Flexible presentations of graded monads

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Joint work with Shin-ya Katsumata, Tarmo Uustalu and Nicolas Wu

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Motivation

- 1. Effects can be modelled using monads
- 2. which often come from presentations
- 3. which induce algebraic operations

[Moggi '89] [Plotkin and Power '02]

[Plotkin and Power '03]

Example:

- 1. Nondeterministic computations can be modelled using the free monoid monad List
- 2. which comes from the presentation of monoids

fail: 0 or: 2 or(fail, x) = x = or(x, fail) or(or(x, y), z) = or(x, or(y, z))

3. which induces algebraic operations

$$fail_X = (\lambda _. []) : 1 \to \text{List} X$$
$$or_X = (\lambda(xs, ys). xs + ys) : \text{List} X \times \text{List} X \to \text{List} X$$

Motivation

1. Effects with quantitative information can be modelled using graded monads

[Katsumata '14]

2. which often come from graded presentations?

[Smirnov '08, Milius et al. '15, Dorsch et al. '19, Kura '20]

3. which induce algebraic operations?

Develop a notion of flexibly graded presentation for graded monads

Each flexibly graded presentation (Σ, E) induces

- 1. a flexibly graded (abstract) clone of terms
- 2. hence an $[\mathbb{E}, Set]$ -monad on GSet
- 3. hence a graded monad

Graded monoids

- A graded monoid A is
 - ▶ a functor $A : \mathbb{N}_{\leq} \rightarrow \mathbf{Set}$
 - with an element $u \in A0$
 - and a natural transformation

$$m_{d_1,d_2}: Ad_1 \times Ad_2 \rightarrow A(d_1 + d_2)$$

such that

$$m_{0,d}(u,x) = x = m_{d,0}(x,u) \qquad m_{d_1+d_2,d_3}(m_{d_1,d_2}(x,y),z) = m_{d_1,d_2+d_3}(x,m_{d_2,d_3}(y,z))$$

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A morphism $f : A \to A'$ of grade $e \in \mathbb{N}$ is a natural transformation

$$f: A \Rightarrow A'(-\cdot e)$$

preserving the structure:

$$f_0(u) = u' \qquad f_{d_1+d_2}(m_{d_1,d_2}(x,y)) = m'_{d_1,d_2}(f_{d_1}(x),f_{d_2}(y))$$

So we get a $[\mathbb{N}_{\leq}, \mathbf{Set}]\text{-category }\mathbf{GMon}, \text{ and } U:\mathbf{GMon} \to \mathbf{GSet}$

Grading via [E, Set]-categories

Let $(\mathbb{E},1,\cdot)$ be a small strict monoidal category of grades

- ▶ for example \mathbb{N}_{\leq} with multiplication
- ▶ $[\mathbb{E}, Set]$ is a monoidal category with Day convolution
- ▶ we work in $[\mathbb{E}, Set]$ -CAT

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The $[\mathbb{E}, Set]$ -category GSet is $[\mathbb{E}, Set]$ enriched over itself:

- objects are graded sets $X : \mathbb{E} \to \mathbf{Set}$
- morphisms of grade e (elements of $\mathbf{GSet}(X, Y)e$) are natural transformations

$$f: X \Longrightarrow Y(-\cdot e)$$

• identities $id_X \in \mathbf{GSet}(X, X)1$

composition

$$(g \circ f) : X \xrightarrow{f} Y(- \cdot e) \xrightarrow{g_{- \cdot e}} Z(- \cdot e \cdot e')$$

where $f \in \mathbf{GSet}(X, Y)e$ and $g \in \mathbf{GSet}(Y, Z)e'$

Relative monads in $[\mathbb{E}, Set]$ -CAT

A $(J : \mathcal{A} \to C)$ -relative monad T is:

[Altenkirch, Chapman, Uustalu '15]

- ▶ a function $T : |\mathcal{A}| \to |C|$
- with a morphism $\eta_X : JX \to TX$ for each $X \in |\mathcal{A}|$
- and a natural transformation

$$(-)^{\dagger}: \mathcal{C}(JX, TY) \Rightarrow \mathcal{C}(TX, TY)$$

for each $X, Y \in |\mathcal{A}|$

such that

$$f^{\dagger} \circ \eta_X = f$$
 $\eta_X^{\dagger} = \mathrm{id}_X$ $(g^{\dagger} \circ f)^{\dagger} = g^{\dagger} \circ f^{\dagger}$

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T has an Eilenberg-Moore [\mathbb{E} , Set]-category, and a forgetful [\mathbb{E} , Set]-functor $U_T : \mathbf{EM}(T) \to C$ Relative monads in $[\mathbb{E}, Set]$ -CAT

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We rely heavily on some general results about relative monads (jww Nathanael Arkor)

Graded monads

For each set X, define

 $JX = \mathbb{E}(1, -) \bullet X : \mathbb{E} \to \mathbf{Set}$

so that

 $\mathbf{GSet}(JX, A)e \cong \mathbf{Set}(X, Ae)$

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and form a fully faithful [\mathbb{E}, Set]-functor
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 $J: \mathbf{RSet} \to \mathbf{GSet}$

(RSet is the free [E, Set]-category on Set)

Definition: an \mathbb{E} -graded monad (on Set) is a *J*-relative monad (This is equivalent to the definitions in [Smirnov '08, Melliès '12, Katsumata '14])

Flexibly graded clones

For each finite sequence $d_1, \ldots, d_n \in |\mathbb{E}|$, define

$$K(d_1,\ldots,d_n) = \coprod_i \mathbb{E}(d_i,-) : \mathbb{E} \to \mathbf{Set}$$

so that

$$\mathbf{GSet}(K(d_1,\ldots,d_n),A)e \cong \prod_i A(d_i \cdot e)$$

and form a fully faithful $[\mathbb{E}, Set]$ -functor

 $K : \mathbf{FCtx} \to \mathbf{GSet}$

is a fully faithful $[\mathbb{E}, Set]$ -functor

A flexibly graded (abstract) clone is a K-relative monad

Flexibly graded clones

Explicitly, a flexibly graded clone T:

- ▶ maps (d_1, \ldots, d_n) to a graded set $T(d_1, \ldots, d_n) : \mathbb{E} \to \text{Set}$ (of terms)
- ▶ has tuples var $\in \prod_i T(d_1, ..., d_n)d_i$ (the variables)

corresponding to $\eta: K(d_1, \ldots, d_n) \to T(d_1, \ldots, d_n)$

has natural transformations (substitution)

subst :
$$T(d_1, \ldots, d_n)d' \times \prod_i T \Gamma(d_i \cdot e) \to T \Gamma(d' \cdot e)$$

corresponding to

$$(-)^{\dagger}$$
: **GSet** $(K(d_1,\ldots,d_n),T\Gamma) e \rightarrow$ **GSet** $(T(d_1,\ldots,d_n),T\Gamma) e$

Flexibly graded presentations

 (Σ, E) consists of

- ▶ sets of operators op $\in \Sigma(d_1, ..., d_n; d')$
- ► sets of equations $(t, u) \in E(d_1, ..., d_n; d')$ where $t, u \in \text{Tm}_{\Sigma}(d_1, ..., d_n)d'$

with the sets $\operatorname{Tm}_{\Sigma} \Gamma d'$ of terms over Σ generated inductively by:

•
$$\operatorname{var}_i \in \operatorname{Tm}_{\Sigma}(d_1, \dots, d_n)d_i$$

for each *i*

• op
$$(e; t_1, \ldots, t_n) \in \operatorname{Tm}_{\Sigma} \Gamma(d' \cdot e)$$

for each op $\in \Sigma(d_1, \ldots, d_n; d')$, $e \in |\mathbb{E}|$, $t \in \prod_i \operatorname{Tm}_{\Sigma} \Gamma(d_i \cdot e)$

►
$$\zeta^* t \in \operatorname{Tm}_{\Sigma} \Gamma d''$$

for each $t \in \operatorname{Tm}_{\Sigma} \Gamma d'$, $\zeta \in \mathbb{E}(d', d'')$

E induces an equivalence relation \equiv on terms, and

$$\operatorname{Tm}_{(\Sigma,E)}(d_1,\ldots,d_n)d' = \operatorname{Tm}_{(\Sigma,E)}(d_1,\ldots,d_n)d' / \equiv$$

forms a flexibly graded clone $Tm_{(\Sigma, E)}$

Presenting graded monoids

Grades:

$$\mathbb{E}=(\mathbb{N}_{\leq},1,\cdot)$$

Operators:

$$u \in \Sigma(; 0)$$
 $m_{d_1, d_2} \in \Sigma(d_1, d_2; (d_1 + d_2))$ (for each $d_1, d_2 \in \mathbb{N}$)

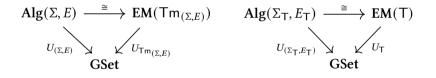
Equations:

$$\begin{split} m_{0,d}(1;u,\mathsf{var}_1) &= \mathsf{var}_1\\ \mathsf{var}_1 &= m_{d,0}(1;\mathsf{var}_1,u)\\ m_{d_1+d_2,d_3}(1;m_{d_1,d_2}(1;\mathsf{var}_1,\mathsf{var}_2),\mathsf{var}_3) &= m_{d_1,d_2+d_3}(1;\mathsf{var}_1,m_{d_2,d_3}(1;\mathsf{var}_2,\mathsf{var}_3))\\ m_{d_1',d_2'}(1;(d_1 \leq d_1')^*\mathsf{var}_1,(d_2 \leq d_2')^*\mathsf{var}_2) &= ((d_1 + d_2) \leq (d_1' + d_2'))^*(m_{d_1,d_2}(1;\mathsf{var}_1,\mathsf{var}_2))\\ m_{d_1,d_2}(d;\mathsf{var}_1,\mathsf{var}_2) &= m_{d_1\cdot e,d_2\cdot e}(1;\mathsf{var}_1,\mathsf{var}_2) \end{split}$$

There is an equivalence

flexibly graded presentations
$$\xrightarrow[(\Sigma_T, E_T) \leftrightarrow T]{}$$
 flexibly graded clones

satisfying



K as a cocompletion

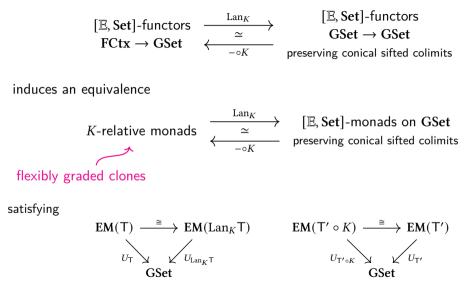
- \blacktriangleright A small category $\mathbb I$ is sifted when $\mathbb I\text{-colimits}$ commute with finite products in Set
- A conical sifted colimit in an $[\mathbb{E}, Set]$ -category C is a conical colimit of a sifted diagram $\mathbb{I} \to \underline{C}$
- $K : \mathbf{FCtx} \to \mathbf{GSet}$ is the free completion of \mathbf{FCtx} under conical sifted colimits: If C has conical sifted colimits, then

$$\begin{array}{ccc} [\mathbb{E}, \mathbf{Set}]\text{-functors} & \xrightarrow{\operatorname{Lan}_K} & [\mathbb{E}, \mathbf{Set}]\text{-functors} \\ & & & & \\ & & &$$

because

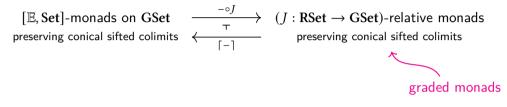
 $\operatorname{Lan}_{K}FX \cong \operatorname{Lan}_{\underline{K}}\underline{F}X$ and $\underline{K}: \mathbb{E}^{*} \to [\mathbb{E}, \operatorname{Set}]$ is a completion under sifted colimits

The equivalence



Constructing a graded monad

There is an adjunction



with a functor $R_{\mathsf{T}'}: \mathsf{EM}(\mathsf{T}') \to \mathsf{EM}(\mathsf{T}' \circ J)$ for each [\mathbb{E} , Set]-monad T' , satisfying



 $R_{T'}$ is not in general an isomorphism

▶ there is no graded monad T'' such that $EM(T'') \cong GMon$ over GSet

Presenting graded monads

Theorem

For every flexibly graded presentation (Σ, E) , there is

- ► a graded monad $T_{(\Sigma,E)}$
- ▶ and functor $R_{(\Sigma,E)}$: Alg (Σ, E) → EM $(\mathsf{T}_{(\Sigma,E)})$ over GSet

such that

For every graded monad T' and functor R' : Alg(Σ, E) → EM(T') over GSet, there is a unique α : T' → T_(Σ,E) such that

$$\operatorname{Alg}(\Sigma, E) \xrightarrow{R_{(\Sigma, E)}} \operatorname{EM}(\mathsf{T}_{(\Sigma, E)}) \qquad \operatorname{T}_{(\Sigma, E)}$$

$$\xrightarrow{R'} \qquad \qquad \downarrow^{\operatorname{EM}(\alpha)} \qquad \qquad \uparrow^{\alpha}_{i}$$

$$\operatorname{EM}(\mathsf{T}') \qquad \qquad \mathsf{T}'$$

► the free $T_{(\Sigma,E)}$ -algebra on a set X is the free (Σ, E) -algebra on $\mathbb{E}(1, -) \bullet X$