{op}(t_1, \ldots, t_k)}$$

Example: monoids have multiplication $(\cdot) : s^2 \Rightarrow s$, unit $\mathsf{e} : 1 \Rightarrow s$

$$\frac{\Gamma \vdash t_1 \qquad \Gamma \vdash t_2}{\Gamma \vdash t_1 \cdot t_2} \qquad\qquad \frac{}{\Gamma \vdash \mathsf{e}}$$

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Non-example: the untyped λ-calculus

$$\frac{\Gamma \vdash t_1 \qquad \Gamma \vdash t_2}{\Gamma \vdash \mathsf{app}(t_1, t_2)} \qquad\qquad \frac{\Gamma, x : s \vdash t}{\Gamma \vdash \mathsf{abs}(x.\, t)}$$

- First-order theories
  - ✓ Presentations/equational logic
  - ✓ Lawvere theories
  - ✓ Finitary monads on $\mathbf{Set}$
  - ✓ Abstract clones, monoids in $[\mathbb{F}, \mathbf{Set}]$
  - ✓ Various constructions, generalizations, metatheory

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First-order presentations

A (monosorted) first-order presentation is a signature with a set of equations, where:

- First-order arities are natural numbers k
- Signatures Σ are sets of operators op with arities
- Contexts $\Gamma = x_1, \ldots, x_n$ are lists of variables
- Terms t are generated by

$$\frac{x \in \Gamma}{\Gamma \vdash x} \qquad \frac{(\mathsf{op} : k) \in \Sigma \qquad \Gamma \vdash t_1 \qquad \cdots \qquad \Gamma \vdash t_k}{\Gamma \vdash \mathsf{op}(t_1, \ldots, t_k)}$$

- Equations $\Gamma \vdash t \equiv t'$

Example: monoids have operators $(\cdot) : 2$ and $e : 0$, and equations

$$x \vdash e \cdot x \equiv x \qquad x \vdash x \equiv x \cdot e$$

$$x_1, x_2, x_3 \vdash (x_1 \cdot x_2) \cdot x_3 \equiv x_1 \cdot (x_2 \cdot x_3)$$

Second-order presentations [Fiore and Hur '10, Fiore and Mahmoud '10]

A (monosorted) second-order presentation is a signature with a set of equations, where:

- Second-order arities are lists $(n_1, \ldots, n_k)$ of natural numbers
- Signatures Σ are sets of operators op with arities
- Variable contexts Γ and metavariable contexts Θ:

$$\Gamma = x_1, \ldots, x_n \qquad \Theta = \mathrm{M}_1 : m_1, \ldots, \mathrm{M}_p : m_p$$

- Terms t are generated by

$$\frac{x \in \Gamma}{\Theta \mid \Gamma \vdash x} \qquad \frac{(\mathrm{M} : m) \in \Theta \qquad \Theta \mid \Gamma \vdash t_1 \qquad \Theta \mid \Gamma \vdash t_m}{\Theta \mid \Gamma \vdash \mathrm{M}(t_1, \ldots, t_m)}$$

$$\frac{(\mathsf{op} : (n_1, \ldots, n_k)) \in \Sigma \qquad \Theta \mid \Gamma, \vec{x_1} \vdash t_1 \qquad \cdots \qquad \Theta \mid \Gamma, \vec{x_k} \vdash t_k}{\Theta \mid \Gamma \vdash \mathsf{op}(\vec{x_1}.\, t_1, \ldots, \vec{x_k}.\, t_k)}$$

- Equations $\Theta \mid \Gamma \vdash t \equiv t'$

Second-order presentations [Fiore and Hur '10, Fiore and Mahmoud '10]

$$\frac{(\mathsf{op} : (n_1, \ldots, n_k)) \in \Sigma \qquad \Theta \mid \Gamma, \vec{x_1} \vdash t_1 \qquad \cdots \qquad \Theta \mid \Gamma, \vec{x_n} \vdash t_n}{\Theta \mid \Gamma \vdash \mathsf{op}(\vec{x_1}.\, t_1, \ldots, \vec{x_n}.\, t_n)}$$

Example: untyped λ-calculus has operators app : $(0, 0)$ and abs : (1)

$$\frac{\Theta \mid \Gamma \vdash t_1 \qquad \Theta \mid \Gamma \vdash t_2}{\Theta \mid \Gamma \vdash \mathsf{app}(t_1, t_2)} \qquad\qquad \frac{\Theta \mid \Gamma, x \vdash t}{\Theta \mid \Gamma \vdash \mathsf{abs}(x.\, t)}$$

and equations

$$\mathrm{M}_1 : 1, \mathrm{M}_2 : 0 \mid \diamond \vdash \mathsf{app}(\mathsf{abs}(x.\, \mathrm{M}_1(x)), \mathrm{M}_2()) \equiv \mathrm{M}_1(\mathrm{M}_2()) \qquad (\beta)$$

$$\mathrm{M} : 0 \mid \diamond \vdash \mathsf{abs}(x.\, \mathsf{app}(\mathrm{M}(), x)) \equiv \mathrm{M}() \qquad (\eta)$$

Moving to higher orders

Both first-order and second-order presentations use part of STLC

- First-order: no functions
- Second-order: only first-order functions (argument is s^m)

Instead of

$$\frac{(\mathsf{op} : (k)) \in \Sigma \qquad \Theta \mid \Gamma, \vec{x} \vdash t}{\Theta \mid \Gamma \vdash \mathsf{op}(\vec{x}.\, t)}$$

have

$$\frac{(\mathsf{op} : (s^k \to s) \Rightarrow s) \in \Sigma \qquad \Theta \mid \Gamma, \overrightarrow{x : s} \vdash t : s}{\Theta \mid \Gamma \vdash \mathsf{op}(\lambda\vec{x}.\, t) : s}$$

Higher-order presentations

Fix a set S of sorts

$$\begin{aligned} A, B ::= {} & s & \operatorname{ord} s &= 0 \\ \mid {} & 1 & \operatorname{ord} 1 &= -1 \\ \mid {} & A_1 \times A_2 & \operatorname{ord}(A_1 \times A_2) &= \max\{\operatorname{ord} A_1, \operatorname{ord} A_2\} \\ \mid {} & A \to B & \operatorname{ord}(A \to B) &= \max\{\operatorname{ord} A + 1, \operatorname{ord} B\} \end{aligned}$$

Definition

For $n \in \mathbb{N} \cup \{\omega\}$, an nth-order signature Σ is:

- a set of operators op
- each with an arity $A \Rightarrow s$ such that $\operatorname{ord} A < n$.

Example ($S = \{s\}$, $n = 2$):

$$\begin{aligned} \Sigma = \{ & \mathsf{app} : s \times s \Rightarrow s, \\ & \mathsf{abs} : (s \to s) \Rightarrow s\} \end{aligned}$$

Higher-order presentations

Given an nth-order signature, generate STLC terms t in the usual way

$$\frac{(\mathsf{op} : A \Rightarrow s) \in \Sigma \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathsf{op}\, t : s}$$

Definition

An nth-order presentation consists of:

- An nth-order signature Σ
- A set of equations

$$x_1 : A_1, \ldots, x_n : A_n \vdash t \equiv t' : s$$

such that $\max\{\mathrm{ord}\, A_1, \ldots, \mathrm{ord}\, A_n\} < n$.

Examples

Monoids are first-order, with $S = \{s\}$

- Operators

$$(\cdot) : s \times s \Rightarrow s \qquad e : 1 \Rightarrow s$$

- Equations

$$x : s, y : s, z : s \ \vdash (x \cdot y) \cdot z \ \equiv \ x \cdot (y \cdot z) \ : \ s$$

$$x : s \vdash e \cdot x \equiv x : s \qquad\qquad x : s \vdash x \cdot e \equiv x : s$$

Examples

Untyped λ-calculus is second-order, with $S = \{s\}$

- Operators

$$\mathsf{app} : s \times s \Rightarrow s \qquad \mathsf{abs} : (s \to s) \Rightarrow s$$

- Equations

$$\begin{aligned} f : s \to s,\ a : s \vdash \quad \mathsf{app}\,(\mathsf{abs}\, f, a) &\equiv f\, a \ : \ s && (\beta) \\ f : s \vdash \ \mathsf{abs}\,(\lambda x : s.\, \mathsf{app}\,(f, x)) &\equiv f \quad : \ s && (\eta) \end{aligned}$$

Examples

Untyped λ-calculus is second-order, with $S = \{s\}$

- Operators

$$\mathsf{app} : s \times s \Rightarrow s \qquad \mathsf{abs} : (s \to s) \Rightarrow s$$

- Equations

$$\begin{aligned} f : s \to s,\ a : s \vdash \quad \mathsf{app}\,(\mathsf{abs}\, f, a) &\equiv f\, a \ : \ s && (\beta) \\ f : s \vdash \ \mathsf{abs}\,(\lambda x : s.\, \mathsf{app}\,(f, x)) &\equiv f \ : \ s && (\eta) \end{aligned}$$

Can do typed λ-calculus with more sorts:

$$s ::= b \mid \mathsf{Fun}(s, s')$$

Examples

Propositional logic/boolean algebras, with $S = \{\mathsf{prop}\}$

- Operators

$$
\begin{aligned}
\top\ \bot &: 1 \Rightarrow \mathsf{prop} && \text{(zeroth-order)}\\
\wedge\ \vee &: \mathsf{prop} \times \mathsf{prop} \Rightarrow \mathsf{prop} && \text{(first-order)}\\
\neg &: \mathsf{prop} \Rightarrow \mathsf{prop} && \text{(first-order)}
\end{aligned}
$$

- Many equations

Examples

First-order logic, with $S = \{\mathsf{prop}, \mathsf{thing}\}$

- Operators

$$\begin{aligned} \top\ \bot &: 1 \Rightarrow \mathsf{prop} && \text{(zeroth-order)} \\ \wedge\ \vee &: \mathsf{prop} \times \mathsf{prop} \Rightarrow \mathsf{prop} && \text{(first-order)} \\ \neg &: \mathsf{prop} \Rightarrow \mathsf{prop} && \text{(first-order)} \\ \forall &: (\mathsf{thing} \to \mathsf{prop}) \Rightarrow \mathsf{prop} && \text{(second-order)} \end{aligned}$$

- Many equations

Examples

Second-order logic, with $S = \{\mathsf{prop}, \mathsf{thing}\}$

- Operators

$$\begin{aligned}
\top\ \bot &: 1 \Rightarrow \mathsf{prop} && \text{(zeroth-order)}\\
\wedge\ \vee &: \mathsf{prop} \times \mathsf{prop} \Rightarrow \mathsf{prop} && \text{(first-order)}\\
\neg &: \mathsf{prop} \Rightarrow \mathsf{prop} && \text{(first-order)}\\
\forall &: (\mathsf{thing} \to \mathsf{prop}) \Rightarrow \mathsf{prop} && \text{(second-order)}\\
\forall_2 &: ((\mathsf{thing} \to \mathsf{prop}) \to \mathsf{prop}) \Rightarrow \mathsf{prop} && \text{(third-order)}
\end{aligned}$$

- Many equations

Formula $\forall P. \forall x. (Px) \vee \neg(Px)$ encoded as

$$\forall_2 \, (\lambda P : \mathsf{thing} \to \mathsf{prop}. \, \forall \, (\lambda x : \mathsf{thing}. \, Px \vee \neg(Px)))$$

Examples

- Parameterized algebraic theories [Staton '13] are two-sorted second-order theories
- Partial differentiation has a monosorted second-order presentation [Plotkin '20]

First-order Lawvere theories

For $S = \{s\}$, first-order arities form a category $\mathcal{A}_1$, which:

- is the opposite of a skeleton of $\mathbf{FinSet}$
- is the free strict cartesian category on S
- has objects s^k for $k \in \mathbb{N}$, morphisms $t : s^k \to s^m$ are STLC terms $x : s^k \vdash t : s^m$ up to $\beta\eta$ (with no operators)

A first-order Lawvere theory is a strict cartesian identity-on-objects functor

$$L : \mathcal{A}_1 \to \mathcal{L}$$

An element of $t \in \mathcal{L}(s^k, s^m)$ "is" a term

$$x : s^k \vdash t : s^m$$

(possibly with operators)

Higher-order Lawvere theories

Category of n-order arities $\mathcal{A}_n$, for $n \in \mathbb{N}_+ \cup \{\omega\}$:

- Objects are some representative subset of types A such that $\operatorname{ord} A < n$, with strict products and exponentials:

$$\begin{gathered} 1 \times A = A = A \times 1 \qquad (A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3) \\ 1 \Rightarrow A = A \qquad A \Rightarrow (A' \Rightarrow A'') = A \times A' \Rightarrow A'' \\ A \Rightarrow 1 = 1 \qquad A \Rightarrow (B_1 \times B_2) = (A \Rightarrow B_1) \times (A \Rightarrow B_2) \end{gathered}$$

- Morphisms $A \to B$ are STLC terms $x : A \vdash t : B$ up to $\beta\eta$

Some facts:

- $\mathcal{A}_{n+1}$ is the "free strict cartesian category on S in which S is exponentiable n times"
- $\mathcal{A}_1$ is the free strict cartesian category on S
- $\mathcal{A}_\omega$ is the free strict CCC on S
- $\mathcal{A}_{n+1}$ has $A \Rightarrow -$ for $A \in \mathcal{A}_n$, corresponding to adding a free variable of type A

Higher-order Lawvere theories

Definition

For $n \in \mathbb{N}_+ \cup \{\omega\}$, an nth-order Lawvere theory is a strict structure-preserving identity-on-objects functor

$$L : \mathcal{A}_n \to \mathcal{L}$$

Morphisms $F : L \to L'$ are commuting triangles

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "F"] & & \mathcal{L}' \\ & \mathcal{A}_n \arrow[ul, "L"] \arrow[ur, "L'"'] & \end{tikzcd}$$

Form a category $\mathbf{Law}_n$.

Theories from presentations

Given an nth-order presentation (Σ, E), have an nth-order Lawvere theory $L : \mathcal{A}_n \to \mathcal{L}$:

- Objects of $\mathcal{L}$ are same as $\mathcal{A}_n$
- Morphisms $t : A \to B$ in $\mathcal{L}$ are terms

$$x : A \vdash t : B$$

  up to equivalence relation generated by E
- $L_{A,B} : \mathcal{A}_n(A, B) \to \mathcal{L}(A, B)$ is inclusion

Coreflective subcategories of theories

$$\begin{tikzcd} \mathbf{Set}^S \arrow[r, shift left, "\lceil - \rceil"] \arrow[r, phantom, "\perp"] & \mathbf{Law}_1 \arrow[l, shift left, "\lfloor - \rfloor"] \end{tikzcd}$$

$\mathbf{Set}^S$ is a coreflective subcategory of $\mathbf{Law}_1$:

- If $L : \mathcal{A}_1 \to \mathcal{L}$ is a first-order Lawvere theory, then

$$\lfloor L \rfloor_s = \mathcal{L}(1, s)$$

- If $X \in \mathbf{Set}^S$, then $\lceil L \rceil$ has nullary operators $\mathrm{op}_x : 1 \Rightarrow s$ for each $x \in X_s$
- $\lceil - \rceil$ is fully faithful

Coreflective subcategories of theories

$$\mathbf{Set}^S \underset{\lfloor - \rfloor}{\overset{\lceil - \rceil}{\underset{\perp}{\rightleftarrows}}} \mathbf{Law}_1 \underset{\lfloor - \rfloor}{\overset{\lceil - \rceil}{\underset{\perp}{\rightleftarrows}}} \cdots \underset{\lfloor - \rfloor}{\overset{\lceil - \rceil}{\underset{\perp}{\rightleftarrows}}} \mathbf{Law}_n \underset{\lfloor - \rfloor}{\overset{\lceil - \rceil}{\underset{\perp}{\rightleftarrows}}} \cdots \underset{\lfloor - \rfloor}{\overset{\lceil - \rceil}{\underset{\perp}{\rightleftarrows}}} \mathbf{Law}_\omega$$

$\mathbf{Law}_n$ is a coreflective subcategory of $\mathbf{Law}_{n'}$ for $n, n' \in \mathbb{N}_+ \cup \{\omega\}$, $n \leq n'$:

- If $L : \mathcal{A}_{n'} \to \mathcal{L}$ is an n'th-order Lawvere theory, then $\lfloor L \rfloor : \mathcal{A}_n \to \lfloor \mathcal{L} \rfloor$ has

$$\lfloor \mathcal{L} \rfloor (A, B) \;=\; \mathcal{L}(A, B)$$

- If $L \in \mathbf{Law}_n$, then $\lceil L \rceil \in \mathbf{Law}_{n'}$ is given by "freely adding some exponentials"
- $\lceil - \rceil$ is fully faithful

Theories from arities

For $S = \{s\}$,

$$p : \mathcal{A}_1^{\mathrm{op}} \simeq \mathbf{FinSet} \hookrightarrow \mathbf{Set}$$

Theories from arities

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$$p : \mathcal{A}_1^{\mathrm{op}} \hookrightarrow \mathbf{Set}^S$$

For $n \in \mathbb{N}_+ \cup \{\omega\}$, define $p : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$ so that

$$pA(B, B') = \mathcal{A}_{n+1}(A \times B, B')$$

- Objects of $A \in \mathcal{A}_{n+1}$ "are" finite nth-order signatures:

$$\begin{pmatrix} (s \times s \Rightarrow s) \times \\ ((s \Rightarrow s) \Rightarrow s) \end{pmatrix} \in \mathcal{A}_3 \quad \text{is} \quad \begin{aligned} \Sigma = \{&\mathsf{app} : s \times s \Rightarrow s, \\ &\mathsf{abs} : (s \to s) \Rightarrow s\} \end{aligned}$$

- p satisfies a universal property:

$$\mathbf{Law}_n(pA, L) \;\cong\; \lceil \mathcal{L} \rceil (1, A)$$

Local presentability

$p : \mathcal{A}_{n+1}^{\mathrm{op}} \longrightarrow \mathbf{Law}_n$ has many useful properties:

- $\mathbf{Law}_n(pA,\ L)\ \cong\ \lceil\mathcal{L}\rceil(1, A)$
- p is fully faithful
- p is dense ($L \mapsto \mathbf{Law}_n(p-, L)$ is fully faithful)
- pA is finitely presentable ($\mathbf{Law}_n(pA, -)$ preserves filtered colimits)

And $\mathbf{Law}_n$ has limits and filtered colimits:

$$(\lim_i \mathcal{L}_i)(A, B) = \lim_i(\mathcal{L}_i(A, B))$$

So:

Theorem

$\mathbf{Law}_n$ *is locally presentable for* $n \in \mathbb{N}_+ \cup \{\omega\}$.

Monad–theory correspondence

(monosorted) first-order Lawvere theories

$\simeq$ finitary monads on $\mathbf{Set}$

Monad–theory correspondence

(monosorted) first-order Lawvere theories

$\simeq$ relative monads on $p : \mathcal{A}_1^{\mathrm{op}} \simeq \mathbf{FinSet} \hookrightarrow \mathbf{Set}$

$\simeq$ finitary monads on $\mathbf{Set}$

Relative monads [Altenkirch, Chapman, Uustalu '10]

Definition

A relative monad T on $J : \mathbf{J} \to \mathbf{C}$ consists of

- An object $TX \in \mathbf{C}$ for each $X \in \mathbf{J}$
- A morphism $\eta_X : JX \to TX$ for each $X \in \mathbf{J}$
- A morphism $f^\dagger : TX \to TY$ for each $f : JX \to TY$

Subject to some laws

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Subject to some laws

If $p : \mathbf{C}_\mathrm{f} \to \mathbf{C}$ is locally finitely presentable, then

$$[\mathbf{C}_\mathrm{f}, \mathbf{C}] \underset{- \circ p}{\overset{\mathrm{Lan}_p}{\underset{\simeq}{\rightleftarrows}}} [\mathbf{C}, \mathbf{C}]_\mathrm{f}$$

extends to an equivalence

relative monads on $p \quad \simeq \quad$ finitary monads on $\mathbf{C}$

Relative monads from theories

Given a monosorted first-order Lawvere theory $L : \mathcal{A}_1 \to \mathcal{L}$, define

$$\begin{array}{rccc} T_L : & \mathcal{A}_1^{\mathrm{op}} & \to & \mathbf{Set} \\ & A & \mapsto & \mathcal{L}(A, s) \end{array}$$

Then

$$\mathbf{Kl}(T_L)(B, A) \;=\; \mathbf{Set}\;(pB, T_L A) \;\cong\; \mathcal{L}(A, B)$$

so T_L forms a relative monad on $p : \mathcal{A}_1^{\mathrm{op}} \to \mathbf{Set}$:

$$\dfrac{\mathrm{id}_A : A \to A \text{ in } \mathcal{L}}{\eta_A : pA \to T_L A \text{ in } \mathbf{Set}} \qquad\qquad \dfrac{\dfrac{F : pB \to T_L A \text{ in } \mathbf{Set}}{A \to B \text{ in } \mathcal{L}}}{F^{\dagger} : T_L B \to T_L A \text{ in } \mathbf{Set}}$$

Relative monads from theories

Given a ~~monosorted~~ first-order Lawvere theory $L : \mathcal{A}_1 \to \mathcal{L}$, define

$$\begin{array}{rccc} T_L : & \mathcal{A}_1^{\mathrm{op}} & \to & \mathbf{Set}^S \\ & A & \mapsto & \mathcal{L}(A, -) \end{array}$$

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$$\frac{\mathrm{id}_A : A \to A \text{ in } \mathcal{L}}{\overline{\eta_A : pA \to T_L A \text{ in } \mathbf{Set}^S}} \qquad\qquad \dfrac{\dfrac{F : pB \to T_L A \text{ in } \mathbf{Set}^S}{\overline{A \to B \text{ in } \mathcal{L}}}}{F^\dagger : T_L B \to T_L A \text{ in } \mathbf{Set}^S}$$

Relative monads from theories

Given an $(n+1)$th-order Lawvere theory $L : \mathcal{A}_{n+1} \to \mathcal{L}$, define

$$
\begin{array}{rcl}
T_L : \mathcal{A}_{n+1}^{\mathrm{op}} & \to & \mathbf{Law}_n \\
T_L A\,(B, B') & = & \mathcal{L}(A \times B,\ B') \\
& \cong & \mathcal{L}(A,\ B \Rightarrow B')
\end{array}
$$

Then

$$
\mathbf{Kl}(T_L)(B, A) \;=\; \mathbf{Law}_n(pB, T_L A) \;\cong\; \mathcal{L}(A, B)
$$

so T_L forms a relative monad on $p : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$:

$$
\frac{\mathrm{id}_A : A \to A \text{ in } \mathcal{L}}{\eta_A : pA \to T_L A \text{ in } \mathbf{Law}_n}
\qquad
\dfrac{\dfrac{F : pB \to T_L A \text{ in } \mathbf{Law}_n}{A \to B \text{ in } \mathcal{L}}}{F^\dagger : T_L B \to T_L A \text{ in } \mathbf{Law}_n}
$$

Monads from theories

Given an $(n+1)$th-order Lawvere theory $L : \mathcal{A}_{n+1} \to \mathcal{L}$, have a fintary monad $\hat{T}_L = \mathrm{Lan}_p T_L : \mathbf{Law}_n \to \mathbf{Law}_n$, where

$$(\hat{T}_L\, L')(B, B') \;\cong\; \int^{A \in \mathcal{A}_{n+1}} \mathcal{L}(A, B \Rightarrow B') \times \lceil \mathcal{L}' \rceil(1, A)$$

If L is the second-order theory of untyped lambda calculus, then $L'' = \hat{T}_L L' \in \mathbf{Law}_1$ freely adds app and abs to L':

$$\begin{aligned} \eta \;&:\; \mathcal{L}'(A,s) \to \mathcal{L}''(A,s) \\ [\![\mathsf{app}]\!] \;&:\; \mathcal{L}''(A,s) \times \mathcal{L}''(A,s) \to \mathcal{L}''(A,s) \\ [\![\mathsf{abs}]\!] \;&:\; \mathcal{L}''(A \times s, s) \to \mathcal{L}''(A,s) \end{aligned}$$

Theories from relative monads

The Kleisli category of a relative monad $T : \mathbf{J} \to \mathbf{C}$ has

- Objects $X \in \mathbf{J}$
- Morphisms $\mathbf{Kl}(T)(X, Y) = \mathbf{C}(JX, TY)$

Theories from relative monads

If $T : \mathcal{A}_1^{\mathrm{op}} \to \mathbf{Set}^S$ is a relative monad on $p : \mathcal{A}_1^{\mathrm{op}} \to \mathbf{Set}^S$, then

$$\begin{array}{rccc} L_T : & \mathcal{A}_1 & \to & \mathbf{Kl}(T)^{\mathrm{op}} \\ & A & \mapsto & A \\ & f & \mapsto & \eta \circ pf \end{array}$$

is a first-order Lawvere theory.

Theories from relative monads

If $T : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$ is a relative monad on $p : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$, then

$$\begin{array}{rccc} L_T : & \mathcal{A}_{n+1} & \to & \mathbf{Kl}(T)^{\mathrm{op}} \\ & A & \mapsto & A \\ & f & \mapsto & \eta \circ pf \end{array}$$

is a strict cartesian identity-on-objects functor.

Want $\mathbf{Kl}(T)(B \Rightarrow B', A) \cong \mathbf{Kl}(T)(B', A \times B)$ for $B \in \mathcal{A}_n$:

$$\frac{p(B \Rightarrow B') \to TA \text{ in } \mathbf{Law}_n}{\overline{pB' \to T(A \times B) \text{ in } \mathbf{Law}_n}}$$

Theories from relative monads

If $T : \mathcal{A}_{n+1}^{\mathrm{op}} \longrightarrow \mathbf{Law}_n$ is a relative monad, and

$$TA + pB \cong T(A \times B) \qquad\qquad (\text{for all } A \in \mathcal{A}_{n+1},\ B \in \mathcal{A}_n)$$

then $L_T : \mathcal{A}_{n+1} \longrightarrow \mathbf{Kl}(T)^{\mathrm{op}}$ is an $(n+1)$th-order Lawvere theory.

If $T = T_L$ for $L \in \mathbf{Law}_{n+1}$:

$$\frac{\dfrac{\dfrac{B' \longrightarrow B'' \text{ in } TA + pB}{B \times B' \longrightarrow B'' \text{ in } TA}}{A \times B \times B' \longrightarrow B'' \text{ in } \mathcal{L}}}{B' \longrightarrow B'' \text{ in } T(A \times B)}$$

Theorem

There are equivalences between

- $(n+1)$*th-order Lawvere theories*
- *Relative monads* T *on* $p : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$ *such that*

$$TA + pB \cong T(A \times B) \qquad \text{(for all } A \in \mathcal{A}_{n+1},\ B \in \mathcal{A}_n\text{)}$$

- *Finitary monads* $T : \mathbf{Law}_n \to \mathbf{Law}_n$ *such that*

$$TL + pB \cong T(L + pB) \qquad \text{(for all } L \in \mathbf{Law}_n,\ B \in \mathcal{A}_n\text{)}$$

Algebras and models

If $L \in \mathbf{Law}_{n+1}$ corresponds to $T : \mathbf{Law}_n \to \mathbf{Law}_n$, there is an equivalence between:

- Cartesian functors $\mathcal{L} \to \mathbf{Set}$
- T-algebras $(M, m : TM \to M)$

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Each cartesian $M : \mathcal{L} \to \mathbf{Set}$ induces a Lawvere theory

$$L_M : \mathcal{A}_n \to \mathcal{L}_M$$

with $\mathcal{L}_M(A, B) = M(A \Rightarrow B)$. For products and exponentials:

$$\mathcal{L}_M(A, \textstyle\prod_i B_i) \;=\; M(\textstyle\prod_i (A \Rightarrow B_i)) \;\cong\; \textstyle\prod_i \mathcal{L}_M(A, B_i)$$
$$\mathcal{L}_M(A, B \Rightarrow B') = M(A \Rightarrow (B \Rightarrow B')) = M(A \times B \Rightarrow B') = \mathcal{L}_M(A \times B, B')$$

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But a model of L in $\mathcal{C}$ is a cartesian-closed functor $M : \mathcal{L} \to \mathcal{C}$?

Conclusions

Have a notion of nth-order Lawvere theory for $n \in \mathbb{N}_+ \cup \{\omega\}$

- Form a chain of coreflective subcategories
- $\mathbf{Law}_n$ is locally presentable
- Equivalences between
  - $(n+1)$th-order Lawvere theories
  - Relative monads T on $p : \mathcal{A}_{n+1}^{\mathrm{op}} \to \mathbf{Law}_n$ such that

$$TA + pB \cong T(A \times B) \qquad \text{(for all } A \in \mathcal{A}_{n+1},\ B \in \mathcal{A}_n\text{)}$$

  - Finitary monads $T : \mathbf{Law}_n \to \mathbf{Law}_n$ such that

$$TL + pB \cong T(L + pB) \qquad \text{(for all } L \in \mathbf{Law}_n,\ B \in \mathcal{A}_n\text{)}$$

all analogous to the first-order case.

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