On the Cartwright-Felleisen-Wadler conjecture

Ohad Kammar
University of Oxford

Dylan McDermott
University of Cambridge

HOPE 2017
Extensible semantics

- Traditional semantics: the meaning of a program changes when we add something to the language
- Extensible semantics [Reynolds ’74, Cartwright and Felleisen ’94]: meaning should be stable under language extension
Extensible semantics

- Traditional semantics: the meaning of a program changes when we add something to the language

- *Extensible* semantics [Reynolds ’74, Cartwright and Felleisen ’94]: meaning should be stable under language extension

Can we do extensible monadic semantics?
Extensible monadic semantics [Wadler '98]

- Given a signature $\Sigma$
  - e.g. $\{\text{read} : 1 \to V, \text{write} : V \to 1\}$
- A monad $T_\varepsilon$ for $\varepsilon \subseteq \Sigma$
  - And a monad morphism $T_\varepsilon \to T_{\varepsilon'}$ for $\varepsilon \subseteq \varepsilon'$

Interpret terms $\Gamma \vdash M : A$ that only use effects in $\varepsilon$ as

$$\varepsilon [M] : [\Gamma] \to T_\varepsilon [A]$$

Adding to $\Sigma$ doesn’t change the semantics of a given program!
This talk: constructing extensible semantics

Given a non-extensible semantics (with monad $T$), can we give an extensible semantics?
This talk: constructing extensible semantics

Given a non-extensible semantics (with monad $T$), can we give an extensible semantics?

Yes: just choose $T_\epsilon = T$.

- Doesn’t help us reason about smaller parts of the language
This talk: constructing extensible semantics

Given a non-extensible semantics (with monad $T$), can we give an extensible semantics?

Yes: just choose $T_\varepsilon = T$.

- Doesn’t help us reason about smaller parts of the language

If $\varepsilon$ is smaller than $\Sigma$ then $T_\varepsilon$ should be “simpler” than $T$.

Example: if $T$ is the state+continuations monad

$$(V \Rightarrow - \Rightarrow R) \Rightarrow V \Rightarrow R$$

$T\{\text{write}\}$ should be

$$(1 + V) \times -$$

(if $|V|, |R| > 1$)
This talk: constructing extensible semantics

Goal
Give a construction that:

▶ Gives us the best possible monads $T_\varepsilon$
▶ Is general: works for as many effects as possible
▶ Constructs a model with the right behaviour:

$$\varepsilon[M] = \varepsilon[N] \iff [M] = [N]$$
Related work

- Cartwright and Felleisen ’94: non-monadic extensible semantics
- Katsumata ’14: give a construction for free monads. Uses a more general notion of effect system
- Kammar ’14: gives a construction for algebraic $T$
  - Based on factorizations of morphisms of Lawvere theories
Factoring monad morphisms

Definition (Factorization system)

A factorization system for the category $\mathcal{C}$ consists of:

- A class $\mathcal{E}$ of morphisms $e : X \rightarrowtail Y$ (" epis")
- A class $\mathcal{M}$ of morphisms $m : X \twoheadrightarrow Y$ (" monos")

such that:

- Every morphism $f : X \rightarrow Y$ can be factored into an epi followed by a mono:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow e & & \downarrow n \\
Z & \xleftarrow{=} & 
\end{array}
$$

- Some other conditions hold
Factoring monad morphisms

Examples of factorization systems

- **Set**: surjections and injections

  \[
  \mathbb{Z} \xrightarrow{|-|} \mathbb{Z} \xrightarrow{=} \mathbb{N}
  \]

  ![Diagram](https://via.placeholder.com/150)

- **ωCpo**: dense epis (closure of image equals domain) and full monos \((n \times \sqsubseteq n y \Rightarrow x \sqsubseteq y)\)

- **Presheaves**: componentwise surjections and componentwise injections
Factoring monad morphisms

**Theorem**

Let \( m : S \to T \) be a strong monad morphism, and factorize \( m \) componentwise:

\[
\begin{array}{c}
SX \\ \\
m_X \\ \\
e_X \\
RX
\end{array}
\quad \begin{array}{c}
= \\ \\
n_X
\end{array}
\begin{array}{c}
TX
\end{array}
\]

If \( \mathcal{E} \) is closed under \( S \) and products then:

- \( R \) is a strong monad
- \( e \) and \( n \) are strong monad morphisms

For every \( \text{op}_S : [A] \to S [B] \) we can define \( \text{op}_R := e_{[B]} \circ \text{op}_S \).
Factoring monad morphisms

**Theorem**

Let $m : S \to T$ be a strong monad morphism, and factorize $m$ componentwise:

$$
\begin{aligned}
SX & \xrightarrow{m_X} TX \\
\downarrow e_X & \quad = \quad \downarrow n_X \\
RX &
\end{aligned}
$$

If $E$ is closed under $S$ and products then:

- $R$ is a strong monad
- $e$ and $n$ are strong monad morphisms

For every $\operatorname{op}_S : [A] \to S [B]$ we can define $\operatorname{op}_R := e_{[B]} \circ \operatorname{op}_S$.

We are given $T$, but what should $S$ and $m$ be?
Using free monads

Want to choose $S$ and $m$ so that $R$:

- Supports exactly the operations in $\varepsilon$
- Behaves like $T$ (i.e. $\varepsilon[\mathcal{M}] = \varepsilon[\mathcal{N}] \iff \mathcal{M} = \mathcal{N}$)
Using free monads

Want to choose \( S \) and \( m \) so that \( R \):

- Supports exactly the operations in \( \varepsilon \)
- Behaves like \( T \) (i.e. \( \varepsilon [M] = \varepsilon [N] \iff [M] = [N] \))

Use the free monad for \( \varepsilon \):

- Epi \( \Rightarrow R \) and \( S \) have exactly the same operations
- Mono \( \Rightarrow R \) behaves like \( T \)?
  - This depends on the factorization system

We need the free monad to preserve epis.
Using free monads

Theorem
Suppose that $\mathcal{C}$ has directed colimits and $F : \mathcal{C} \to \mathcal{C}$ preserves them. If $F$ also preserves $\mathcal{E}$-morphisms then the free monad preserves epis.

Use

$$F = \sum_{(\text{op} : A \to B) \in \mathcal{E}} A \times (B \Rightarrow (\neg))$$

to get the free monad we want

Get $m$ from initiality of the free monad
Examples

In \textbf{Set}: 

- If $T$ is state+continuations:

\[ T_\emptyset = \text{Id} \quad T_{\{\text{read,write}\}} = V \Rightarrow V \times - \]
\[ T_{\{\text{read}\}} = V \Rightarrow - \quad T_{\{\text{write}\}} = (1 + V) \times - \]

- Non-example: can’t get writer+nondeterminism from state+nondeterminism

In \textbf{ωCpo}:

- If $T$ is exceptions+partiality then $T_{\{\text{diverge}\}}$ is partiality

Presheaves:

- If $T$ is local state [Plotkin and Power ’02] then

\[ T_{\{\text{read,write}\}} nX = V^n \Rightarrow V^n \times X^n \]
Correctness

We want $\varepsilon[M] = \varepsilon[N] \iff [M] = [N]$

Need a notion of predicate

▶ Factorization systems of interest induce fibrations [Hughes and Jacobs ’03]
▶ What does a suitable factorization system look like?

Anything else?

▶ Reynolds uses projection theorems
▶ Partial maps between non-extensible and extensible semantics
How general is the construction?

- The category needs enough structure, including a suitable factorization system
  - All of our examples have this (but others might not!)
- We don’t assume anything about $T$
  - This works for arbitrary effects
  - But $\Sigma$ contains only Kleisli arrows
- We only consider effects
  - Can’t add/remove other language features (e.g. linear types)
- More interesting effect systems?
Future work

- Correctness proofs using fibrations
- How easy is it to use the construction?
- More examples: full ground state, probability, ...
Conclusions

- Can construct extensible semantics from non-extensible semantics
- Construction is general
  - No restrictions on $T$
- Still work in progress – don’t know if the extensible semantics is correct!