Brief Anouncement:
Tight Bounds for Repeated Balls-into-Bins

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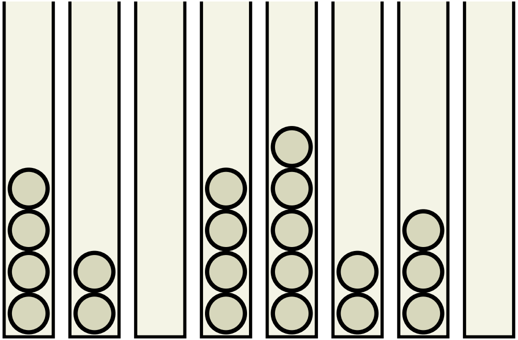

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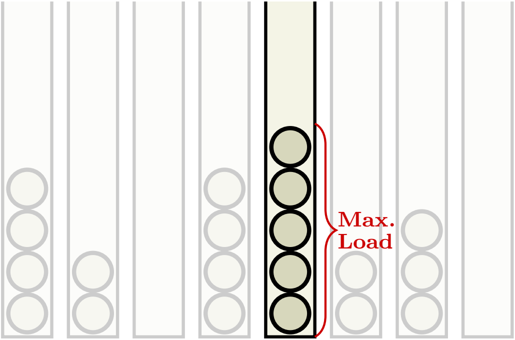

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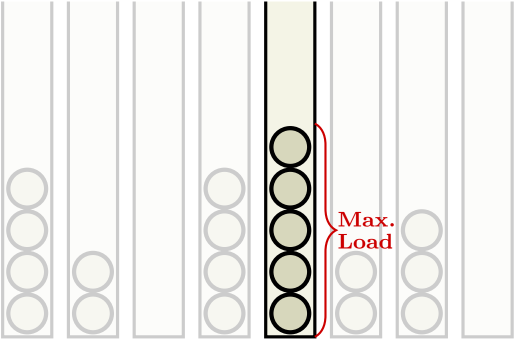

- (One-Choice) Allocating each ball uniformly at random for $m = \Omega(n \log n)$ gives a maximum load of: $\frac{m}{n} + \mathcal{O}(\sqrt{\frac{m}{n} \cdot \log n})$.

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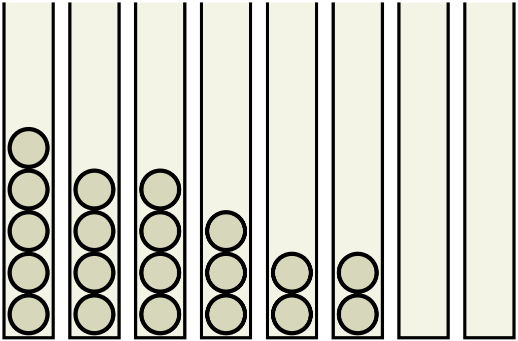

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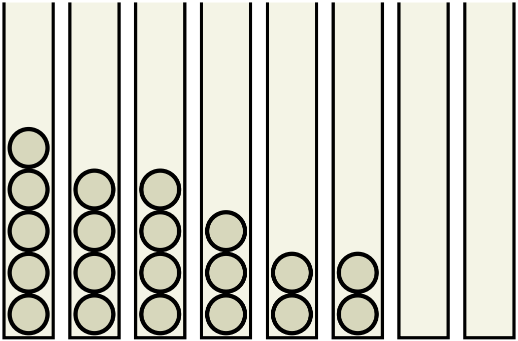

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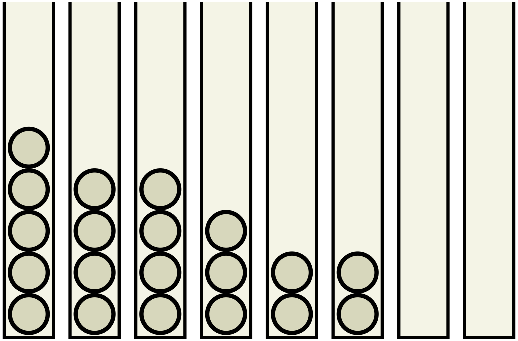

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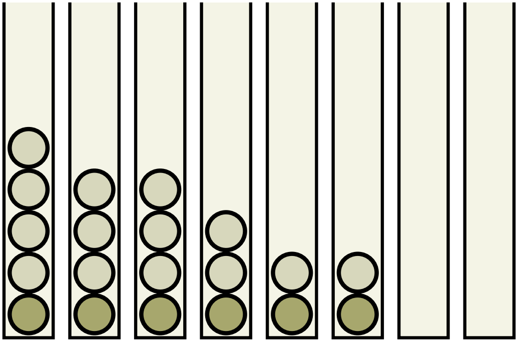

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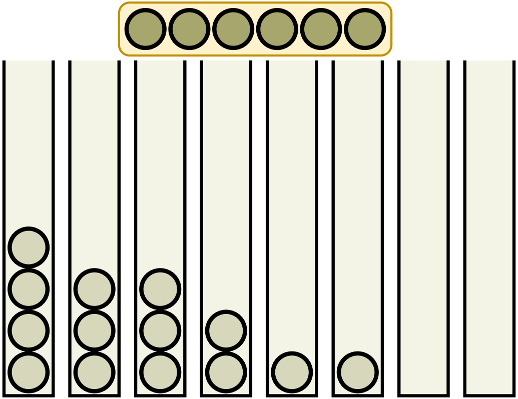

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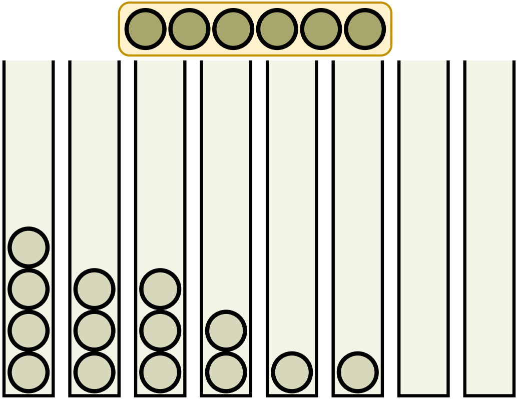

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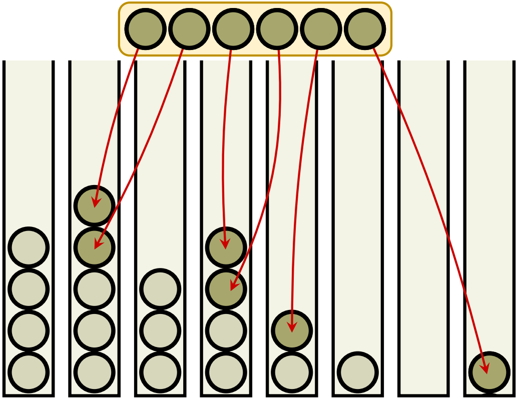

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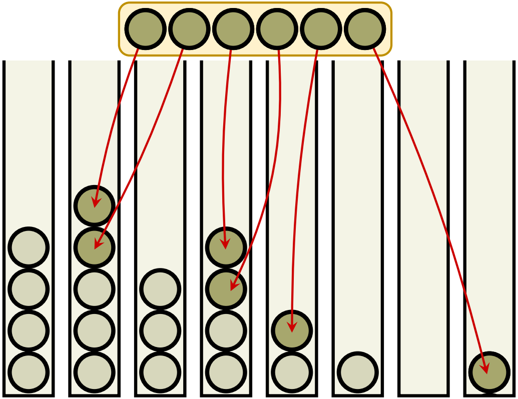
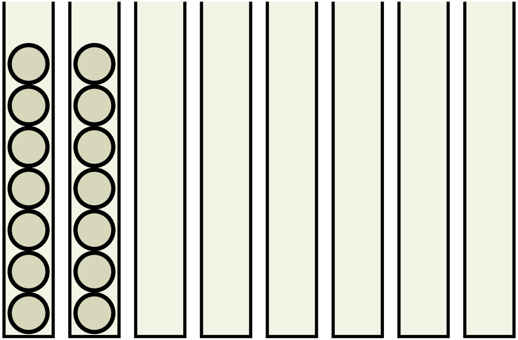


Number of balls is always exactly m.

RBB in action



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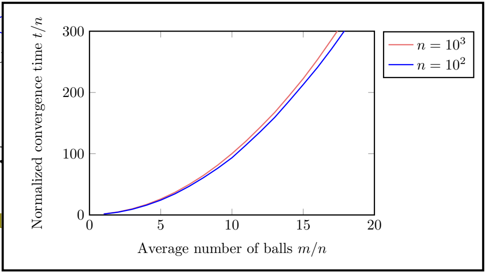

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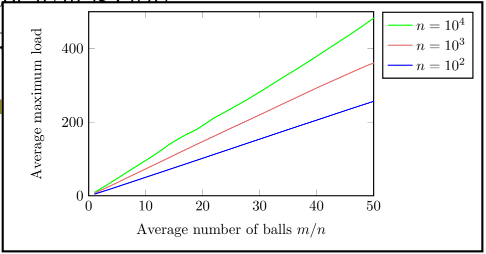

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- *How many rounds for all balls to traverse all bins?*
  - For $m = n$, traversal time is $\Omega(n \log n)$ and $\mathcal{O}(n \log^2 n)$ [BCN+19].
  - For $m = \text{poly}(n)$, traversal time is $\Theta(m \log n)$.

Future work

- Explore the process in the graphical setting.

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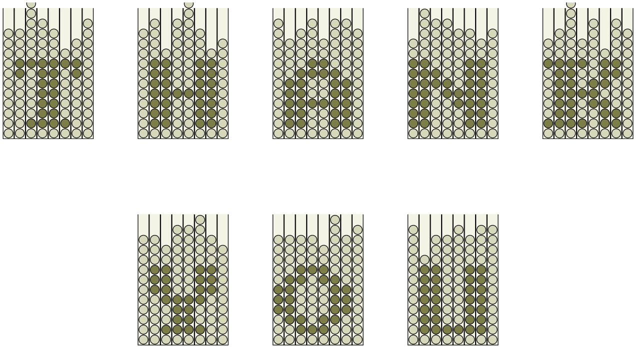
- Explore versions of the process with continuous loads.

Questions?



Bibliography I

- Luca Becchetti, Andrea Clementi, Emanuele Natale, Francesco Pasquale, and Gustavo Posta, *Self-stabilizing repeated balls-into-bins*, Proceedings of the 27th ACM Symposium on Parallelism in Algorithms and Architectures (New York, NY, USA), SPAA '15, Association for Computing Machinery, 2015, p. 332–339.
- Luca Becchetti, Andrea E. F. Clementi, Emanuele Natale, Francesco Pasquale, and Gustavo Posta, *Self-stabilizing repeated balls-into-bins*, Distributed Comput. **32** (2019), no. 1, 59–68.