Flexibly graded monads and graded algebras

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Computational effects are often modelled, following Moggi [8, 9], using (strong) monads. Every monad T induces a class of algebraic structures (the T-*algebras*), and the monads we typically use to model computational effects have the important (informal) property that the corresponding algebraic structures have a natural meaning for programming. Conversely, algebraic structures give rise to monads, often via *presentations* [11]. For example, the algebras of the list monad on **Set** for finite nondeterminism are monoids, and the list monad arises from the standard presentation of monoids. The unit and multiplication of a monoid correspond to nullary and binary nondeterministic choice. The structure of the *free* T-algebras is especially important, since this provides *algebraic operations* in the sense of Plotkin and Power [12]. These provide the interpretations of effect-causing program constructs.

This work is about the analogous situation for *graded* monads [13, 7, 3], which track quantitative information in models of computational effects [3, 10]. It turns out that grading changes the algebra story significantly. Several computationally natural graded algebraic structures, such as graded monoids, are not the algebras of *any* graded monad. Despite this, for a given class of graded algebraic structures, there usually *is* a graded monad for which the free algebras—the algebras we care most about—form elements of this class. In fact, there is usually a *canonical* graded monad with this property. For example, the notion of graded monoid canonically induces the graded list monad, along with a graded monoid structure on each of the free algebras. The graded monoid structure provides interpretation of nondeterministic computations, and is similar to the monoid structure on each of the free algebras of the list monad.

We make three related contributions. First, we explain the phenomenon outlined in the previous paragraph, in particular by introducing a notion of *flexibly graded monad*. It is the case that the graded algebraic structures we consider are algebras for flexibly graded monads, and every flexibly graded monad induces a canonical (ordinary) graded monad. The graded list monad, and the graded monoid structure of the free algebras, arise in this way. Second, we use this construction to show that certain graded algebraic structures, such as graded monoids, are not the algebras for any graded monad. Finally, we carry out the development using *locally graded categories* [14, 5], an instance of enriched categories, showing that they provide a useful setting for work on grading. We show that both ordinary and flexibly graded monads are special cases of enriched (relative) monads [1]. This enables us to import results about enrichment and relative monads in general and apply them to grading; we rely heavily on these results here.

This is an extended abstract of an MPC 2022 paper [6]. Part of our motivation is to provide a theoretical basis for a notion of *flexibly graded presentation* suitable for computational effects; we describe this notion of presentation in an ICFP 2022 paper [4].

Locally graded categories We assume a small strict monoidal category $(\mathbb{E}, 1, \cdot)$. The *grades* $e \in |\mathbb{E}|$ quantify the effect of computations; the morphisms provide subgrading useful for overapproximation, the unit 1 is the grade of a just-returning computation, and the multiplication $\cdot$ provides the grade of a sequence of two computations. For example, the grades could be natural numbers upper-bounding the number of outcomes from a nondeterministic computation; then $(\mathbb{E}, 1, \cdot)$ would be natural numbers with their usual ordering and multiplication. A *locally graded category* [14, 5] $\mathcal{C}$ is a category enriched over $[\mathbb{E}, \mathbf{Set}]$ with the Day convolution monoidal structure. Explicitly, $\mathcal{C}$ has a class $|\mathcal{C}|$ of objects, and sets $\mathcal{C}(X,Y)e$ of *morphisms from* X *to* Y *of grade* e (where $X, Y \in |\mathcal{C}|$ and $e \in |\mathbb{E}|$); we write $f : X \mathrel{-e\!\!\rightarrow} Y$ to indicate $f \in \mathcal{C}(X,Y)e$. Composition involves the monoidal structure of $\mathbb{E}$. Enriched category theory provides a notion of *functor* between locally graded categories. The locally graded category $\mathbf{GSet}_{\mathbb{E}}$ has as objects the *graded sets*, i.e. functors $\mathbb{E} \to \mathbf{Set}$, and as morphisms $X \mathrel{-e\!\!\rightarrow} Y$ the natural transformations $X \Rightarrow Y(- \cdot e)$. Another locally graded category $\mathbf{RSet}_{\mathbb{E}}$ whose objects are sets embeds into $\mathbf{GSet}_{\mathbb{E}}$ via a functor K that sends a set X to the graded set $\mathbb{E}(1,-) \times X$.

Relative monads *Relative monads* [1] are similar to monads, but with restricted domain. For $J : \mathcal{J} \to \mathcal{C}$ a functor between locally graded categories, a J*-relative monad* T consists of a function $T : |\mathcal{J}| \to |\mathcal{C}|$, *unit* morphisms $\eta_X : JX \mathrel{-1\!\!\rightarrow} TX$, and a *Kleisli extension* operator that maps morphisms $f : JX \mathrel{-e\!\!\rightarrow} TY$ to morphisms $f^\dagger : TX \mathrel{-e\!\!\rightarrow} TY$, satisfying some laws. Many of the usual constructions on monads adapt to relative monads. There is a notion of T-*algebra*, forming a locally graded category $\mathbf{EM}(\mathsf{T})$, together with *relative adjoint* free and forgetful functors $\mathcal{J} \xrightarrow{F_\mathsf{T}} \mathbf{EM}(\mathsf{T}) \xrightarrow{U_\mathsf{T}} \mathcal{C}$. The free algebra $F_\mathsf{T} X$ has TX as carrier. Relative monads are completely determined by their algebras, in the sense that every isomorphism $\mathbf{EM}(\mathsf{T}') \cong \mathbf{EM}(\mathsf{T})$ over $\mathcal{C}$ (i.e. commuting with the forgetful functors) induces an isomorphism $\mathsf{T} \cong \mathsf{T}'$. Graded monads are one instance: K-relative monads are graded monads (on $\mathbf{Set}$), and their algebras are graded algebras in the sense of [2]. Another is flexibly graded monads, which by definition are $\mathrm{Id}_{\mathbf{GSet}_{\mathbb{E}}}$-relative monads (monads on $\mathbf{GSet}_{\mathbb{E}}$).

Constructing graded monads Graded algebraic structures are often the algebras of a flexibly graded monad, More precisely, if they form a locally graded category $\mathcal{A}$ with a forgetful functor $\mathcal{A} \to \mathbf{GSet}_{\mathbb{E}}$, there is often a (necessarily unique) flexibly graded monad T such that $\mathcal{A} \cong \mathbf{EM}(\mathsf{T})$ over $\mathbf{GSet}_{\mathbb{E}}$. Every flexibly graded T restricts to a graded monad $\lfloor \mathsf{T} \rfloor$, given on objects by $\lfloor T \rfloor X = T(KX)$, and induces a functor $R_\mathsf{T} : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\lfloor \mathsf{T} \rfloor)$. This $\lfloor \mathsf{T} \rfloor$ satisfies a universal property: for every other graded monad T' and functor $R' : \mathbf{EM}(\mathsf{T}) \to \mathbf{EM}(\mathsf{T}')$ over $\mathbf{GSet}_{\mathbb{E}}$, there is a unique morphism $\alpha : \mathsf{T}' \to \lfloor \mathsf{T} \rfloor$ of graded monads such that $R' = \mathbf{EM}(\alpha) \circ R_\mathsf{T}$. Thus, while $\lfloor \mathsf{T} \rfloor$ may not exactly capture $\mathcal{A}$ (usually R_T is not an isomorphism), it is as close as we can get with a graded monad. The functor $F_{\lfloor \mathsf{T} \rfloor}$ factors as $\mathbf{RSet}_{\mathbb{E}} \xrightarrow{F_\mathsf{T}} \mathbf{EM}(\mathsf{T}) \xrightarrow{R_\mathsf{T}} \mathbf{EM}(\lfloor \mathsf{T} \rfloor)$, so free $\lfloor \mathsf{T} \rfloor$-algebras form objects of $\mathcal{A}$ even though general $\lfloor \mathsf{T} \rfloor$-algebras do not.

Example: graded monoids and lists A *graded monoid* is a graded object $A : \mathbb{N}_{\leq} \to \mathbf{Set}$, with an associative natural transformation $m_{e_1,e_2} : Ae_1 \times Ae_2 \to A(e_1 + e_2)$ (multiplication) that has a unit $u \in A0$. Graded monoids form a locally graded category $\mathbf{GMon}$ that forgets into $\mathbf{GSet}_{\mathbb{N}^{\times}_{\leq}}$, where $\mathbb{N}^{\times}_{\leq}$ is natural numbers with multiplication. There is a flexibly graded monad T, similar to the ordinary list monad, whose algebras are graded monoids ($\mathbf{GMon} \cong \mathbf{EM}(\mathsf{T})$ over $\mathbf{GSet}_{\mathbb{E}}$). Informally, TXe contains lists over the graded set X, with elements whose grades sum to at most $e \in \mathbb{N}$. T induces a graded monad $\mathsf{List} = \lfloor \mathsf{T} \rfloor$, which is the standard graded list monad. The elements of $\mathrm{List} Xe = T(KX)e$ are lists over the set X with length at most e, because KX is generated by assigning the grade 1 to elements of X. The above provides a universal property for List, stated in terms of $\mathbf{GMon} \cong \mathbf{EM}(\mathsf{T})$. It also provides a graded monoid structure on each of the free List-algebras; the unit is the empty list $[] \in \mathrm{List}X0$, and the multiplication is concatenation $(+\!\!+) : \mathrm{List}Xe_1 \times \mathrm{List}Xe_2 \to \mathrm{List}X(e_1 + e_2)$. Hence, while no graded monad has graded monoids as algebras (Theorem 4 of the preprint [6]), graded monoids do induce a canonical graded monad, and the structure needed to interpret finite nondeterminism.

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