1 Linear Algebra

**Exercise 1 [Properties of dagger]** Prove the following properties of the conjugate transpose, defined for any \(x \in \mathbb{C}^n\) as

\[ x^\dagger = (x_1^*, \ldots, x_n^*). \]

(a) For any two vectors \(x, y \in \mathbb{C}^n\), it holds that \(x^\dagger y = (y^\dagger x)^\dagger\).

(b) For any vector \(x \in \mathbb{C}^n\), it holds that \(x^\dagger x \geq 0\).

*(Answer)*

(a) By writing out the definition of the two vectors, we have that

\[ (y^\dagger x)^\dagger = \left( \sum_{i=1}^{n} y_i^* x_i \right)^\dagger = \sum_{i=1}^{n} (y_i^* x_i)^\dagger = \sum_{i=1}^{n} y_i x_i^* = x^\dagger y. \]

(b) For any vector \(x \in \mathbb{C}^n\),

\[ x^\dagger x = \sum_{i=1}^{n} x_i^* x_i = \sum_{i=1}^{n} |x_i|^2 \geq 0. \]

**Extended Note 1** A matrix \(A \in \mathbb{C}^{n \times n}\) is *Hermitian* if \(A^\dagger = A\). The notation \(A^\dagger\) means the complex conjugate of \(A\), i.e., \(A^\dagger = (A^T)^*\). If you prefer you can attempt the following exercises, by assuming that \(A\) is a matrix with real entries and then \(A^\dagger = A^T = A\), meaning that the matrix is *symmetric*.

**Exercise 2 [Real eigenvalues]** Consider a Hermitian matrix \(A \in \mathbb{R}^{n \times n}\), then all its eigenvalues are real.

*(Answer)* Let \(A\) be a Hermitian matrix. Then, for any vector \(x \in \mathbb{C}^n\), it holds that

\[ (Ax)^\dagger x = x^\dagger A^\dagger x = x^\dagger (Ax). \]  \hspace{1cm} (1)

Assuming that \(x\) is an eigenvector corresponding to eigenvalue \(\lambda\), i.e., \(Ax = \lambda x\). Then, we have that

\[ (Ax)^\dagger x = (\lambda x)^\dagger x = \lambda^* x^\dagger x \]

and

\[ x^\dagger (Ax) = x^\dagger \lambda x = \lambda x^\dagger x \]

By **(1)**, we have that

\[ \lambda x^\dagger x = \lambda^* x^\dagger x \Rightarrow (\lambda - \lambda^*) x^\dagger x = 0. \]

Hence, since \(x^\dagger x > 0\) (as \(x \neq 0\)), we have that \(\lambda = \lambda^*\) and hence \(\lambda\) is real.

**Exercise 3 [Orthogonal eigenvectors]** Consider a Hermitian matrix \(A \in \mathbb{C}^{n \times n}\) and let \(x, y \in \mathbb{C}^n\) be two eigenvectors corresponding to different eigenvalues \(\lambda \neq \lambda'\). Then, \(x \perp y\).

*(Answer)* Since \(A\) is Hermitian, we have that

\[ (Ax)^\dagger y = x^\dagger A^\dagger y = x^\dagger Ay = x^\dagger (Ay). \]
Exercise 4 [Spectral theorem] Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.
(a) Let $k \leq n-1$ and let $x_1, \ldots, x_k$ be orthogonal eigenvectors of $A$. Then, there exists an eigenvector $x_{k+1}$ that is orthogonal to $x_1, \ldots, x_k$.
(b) Prove the spectral theorem,
(c) Argue that $A$ can be written as $XDX^{-1}$ for some matrix $X$ and a diagonal matrix $D$.

Exercise 5 [Inverse matrix] Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with (orthonormal) eigenvectors $x_1, \ldots, x_n$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that:
(a) Let $x \in \mathbb{R}^n$ be arbitrary. By writing $x = (x^T x_1)x_1 + \ldots + (x^T x_n)x_n$, show that $x^T x_1 + \ldots x_n x_n^T = I$.
(b) By writing $Ax = AIx$, show that $A = \lambda_1 x_1 x_1^T + \ldots + \lambda_n x_n x_n^T$.
(c) Show that if $\lambda_1 \neq 0, \ldots, \lambda_n \neq 0$, then $A^{-1} = \frac{1}{\lambda_1} x_1 x_1^T + \ldots + \frac{1}{\lambda_n} x_n x_n^T$.

Exercise 6 [Power of a matrix] Consider a real symmetric matrix $A \in \mathbb{R}^{n \times n}$ with eigenvectors $x_1, \ldots, x_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, for any $k \in \mathbb{N}_{\geq 1}$, the eigenvectors of $A^k$ are $x_1, \ldots, x_n$ and the eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$.

Exercise 7 [Trace of a matrix] The trace of a matrix is defined as $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$.

(a) Show that for any two matrices $X$ and $Y$, we have that $\text{tr}(XY) = \text{tr}(YX)$.
(b) Show that for any real symmetric matrix $A$, $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

(Answer)
(a) Using the definition of the trace, we have that $\text{tr}(XY) = \sum_{i=1}^{n} (XY)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}Y_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} X_{ij}Y_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} Y_{ji}X_{ij} = \text{tr}(YX)$.
(b) By writing $A = XDX^{-1}$, we have that $\text{tr}(XDX^{-1}) = \text{tr}(XX^{-1}D) = \text{tr}(ID) = \text{tr}(D) = \sum_{i=1}^{n} \lambda_i$.

Exercise 8 [Determinant of symmetric matrix] Consider a real symmetric matrix $A$. Show that $\det(A) = \prod_{i=1}^{n} \lambda_i$.

Hint: Use the property that $\det(AB) = \det(A) \cdot \det(B)$.
By writing \( A = XDX^{-1} \) we have that

\[
\det(A) = \det(XDX^{-1}) = \det(X)\det(D)\det(X^{-1}) = \det(D) = \prod_{i=1}^{n} \lambda_i,
\]

using that \( \det(X^{-1}) = (\det(X))^{-1} \) and that the determinant of a diagonal matrix is just the product of the entries of the diagonal.

# 2 Graph matrices

## 2.1 Adjacency matrix

**Exercise 9 [Basic properties]** Consider the adjacency matrix of an undirected graph.

(a) Show that \( \deg(v_i) = \sum_{j=1}^{n} A_{ij} \).

(b) Show that the adjacency matrix of a \( d \)-regular graph has eigenvalue \( d \).

**Exercise 10 [Counting paths]** Consider the adjacency matrix \( A \) of a graph \( G \).

(a) Show that if \((A^k)_{ij} > 0\) for some integer \( k > 0 \) then there is a path of length \( k \) connecting \( i \) and \( j \).

(b) Show that \((A^k)_{ij}\) also gives the number of paths connecting \( i \) and \( j \) with \( k \) hops.

(c) Interpret \( \text{tr}(A^k) > 0 \).

**Exercise 11 [Bipartite graphs]** Show that for any bipartite graph \( G \) with adjacency matrix \( A \), if \( \lambda > 0 \) is an eigenvalue then \( -\lambda \) is also an eigenvalue.

\((Answer)\) Let \( G \) be a bipartite graph where the left part has \( k \) vertices and the right part has \( n - k \). We re-index the \( n \) nodes of the graph such that the vertices of the left part are \( 1, \ldots, k \) and the vertices of the right part are \( k + 1, \ldots, n \). Then, the adjacency matrix can be written as

\[
A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.
\]

Let \( \begin{bmatrix} x \\ y \end{bmatrix} \) be the eigenvector of \( A \) with eigenvalue \( \lambda \), then

\[
A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Bx \\ B^Ty \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.
\]

By considering the vector \( z' = \begin{bmatrix} -x \\ -y \end{bmatrix} \) and \( \lambda' = -\lambda \), then we have that

\[
A z' = A \begin{bmatrix} -x \\ -y \end{bmatrix} = -\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \lambda' z'.
\]

**Exercise 12 [Converse for bipartite graphs]**

(a) Argue that a graph with no odd length cycle is bipartite.

(b) Show that the statement “a graph has no odd length cycle” is equivalent to “for every odd \( k \), \( \text{tr}(A^k) = 0 \).”

(c) Deduce that if for every eigenvalue \( \lambda \) of \( A \), there is another eigenvalue \( -\lambda \), then the graph is bipartite.

\((Answer)\)

(a)

(b)
(c) Consider any odd power \(k\), then
\[
\text{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k = 0.
\]

Hence, by Exercise ?? ??, we have that the graph has no cycle of odd length and so it is bipartite by ??.

**Exercise 13** On Lecture 11/slide 10 (Example 1) we determined the spectrum of the adjacency matrix \(A\) for the complete graph (a.k.a. clique) of size 3. Here we would like to generalise this to any complete graph of size \(n > 3\). Prove that the spectrum consists of eigenvalues \(n - 1\) (with multiplicity 1) and \(-1\) (with multiplicity \(n - 1\)).

**Exercise 14** Consider an undirected, and \(d\)-regular graph \(G = (V, E)\). In this exercise, you will show that the diameter of the graph is at most \(O\left(\frac{\log n}{\Phi}\right)\).

(a) Consider an arbitrary vertex \(u \in V\) and let \(S_0 := \{u\}\) and \(S_i := B_{\leq i}(u)\), the set of nodes at a distance of at most \(i\) from \(u\). Show that for any \(|S_i| \leq n/2\),
\[
|E(S_i, S_i^c)| \geq \Phi \cdot |S_i| \cdot d,
\]
and that
\[
|S_{i+1}| \geq |S_i| \cdot (1 + \Phi).
\]

(b) Using that \(\log(1 + z) \geq (1/2) \cdot z\) (for any \(z \in [0, 1]\)), deduce that \(|S_i| > n/2\) for \(i > 2 \cdot \frac{\log n}{\Phi}\).

(c) Deduce that there is no pair of vertices at a distance \(> 4 \log n/\Phi\). \(\text{Hint: Consider a vertex } u \in V, \text{ and consider } B_{\leq r}(u), \text{ the set of vertices with distance at most } r \text{ to } u. \text{ Then use the definition of the conductance to prove a lower bound on } |B_{\leq r+1}(u)| \text{ in terms of } |B_{\leq r}(u)|.\"

**Further Reading 1** Take a look at this paper for a bound on the diameter of a graph using the \(k\)-th eigenvalue \(\lambda_k\).

**Exercise 15** [Perron-Frobenius] Let \(G\) be a connected graph with adjacency matrix \(A\) with eigenvectors \(x_1, \ldots, x_n\) and eigenvalues \(\lambda_1, \ldots, \lambda_n\), then show that
(a) \(\lambda_1 \geq -\lambda_n\),
(b) \(\lambda_1 > \lambda_2\),
(c) There exists an eigenvector \(x_1\) which has all its entries \(> 0\).

2.2 Laplacian matrix

**Exercise 16** [Factorisation] Consider the unnormalised Laplacian matrix \(L = D - A\) and the incident matrix \(M \in \mathbb{R}^{n \times m}\) defined as \(M_{uv} = 1_{u \in e}\) (i.e., indicates which edges contain which vertices). Show that
\[
L = M^T M.
\]

**Exercise 17** Consider an undirected, \(d\)-regular graph \(G\) and the matrices \(A_G\) and \(L_G\).

(a) Show that the two matrices have the same eigenvectors.

(b) Describe the correspondence between their eigenvalues.

(\text{Answer}) Assume that \(x\) is an eigenvector and \(\lambda\) is its corresponding eigenvalue. Then,
\[
Ax = \lambda x.
\]

Then,
\[
Lx = \left( I - \frac{1}{d} A \right) x = x - \frac{\lambda}{d} x = \left( 1 - \frac{\lambda}{d} \right) x.
\]
So $x$ is still an eigenvector and $1 - \lambda/d$ is an eigenvalue.

**Exercise 18** Consider a $d$-regular graph and its Laplacian matrix $L$.

(a) Using the quadratic form, show that for any vector $x \in \mathbb{R}^n$ (with $x \neq 0$),

\[
\frac{x^T L x}{x^T x} \leq 2.
\]

(b) Deduce that $\lambda_n \leq 2$.

Prove that for any $d$-regular graph, the largest eigenvalue of the Laplacian $L$ satisfies $\lambda_n \leq 2$.

**Exercise 19** Show that if $G$ is an undirected, $d$-regular, connected and bipartite graph, then the largest eigenvalue $\lambda_n$ of the Laplacian matrix satisfies $\lambda_n = 2$ (this proves one direction of the fourth statement in the Lemma from Lecture 11/slide 14).

**Exercise 20** Redo Exercise 18 without assuming that the graph is not $d$-regular.

**Exercise 21** Consider the transition matrix of a lazy random walks $\tilde{P} = (P + I)/2$ on a $d$-regular graph (here $I$ is the $n \times n$ identity matrix and $P$ is the transition matrix of a simple random walk).

(a) Using Exercise 18 and that the eigenvalues of $L$ are in $[0, 2]$, argue that the eigenvalues of $A$ are in $[-d, d]$.

(b) Prove that all eigenvalues of $\tilde{P}$ are non-negative.

### 3 Conductance

**Exercise 22** [Conductance of graphs]

(a) Compute the conductance of the complete graph $K_n$.

(b) Compute the conductance of the cycle $C_n$.

(c) Compute the conductance of a path $P_n$.

(d) Compute the conductance of a 2D grid.

(e) (+) Compute the conductance of a 3D grid.

**Exercise 23**

(a) Prove that for every $n > 2$ there is an unweighted, undirected $n$-vertex graph with conductance 1.

(b) (+) Can you characterise all graphs with that property?

**Exercise 24** Prove that for any $d$-regular graph with $n \to \infty$ being large, the conductance satisfies $\Phi(G) \leq \frac{1}{2} + o(1)$.

*Hint:* Use the probabilistic method to construct a set $S$ with the required conductance. First obtain bounds for $||S| - n/2|$ and then for $|E(S, \bar{S}) - E|/2$. 

5