Balanced Allocations with Heterogeneous Bins: The Power of Memory

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Balanced allocations: Background

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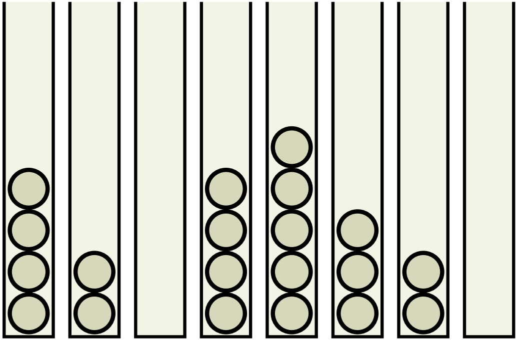

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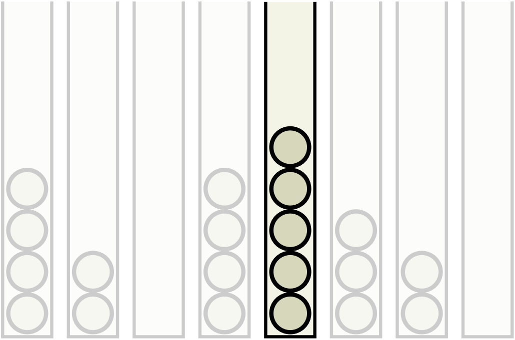

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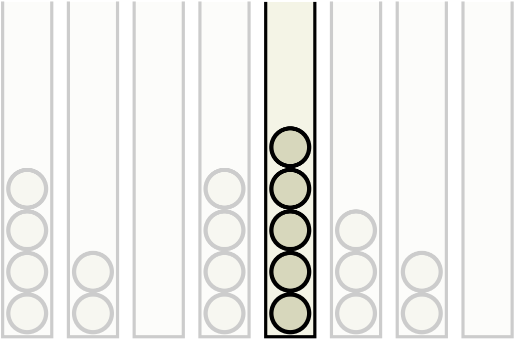

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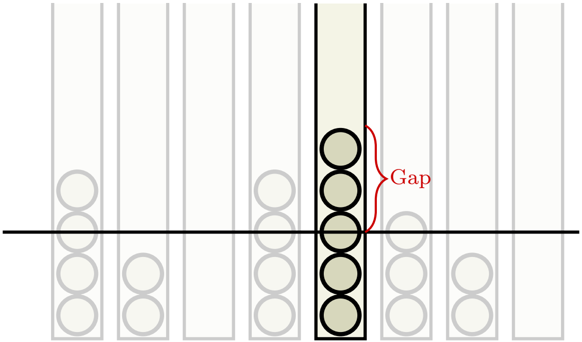

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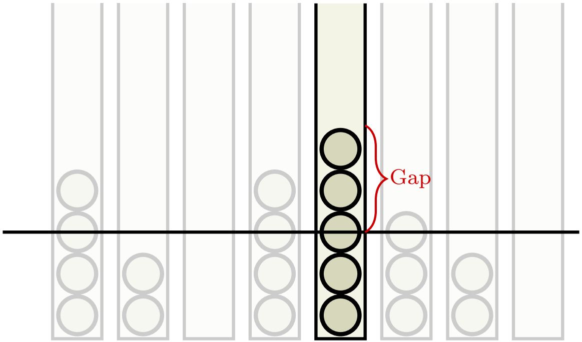

- Applications in hashing [PR01], load balancing [Wie16] and routing [GKK88].

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Meaning with probability
at least $1 - n^{-c}$ for constant $c > 0$.

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d-CHOICE Process:
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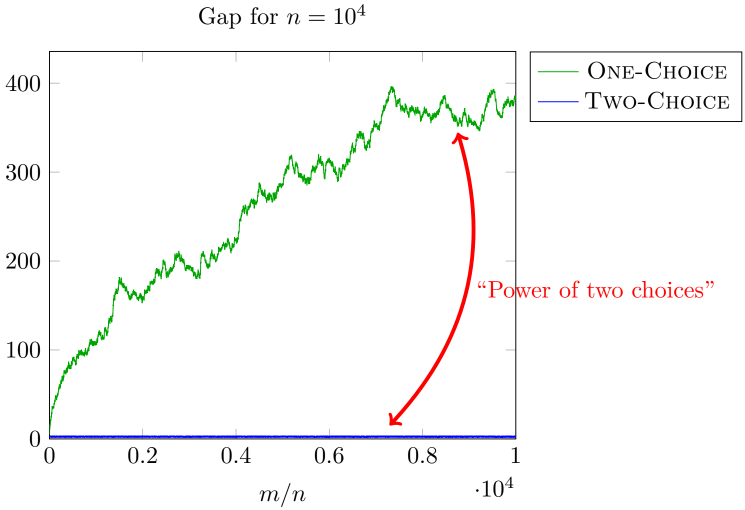
Power of two choices: Visualisation

`Open visualiser`



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MEMORY Process ($M = 1$):

Initialization: Set the cache $c^0 = 1$.

Iteration: For each step $t \geq 0$:

- Sample bins $i_1, \ldots, i_d$ uniformly at random.
- Allocate to bin $j = \operatorname{argmin}_{k \in \{c^t, i_1, \ldots, i_d\}} x_k^t$.
- Update the cache to $c^{t+1} = \operatorname{argmin}_{k \in \{c^t, i_1, \ldots, i_d\}} x_k^{t+1}$.

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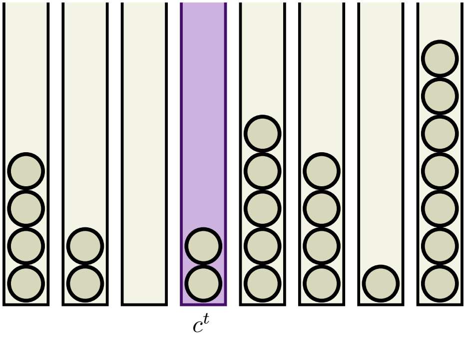

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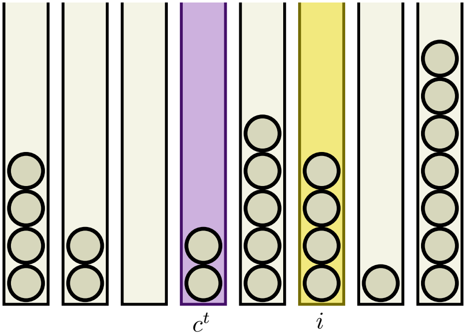

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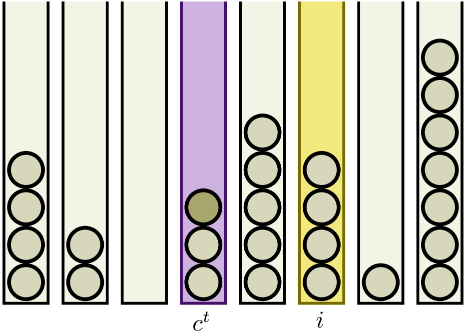

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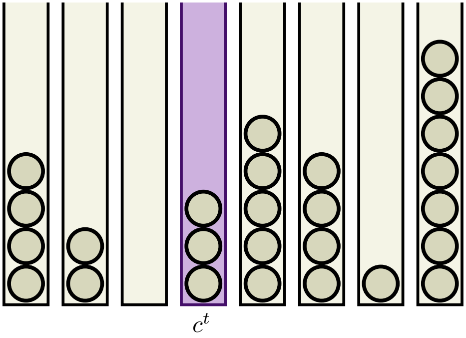

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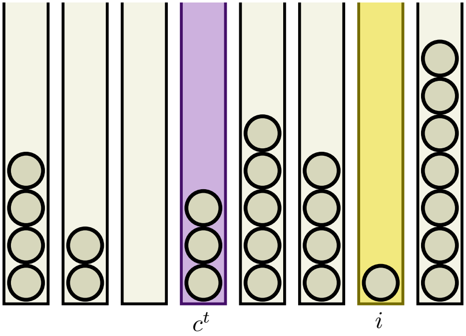

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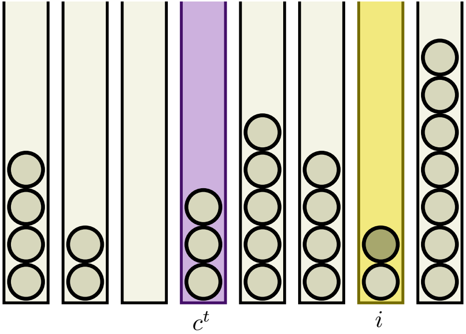

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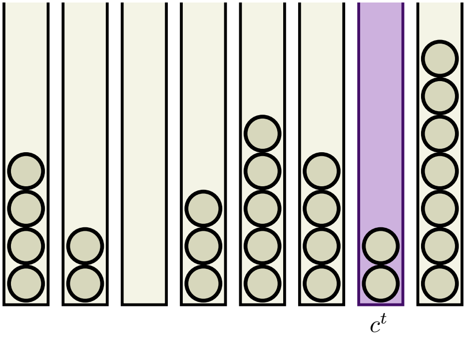

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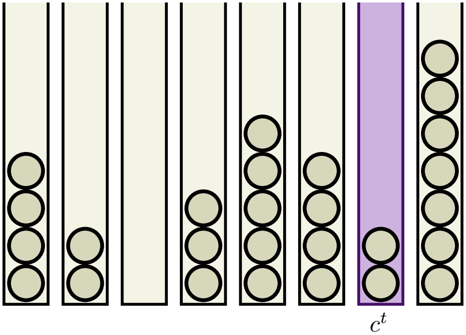

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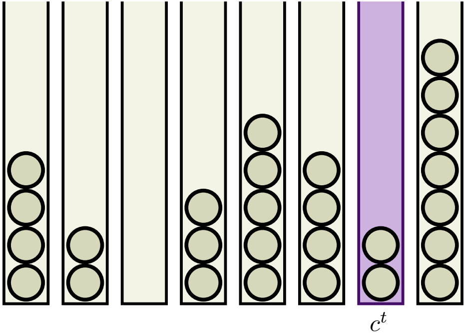

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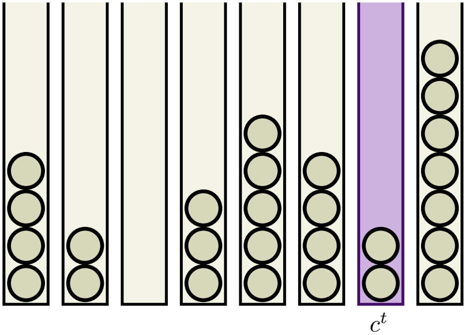

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What happens in the heavily-loaded case ($m \geq n$)?

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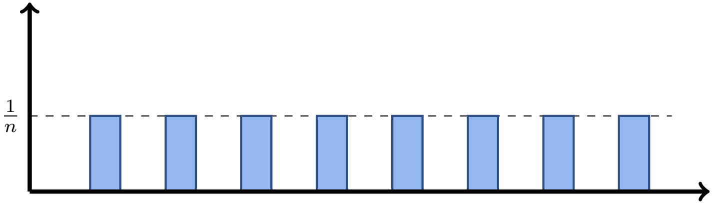

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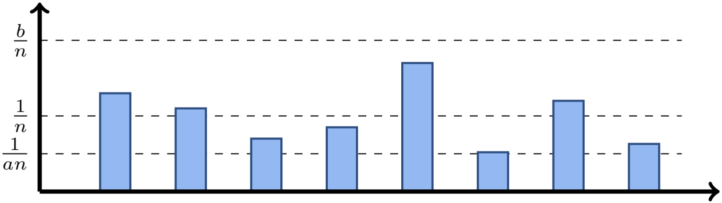

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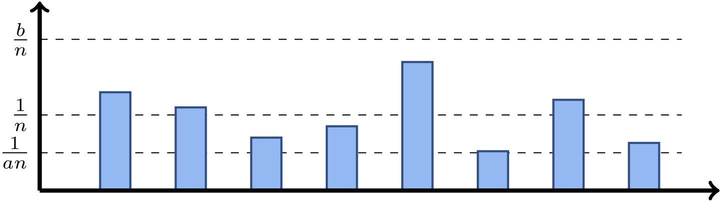

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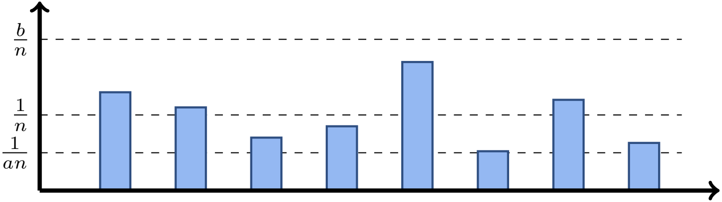

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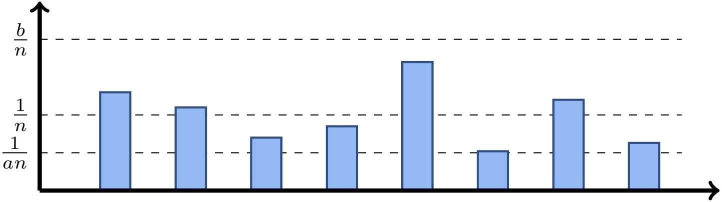

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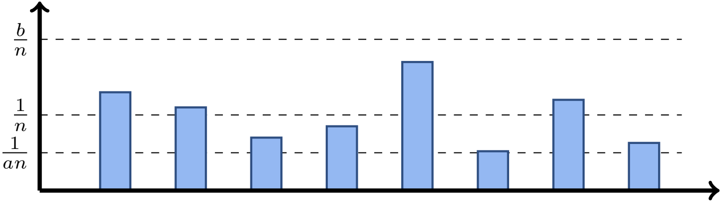

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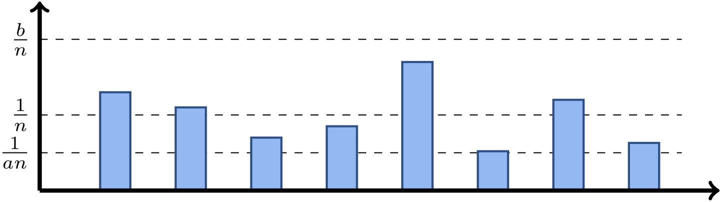

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How does Memory deal with heterogeneous sampling distributions?

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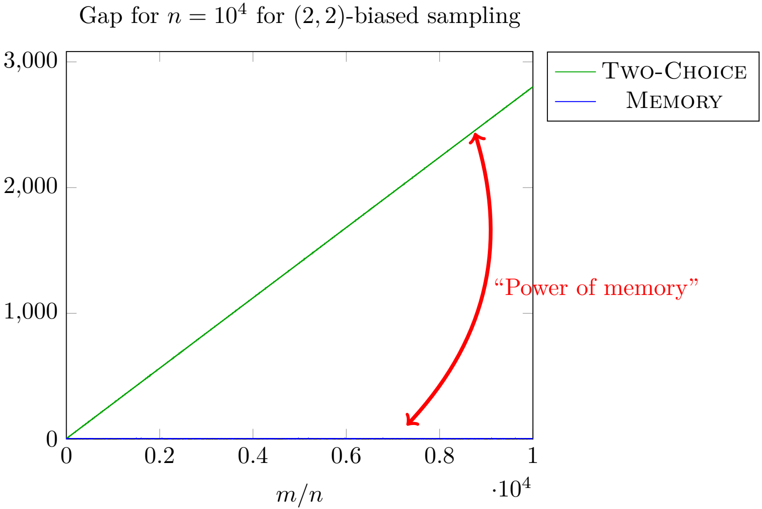
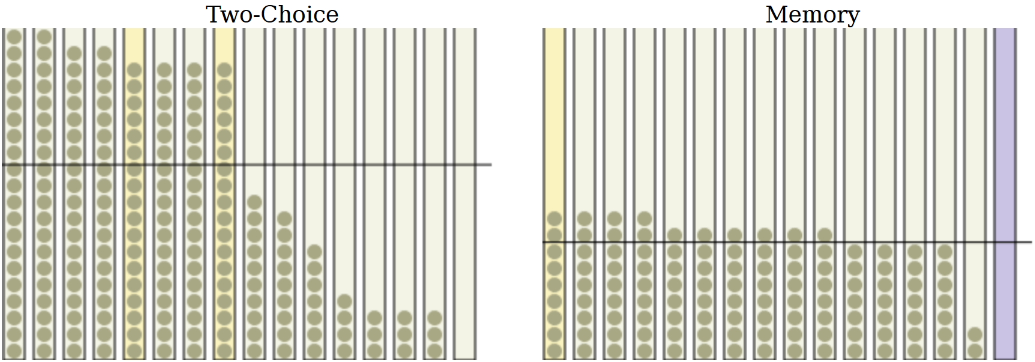
Power of memory: Visualisation

`Open visualiser`



Why MEMORY recovers?



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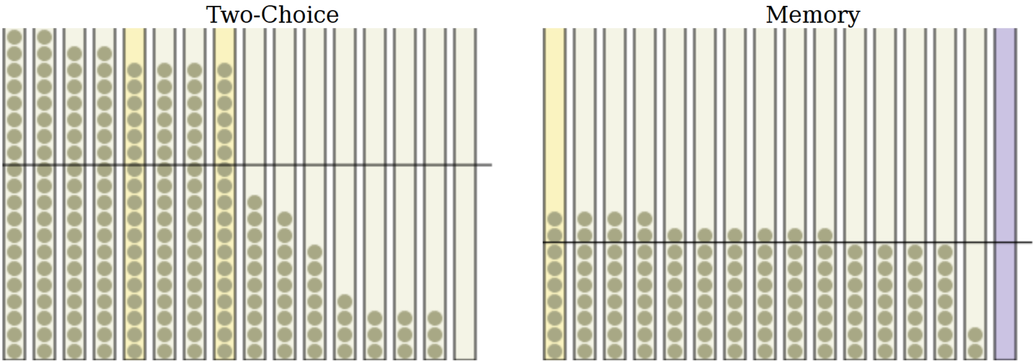

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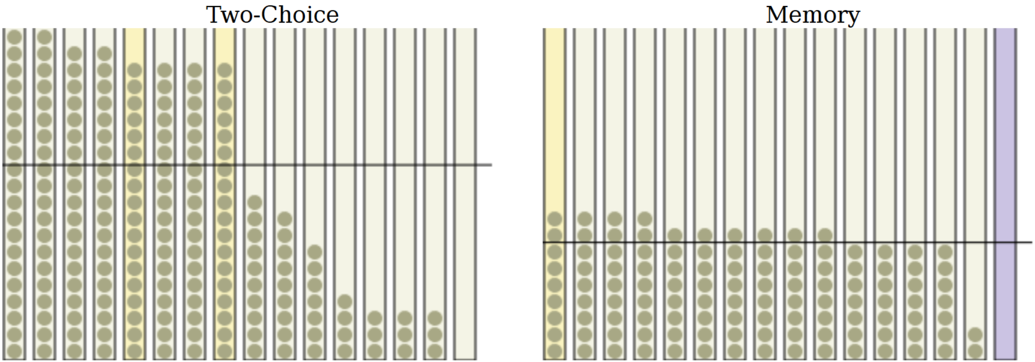

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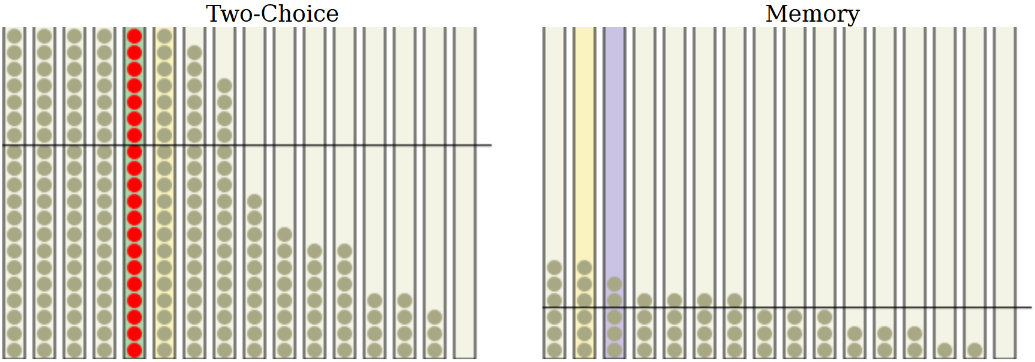

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Upper Bound for Memory

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- The base case follows by an involved analysis of the hyperbolic cosine potential function [PTW15, LS22a].

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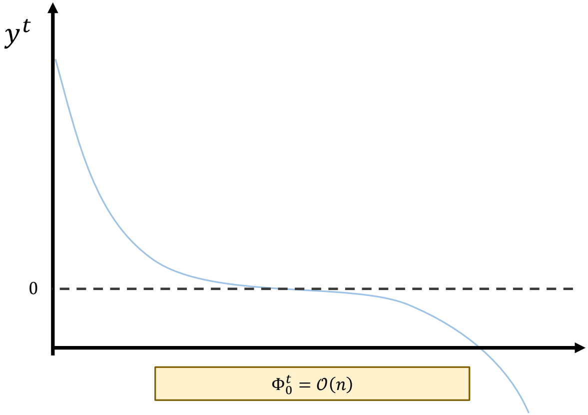

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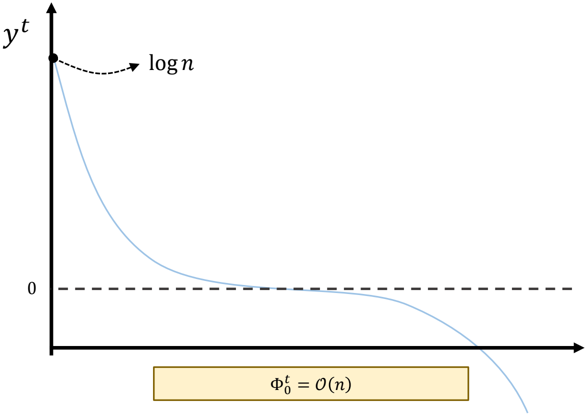

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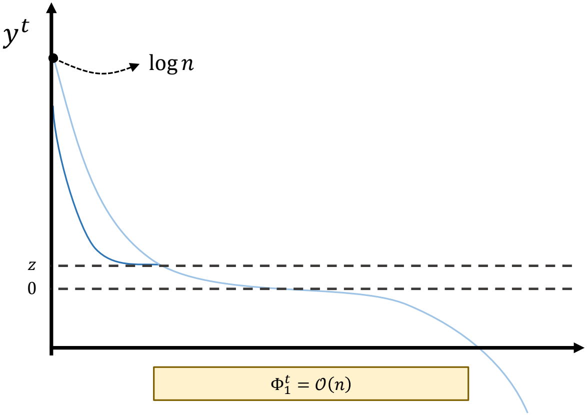

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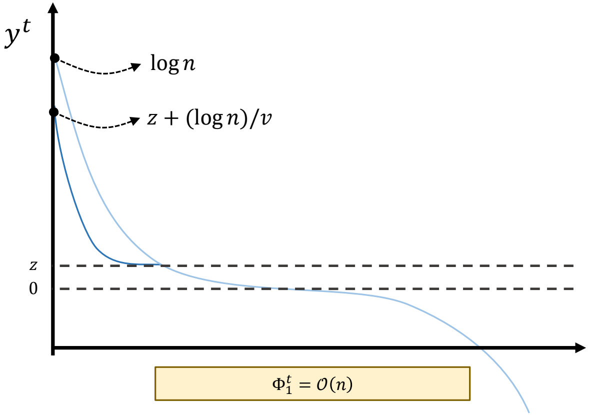

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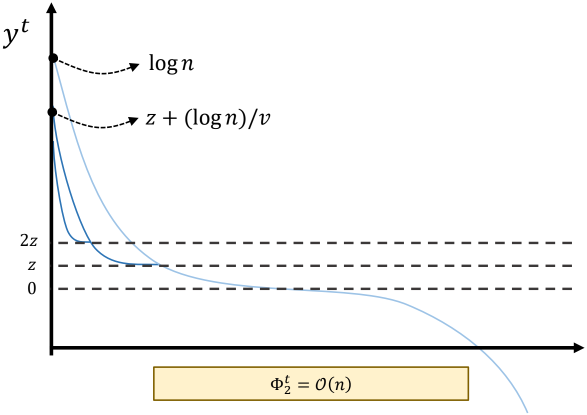

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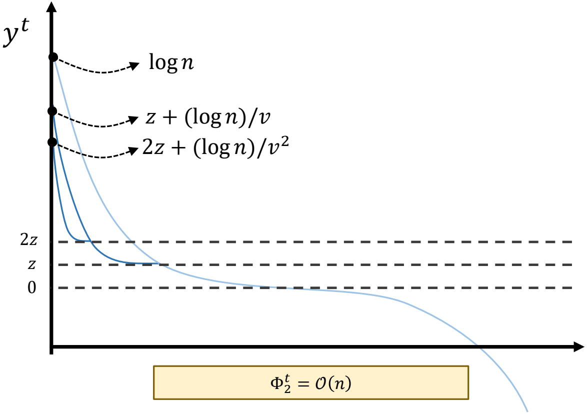

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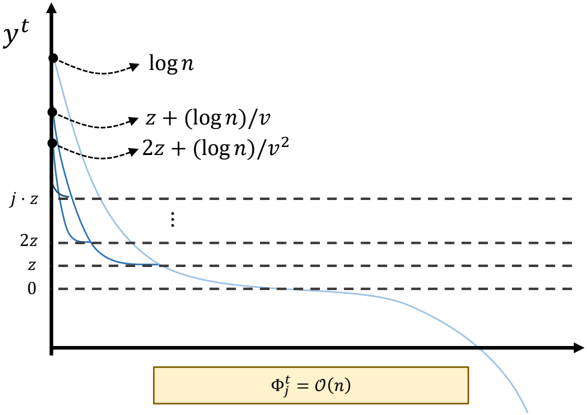

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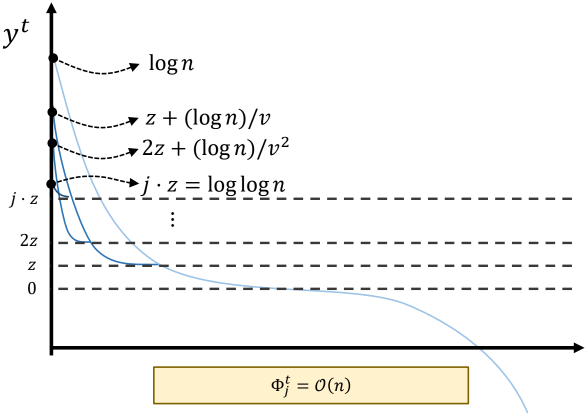

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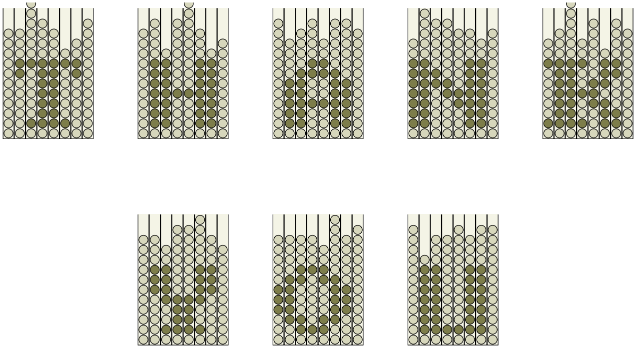
Questions?



More visualisations: `dimitrioslos.com/soda23`

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- And so, by Markov's inequality $\mathbf{Pr}\left[\, \Gamma^m \leq \frac{c}{\alpha\epsilon} \cdot n^3 \,\right] \geq 1 - n^{-2}$.

Hyperbolic cosine potential

- Peres, Talwar and Wieder [PTW15] used the **hyperbolic cosine potential** Γ^t, defined as
$$\Gamma^t := \Phi^t + \Psi^t := \sum_{i=1}^{n} e^{\alpha(x_i^t - t/n)} + \sum_{i=1}^{n} e^{-\alpha(x_i^t - t/n)}.$$
- When $\Gamma^m = \text{poly}(n)$, then $\text{Gap}(m) = \mathcal{O}\left(\frac{\log n}{\alpha}\right)$.
- They showed that for any (δ, ϵ)-smooth probability allocation vector p^t,
$$\mathbf{E}\left[\, \Gamma^{t+1} \mid \mathfrak{F}^t \,\right] \leq \Gamma^t \cdot \left(1 - \frac{\alpha\epsilon}{n}\right) + c.$$
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Problem: p^t for MEMORY may not be (δ, ϵ)-smooth

A generalised drift inequality [LS22a]

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- If for some (δ, ϵ)-smooth probability vector q,

$$\mathbf{E}\left[\, \Phi^{t+1} \,\middle|\, \mathfrak{F}^t \,\right] \leq \Phi^t + \sum_{i=1}^{n} \Phi_i^t \cdot \left(q_i^t - \frac{1}{n} \right) \cdot \alpha + \Phi^t \cdot C \cdot \frac{\alpha^2}{n},$$

$$\mathbf{E}\left[\, \Psi^{t+1} \,\middle|\, \mathfrak{F}^t \,\right] \leq \Psi^t + \sum_{i=1}^{n} \Psi_i^t \cdot \left(\frac{1}{n} - q_i^t \right) \cdot \alpha + \Psi^t \cdot C \cdot \frac{\alpha^2}{n}.$$

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- which is $(1/4, 1/2)$-smooth, implying an $\mathcal{O}(\log n)$ gap for Memory.

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- And so $\text{Gap}(m) = \mathcal{O}((\log n)/\alpha)$ gap.

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