# On the relation between call-by-value and call-by-name

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# Goal

Suppose we have two semantics for a single language

- e.g. call-by-value and call-by-name

How does replacing one with the other affect the behaviour of programs?

# Goal

- Call-by-value: $(\lambda x.\, e)\, e' \rightsquigarrow_{\mathrm{v}}{}^{*} (\lambda x.\, e)\, v \rightsquigarrow_{\mathrm{v}} e[x \mapsto v] \rightsquigarrow_{\mathrm{v}}{}^{*} \cdots$
- Call-by-name: $(\lambda x.\, e)\, e' \rightsquigarrow_{\mathrm{n}} e[x \mapsto e'] \rightsquigarrow_{\mathrm{n}}{}^{*} \cdots$

# Goal

- Call-by-value: $(\lambda x.\, e)\, e' \leadsto_{\mathrm{v}}{}^{*} (\lambda x.\, e)\, v \leadsto_{\mathrm{v}} e[x \mapsto v] \leadsto_{\mathrm{v}}{}^{*} \cdots$
- Call-by-name: $(\lambda x.\, e)\, e' \leadsto_{\mathrm{n}} e[x \mapsto e'] \leadsto_{\mathrm{n}}{}^{*} \cdots$

If we replace call-by-value with call-by-name, then:

- No side-effects: nothing changes
- Only recursion: behaviour changes

$$\text{CBV:} \quad (\lambda x.\, \mathbf{false})\Omega \leadsto_{\mathrm{v}} (\lambda x.\, \mathbf{false})\Omega \leadsto_{\mathrm{v}} \cdots$$

$$\text{CBN:} \quad (\lambda x.\, \mathbf{false})\Omega \leadsto_{\mathrm{n}} \mathbf{false}$$

but if CBV terminates with result $v$, CBN terminates with $v$

# Goal

- Call-by-value: $(\lambda x.\, e)\, e' \rightsquigarrow_{\mathrm{v}}{}^* (\lambda x.\, e)\, v \rightsquigarrow_{\mathrm{v}} e[x \mapsto v] \rightsquigarrow_{\mathrm{v}}{}^* \cdots$
- Call-by-name: $(\lambda x.\, e)\, e' \rightsquigarrow_{\mathrm{n}} e[x \mapsto e'] \rightsquigarrow_{\mathrm{n}}{}^* \cdots$

If we replace call-by-value with call-by-name, then:

- No side-effects: nothing changes
- Only recursion: behaviour changes, but if CBV terminates with result $v$, CBN terminates with $v$
- Only nondeterminism: behaviour also different, but if CBV can terminate with result $v$, then CBN can also terminate with result $v$
- Mutable state: behaviour changes, we can't say much about how

Questions:

- How can we prove these?
- What properties of the side-effects do we need to prove something?

# How to relate different semantics of the same language

1. Define another language that captures both semantics via two sound and adequate translations $(\!|-|\!)^{\mathrm{v}}, (\!|-|\!)^{\mathrm{n}}$

$$\text{(CBV)} \qquad (\!|e|\!)^{\mathrm{v}} \;\longleftarrow\!\!\dashv\; e \;\longmapsto\; (\!|e|\!)^{\mathrm{n}} \qquad \text{(CBN)}$$

5. For programs (closed, ground expressions) $e$

$$(\!|e|\!)^{\mathrm{v}} \preccurlyeq (\!|e|\!)^{\mathrm{n}}$$

# How to relate different semantics of the same language

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**ON THE RELATION BETWEEN DIRECT AND CONTINUATION SEMANTICS†**


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# How to relate different semantics of the same language

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2. Define maps between the two translations

$$\text{CBV translation of } \tau \underset{\Psi_\tau}{\overset{\Phi_\tau}{\rightleftarrows}} \text{CBN translation of } \tau$$

3. Show that $\Phi$, $\Psi$ satisfy nice properties
4. Relate the two translations of (possibly open) expressions $e$

$$(\!|e|\!)^{\mathrm{v}} \preccurlyeq_{\mathrm{ctx}} \Psi_\tau((\!|e|\!)^{\mathrm{n}}[\Phi_\Gamma])$$

5. For programs (closed, ground expressions) $e$

$$(\!|e|\!)^{\mathrm{v}} \preccurlyeq (\!|e|\!)^{\mathrm{n}}$$

# How to relate different semantics of the same language

To relate CBV and CBN:

1. Call-by-push-value [Levy '99] captures CBV and CBN
2. We can define maps $\Phi_\tau, \Psi_\tau$ using the syntax of CBPV
3. $\Phi$ and $\Psi$:
   - behave nicely wrt the CBV and CBN translations, e.g.

     $$\Phi_{\tau_1 \times \tau_2} (\!| e |\!)^{\mathrm{v}} \;=\; (\Phi_{\tau_1} (\!| \mathbf{fst}\, e |\!)^{\mathrm{v}},\; \Phi_{\tau_2} (\!| \mathbf{snd}\, e |\!)^{\mathrm{v}})$$

   - form Galois connections $\Phi_\tau \dashv \Psi_\tau$ (wrt $\preccurlyeq_{\mathrm{ctx}}$) when side-effects are thunkable
4. (3) implies $(\!| e |\!)^{\mathrm{v}} \preccurlyeq_{\mathrm{ctx}} \Psi_\tau((\!| e |\!)^{\mathrm{n}}[\Phi_\Gamma])$
5. (4) is $(\!| e |\!)^{\mathrm{v}} \preccurlyeq (\!| e |\!)^{\mathrm{n}}$ when $e$ is a program

# Example

For recursion and nondeterminism, define

$$M_1 \preccurlyeq M_2 \quad \Leftrightarrow \quad \forall V.\, M_1 \Downarrow \mathbf{return}\, V \;\Rightarrow\; M_2 \Downarrow \mathbf{return}\, V$$

($\Downarrow$ is evaluation in CBPV)

so $M_1 \preccurlyeq_{\text{ctx}} M_2$ means

$$\forall V.\, C[M_1] \Downarrow \mathbf{return}\, V \;\Rightarrow\; C[M_2] \Downarrow \mathbf{return}\, V$$

for closed, ground contexts $C$

Both side-effects are thunkable, so $\Phi$ and $\Phi$ form Galois connections, so

$$(\!|e|\!)^{\text{v}} \preccurlyeq_{\text{ctx}} \Psi_\tau((\!|e|\!)^{\text{n}}[\Phi_\Gamma])$$

# Example

For programs $e$, we have

$$(\!|e|\!)^{\mathrm{v}} \preccurlyeq (\!|e|\!)^{\mathrm{n}}$$

so

$$\begin{aligned} e \rightsquigarrow_{\mathrm{v}}^{*} v &\iff (\!|e|\!)^{\mathrm{v}} \Downarrow \mathbf{return}\,(\!|v|\!) && \text{(soundness)} \\ &\implies (\!|e|\!)^{\mathrm{n}} \Downarrow \mathbf{return}\,(\!|v|\!) && ((\!|e|\!)^{\mathrm{v}} \preccurlyeq (\!|e|\!)^{\mathrm{n}}) \\ &\iff e \rightsquigarrow_{\mathrm{n}}^{*} v && \text{(adequacy)} \end{aligned}$$

# Call-by-push-value [Levy '99]

Split syntax into values and computations

- Values don't reduce, computations do

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Split syntax into values and computations

- Values don't reduce, computations do

Evaluation order is explicit

- Sequencing of computations:

$$\frac{\Gamma \vdash V : A}{\Gamma \,\underline{\vdash}\, \mathbf{return}\, V : \mathbf{F}\, A} \qquad \frac{\Gamma \,\underline{\vdash}\, M_1 : \mathbf{F}\, A \qquad \Gamma, x : A \,\underline{\vdash}\, M_2 : \underline{C}}{\Gamma \,\underline{\vdash}\, M_1 \;\mathbf{to}\; x.\, M_2 : \underline{C}}$$

- Thunks:

$$\frac{\Gamma \,\underline{\vdash}\, M : \underline{C}}{\Gamma \vdash \mathbf{thunk}\, M : \mathbf{U}\, \underline{C}} \qquad \frac{\Gamma \vdash V : \mathbf{U}\, \underline{C}}{\Gamma \,\underline{\vdash}\, \mathbf{force}\, V : \underline{C}}$$

# Call-by-value and call-by-name

Source language types:

$$\tau ::= \mathbf{1} \mid \mathbf{2} \mid \tau \to \tau'$$

CBV and CBN translations into CBPV:

$$\begin{array}{rcl} \tau & \mapsto & \text{value type } (\!|\tau|\!)^{\mathrm{v}} \\ \mathbf{1} & \mapsto & \mathbf{1} \\ \mathbf{2} & \mapsto & \mathbf{2} \\ (\tau \to \tau') & \mapsto & \mathbf{U}((\!|\tau|\!)^{\mathrm{v}} \to \mathbf{F}\,(\!|\tau'|\!)^{\mathrm{v}}) \end{array} \qquad \begin{array}{rcl} \tau & \mapsto & \text{computation type } (\!|\tau|\!)^{\mathrm{n}} \\ \mathbf{1} & \mapsto & \mathbf{F}\ \mathbf{1} \\ \mathbf{2} & \mapsto & \mathbf{F}\ \mathbf{2} \\ (\tau \to \tau') & \mapsto & ((\mathbf{U}\,(\!|\tau|\!)^{\mathrm{n}}) \to (\!|\tau'|\!)^{\mathrm{n}}) \end{array}$$

$$\Gamma, x : \tau \ \mapsto\ (\!|\Gamma|\!)^{\mathrm{v}}, x : (\!|\tau|\!)^{\mathrm{v}} \qquad \Gamma, x : \tau \ \mapsto\ (\!|\Gamma|\!)^{\mathrm{n}}, x : \mathbf{U}\,(\!|\tau|\!)^{\mathrm{n}}$$

$$\Gamma \vdash e : \tau \ \mapsto\ (\!|\Gamma|\!)^{\mathrm{v}} \,\underline{\vdash}\, (\!|e|\!)^{\mathrm{v}} : \mathbf{F}\,(\!|\tau|\!)^{\mathrm{v}} \qquad \Gamma \vdash e : \tau \ \mapsto\ (\!|\Gamma|\!)^{\mathrm{n}} \,\underline{\vdash}\, (\!|e|\!)^{\mathrm{n}} : (\!|\tau|\!)^{\mathrm{n}}$$

# Call-by-value and call-by-name

Define maps between CBV and CBN:

$$\Gamma \vdash M : \mathbf{F}\,(\!|\tau|\!)^{\mathrm{v}} \quad \mapsto \quad \Gamma \vdash \Phi_\tau M : \quad (\!|\tau|\!)^{\mathrm{n}} \qquad \text{(CBV to CBN)}$$

$$\Gamma \vdash N : \quad (\!|\tau|\!)^{\mathrm{n}} \quad \mapsto \quad \Gamma \vdash \Psi_\tau N : \mathbf{F}\,(\!|\tau|\!)^{\mathrm{v}} \qquad \text{(CBN to CBV)}$$

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Example: for $\tau = \mathbf{1} \rightarrow \mathbf{1}$, we have

$$(\!|\mathbf{1} \rightarrow \mathbf{1}|\!)^{\mathrm{v}} = \mathbf{U}\,(\mathbf{1} \rightarrow \mathbf{F}\,\mathbf{1})$$
$$(\!|\mathbf{1} \rightarrow \mathbf{1}|\!)^{\mathrm{n}} = \mathbf{U}\,(\mathbf{F}\,\mathbf{1}) \rightarrow \mathbf{F}\,\mathbf{1}$$

$$M \quad \overset{\Phi_{\mathbf{1}\rightarrow\mathbf{1}}}{\longmapsto} \quad M \ \mathbf{to}\ f.\,\lambda x.\,\mathbf{force}\,x\ \mathbf{to}\ z.\,z\,\text{`}\,\mathbf{force}\,f$$

$$N \quad \overset{\Psi_{\mathbf{1}\rightarrow\mathbf{1}}}{\longmapsto} \quad \mathbf{return}\,(\mathbf{thunk}\,(\lambda x.\,(\mathbf{thunk}\,\mathbf{return}\,x)\,\text{`}\,N))$$

# Galois connection between CBV and CBN?

Since $\Phi$ and $\Psi$ behave nicely wrt translations, e.g.

$$\Phi_{\tau_1\times\tau_2}(\!|e|\!)^{\mathrm{v}} = (\Phi_{\tau_1}(\!|\mathbf{fst}\,e|\!)^{\mathrm{v}},\ \Phi_{\tau_2}(\!|\mathbf{snd}\,e|\!)^{\mathrm{v}})$$

if $(\Phi_\tau, \Psi_\tau)$ is a Galois connection (adjunction) for each $\tau$, i.e.

$$M \preccurlyeq_{\mathrm{ctx}} \Psi_\tau(\Phi_\tau M) \qquad \Phi_\tau(\Psi_\tau N) \preccurlyeq_{\mathrm{ctx}} N$$

then

$$(\!|e|\!)^{\mathrm{v}} \preccurlyeq_{\mathrm{ctx}} \Psi_\tau((\!|e|\!)^{\mathrm{n}}[\Phi_\Gamma])$$

# Galois connection between CBV and CBN?

These do not always hold!

$$M \;\preccurlyeq_{\text{ctx}}\; \Psi_\tau(\Phi_\tau M) \qquad \Phi_\tau(\Psi_\tau N) \;\preccurlyeq_{\text{ctx}}\; N$$

- Don't hold for: exceptions, mutable state

$$\mathbf{raise} \not\preccurlyeq_{\text{ctx}} \mathbf{return}\,(\dots) = \Psi_{\mathbf{1}\to\mathbf{1}}(\Phi_{\mathbf{1}\to\mathbf{1}}\,\mathbf{raise})$$

$$\big(\diamond \,\underline{\vdash}\, \mathbf{raise} : \mathbf{F}\,(\mathbf{U}\,(\mathbf{1} \to \mathbf{F}\,\mathbf{1}))\big)$$

- Do hold for: no side-effects, recursion, nondeterminism

This is where the side-effects matter

# Galois connection between CBV and CBN?

## Definition (Thunkable [Führmann '99])

A computation $\Gamma \vDash M : \mathbf{F}\,A$ is (lax) *thunkable* if

$$M \ \mathbf{to}\ x.\,\mathbf{return}\,(\mathbf{thunk}\,(\mathbf{return}\,x)) \ \ \preccurlyeq_{\text{ctx}} \ \ \mathbf{return}\,(\mathbf{thunk}\,M)$$

- Essentially: we're allowed to suspend the computation $M$
- Implies $M$ commutes with other computations, is (lax) discardable, (lax) copyable

# Galois connection between CBV and CBN?

**Definition (Thunkable [Führmann '99])**

A computation $\Gamma \vDash M : \mathbf{F}\, A$ is (lax) *thunkable* if

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**Lemma**

*If every computation is thunkable, then* $(\Phi_\tau, \Psi_\tau)$ *is a Galois connection.*

# How to relate call-by-value to call-by-name

If every computation is thunkable then

$$(\!|e|\!)^{\mathrm{v}} \;\; \preccurlyeq_{\mathrm{ctx}} \;\; \Psi_\tau((\!|e|\!)^{\mathrm{n}}[\Phi_\Gamma])$$

for each $e$. (And the converse holds for computations of ground type.)

And if $e$ is a program then

$$(\!|e|\!)^{\mathrm{v}} \preccurlyeq (\!|e|\!)^{\mathrm{n}}$$

# Overview

How to relate two different semantics:

1. Translate from source language to intermediate language
2. Define maps between two translations
3. Relate terms:

$$(\!|e|\!)^{\mathrm{v}} \preccurlyeq_{\mathrm{ctx}} \Psi_{\tau}((\!|e|\!)^{\mathrm{n}}[\Phi_{\Gamma}])$$

- Works for call-by-value and call-by-name
- Also works for other things like comparing direct and continuation-style semantics [Reynolds '74], strict and lazy products, etc.