Tight Bounds for Repeated Balls-into-Bins

Dimitrios Los[1], Thomas Sauerwald[1]

[1]University of Cambridge, UK

Balls-into-Bins: Background

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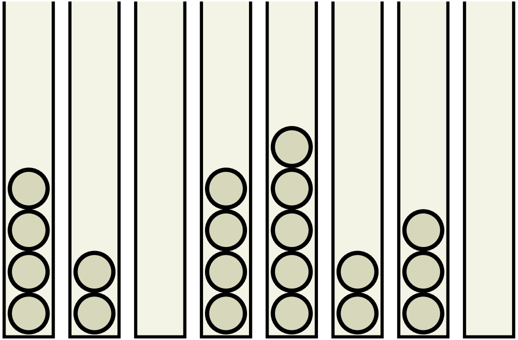

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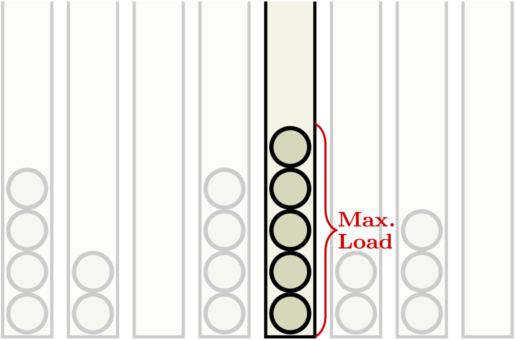

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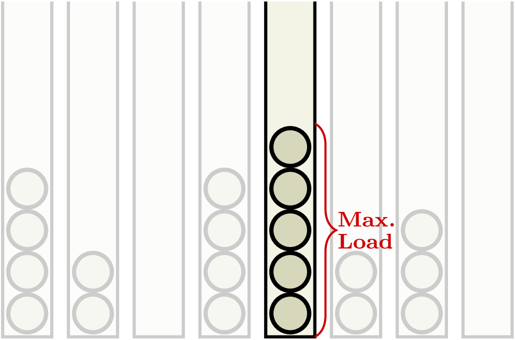

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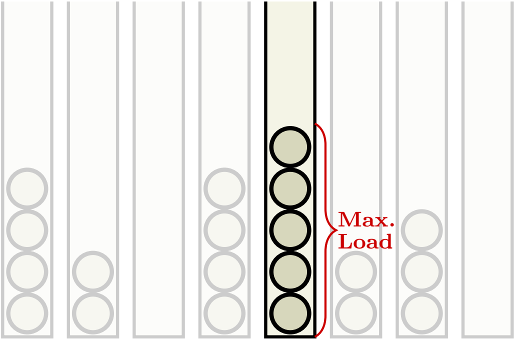

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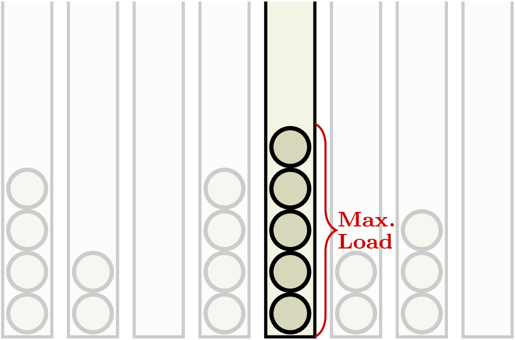

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Meaning with probability at least $1 - n^{-c}$ for constant $c > 0$.

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For $m = n$, we get the $\Theta\left(\frac{\log n}{\log \log n}\right)$ bound.

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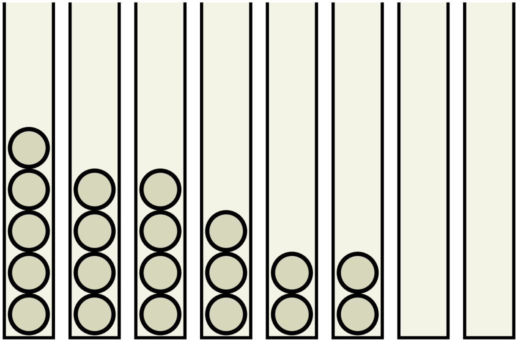

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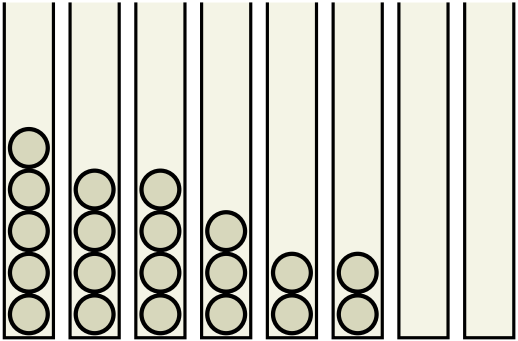

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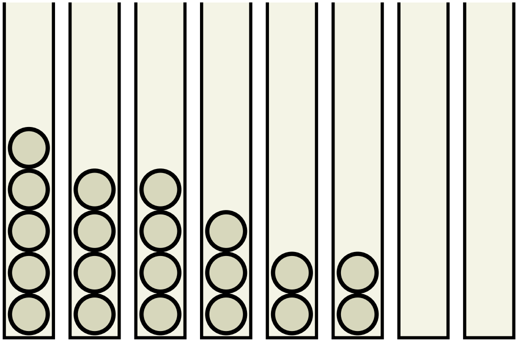

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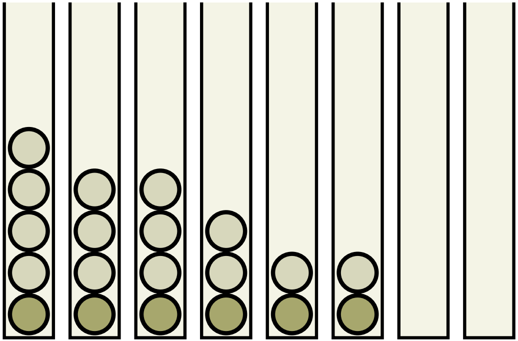

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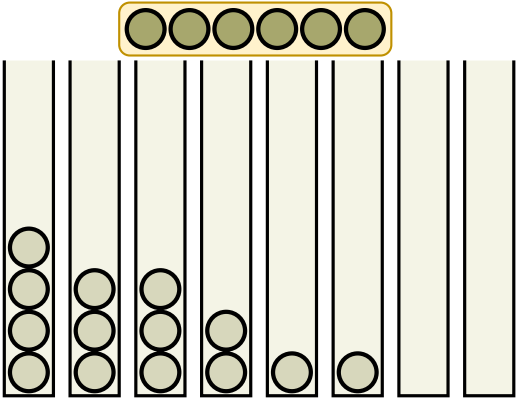

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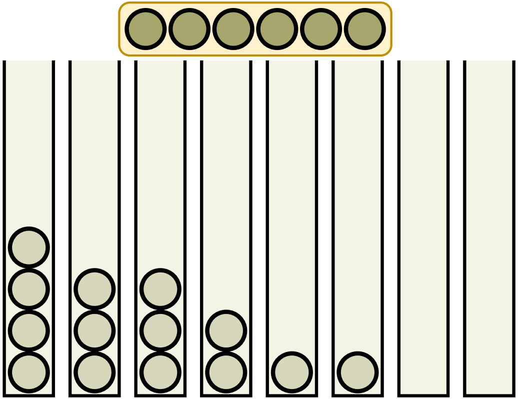

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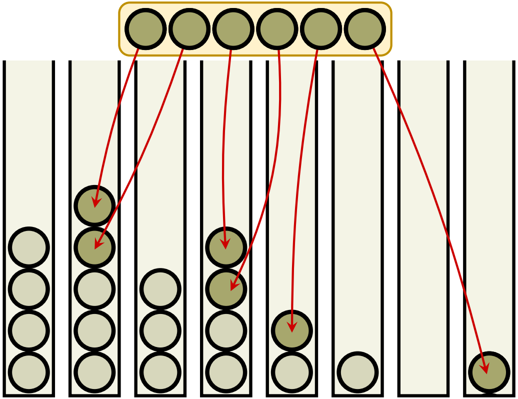

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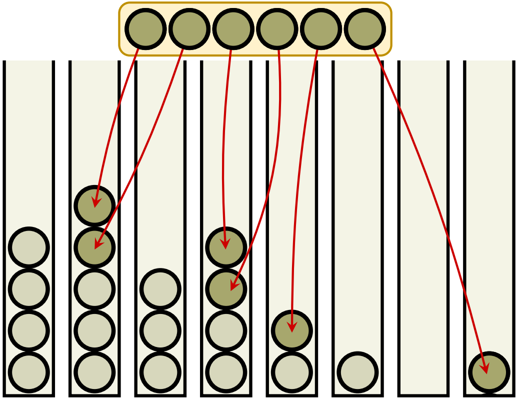

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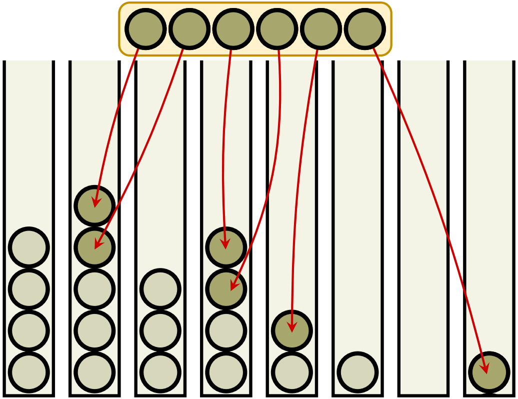

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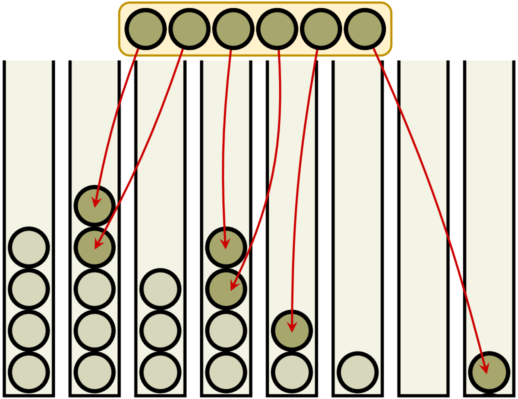

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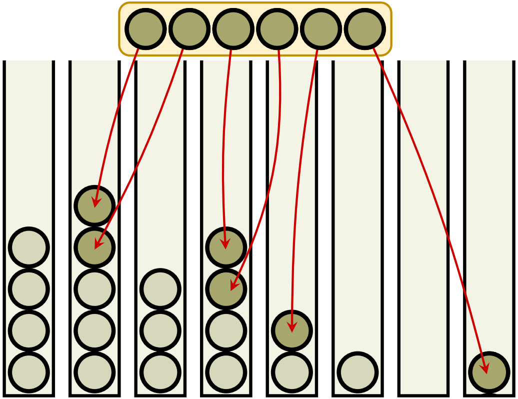

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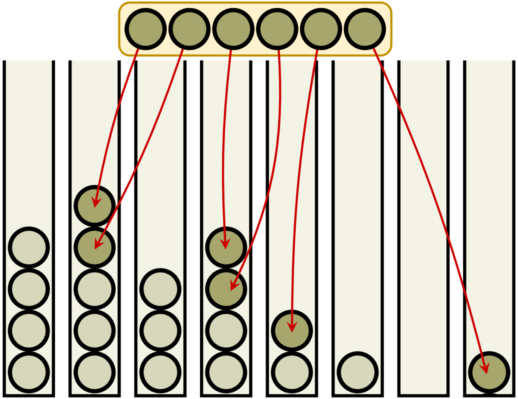

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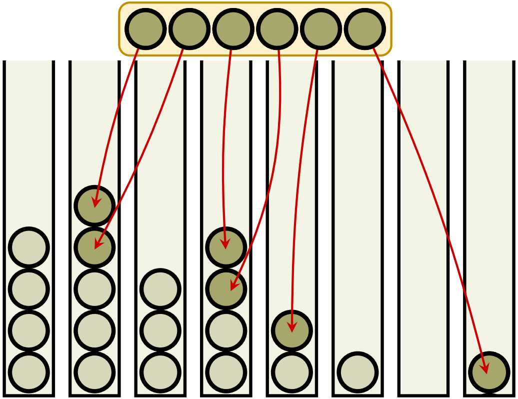

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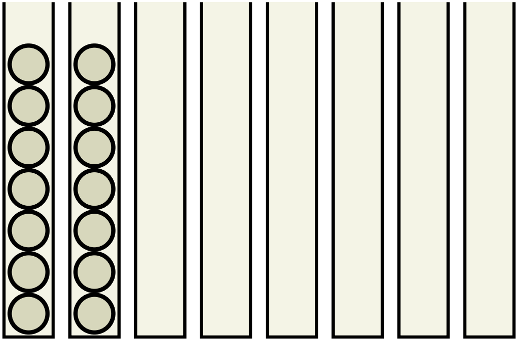
Open in Visualiser.

RBB in action



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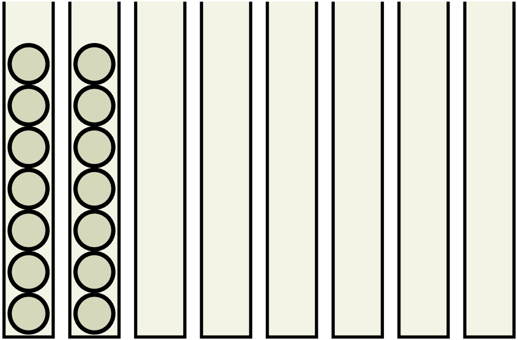
- Starting with an unbalanced configuration, the process eventually stabilises in a balanced configuration.



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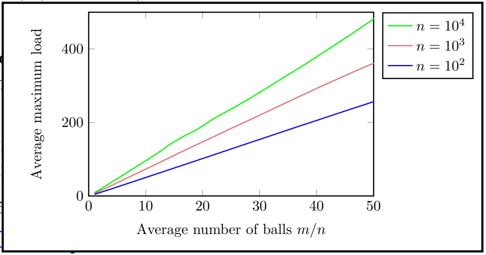

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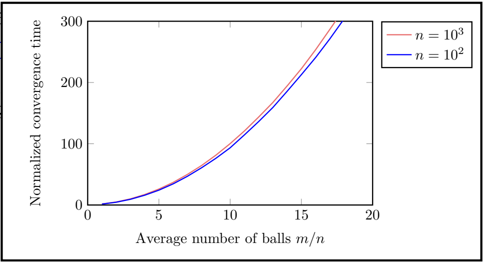

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Techniques

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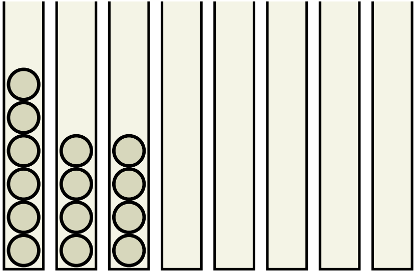

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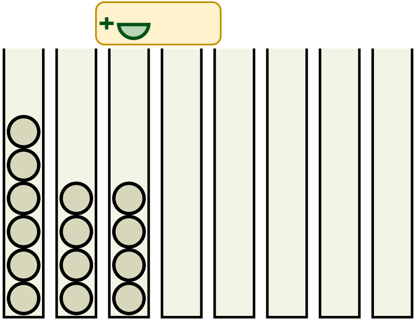

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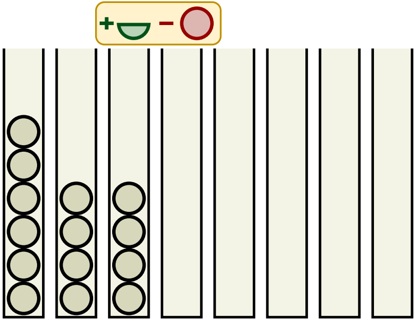

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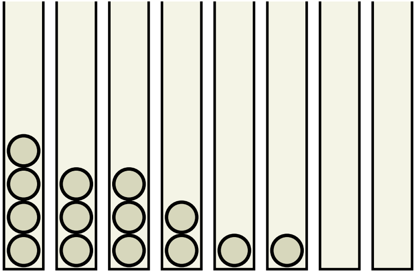

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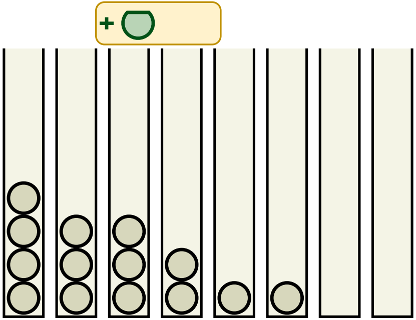

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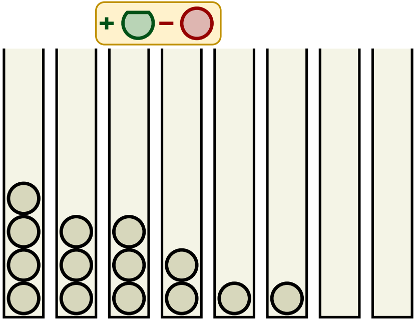

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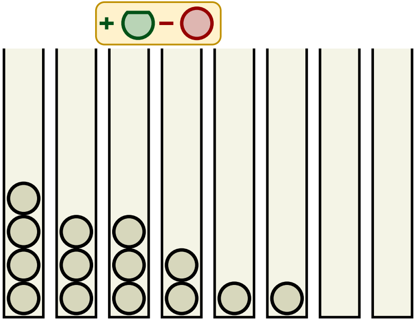

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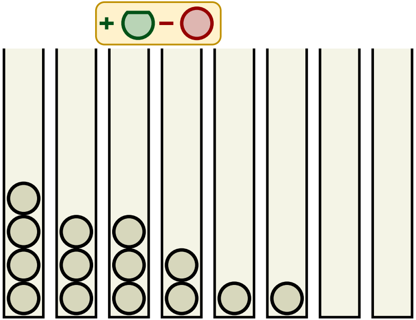

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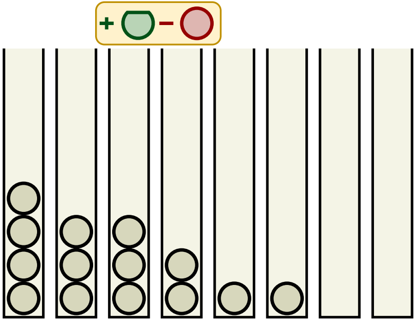

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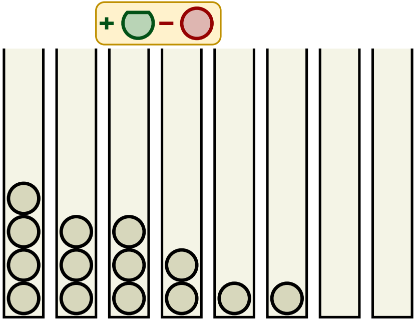

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The $\mathcal{O}(\frac{m}{n} \cdot \log n)$ upper bound

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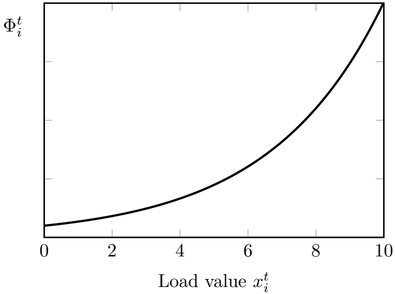

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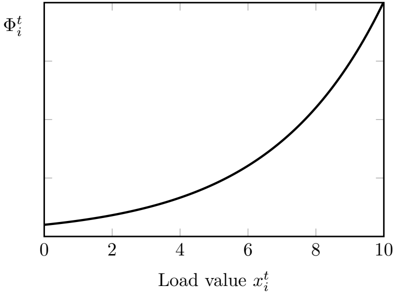

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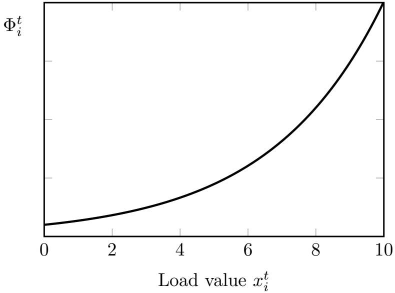

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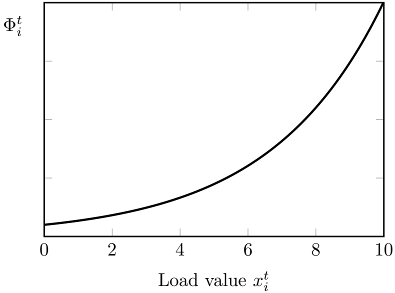

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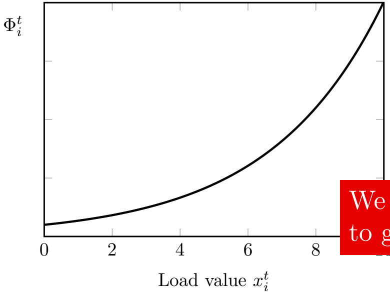

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- By the method of bounded differences, we get the lower bound holds w.h.p.

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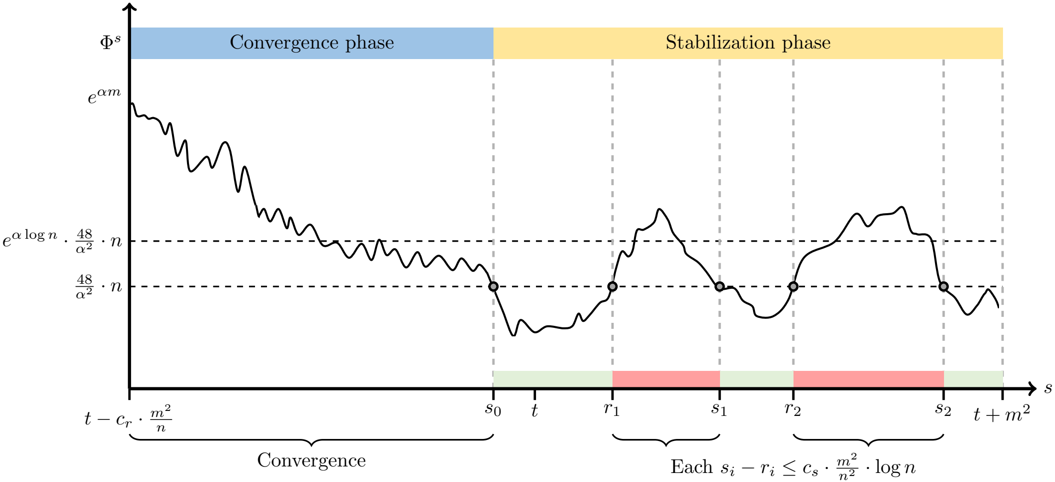

The $\Omega(\frac{m}{n} \cdot \log n)$ lower bound

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- Note that $\Upsilon^t \leq m^2$. So, using a concentration inequality, in any interval of $\Omega(m^2/n)$ length, there can be $\leq \gamma$ fraction of empty bins.

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Actually, this is $\cup_{s \in [0,t]}$ as
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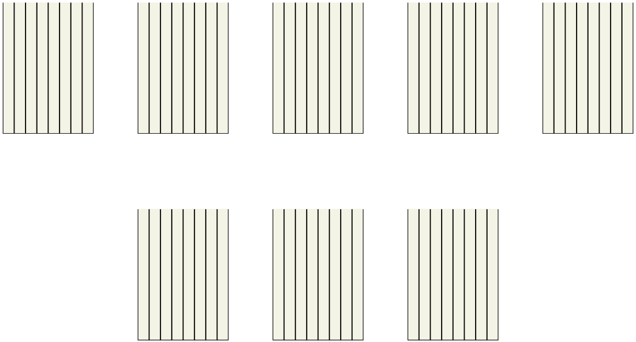
- Explore the process in the graphical setting [BCN+19].
- Explore versions of the process with continuous loads.
- Explore different reallocation rules.
- Relate to the setting where $n \cdot (1 - \Theta(n/m))$ new tasks arrive in each round [BFK+18].

Questions?



More visualisations: `dimitrioslos.com/stacs23`

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