Balanced Allocations with the Choice of Noise

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Balanced allocations: Background

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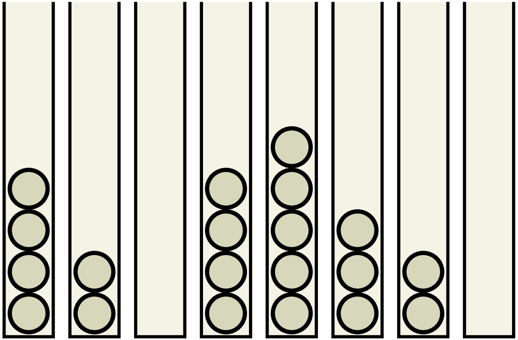

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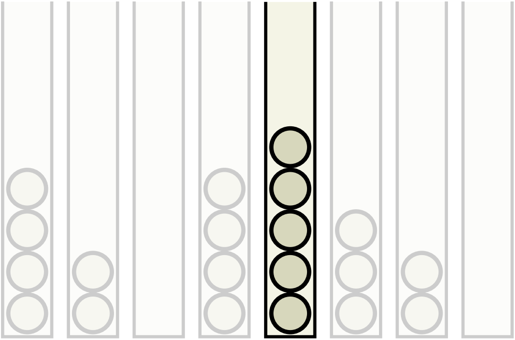

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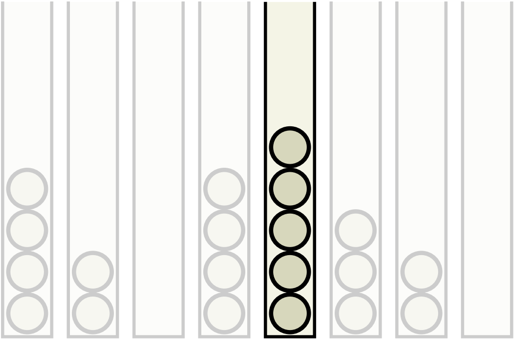

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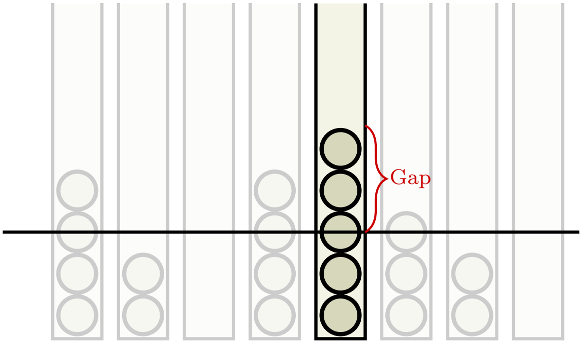

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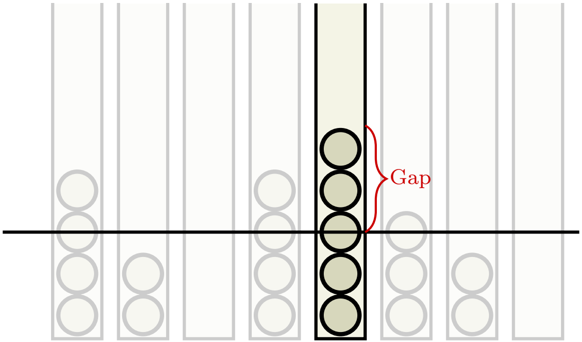

- Applications in hashing [PR01], load balancing [Wie16] and routing [GKK88].

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Meaning with probability at least $1 - n^{-c}$ for constant $c > 0$.

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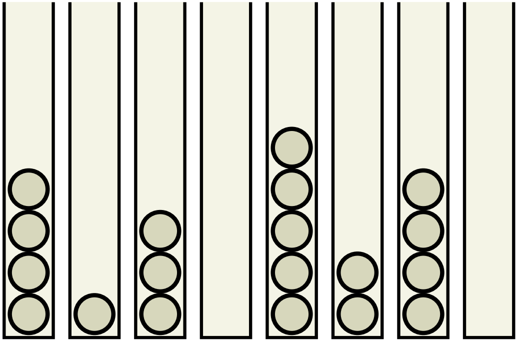

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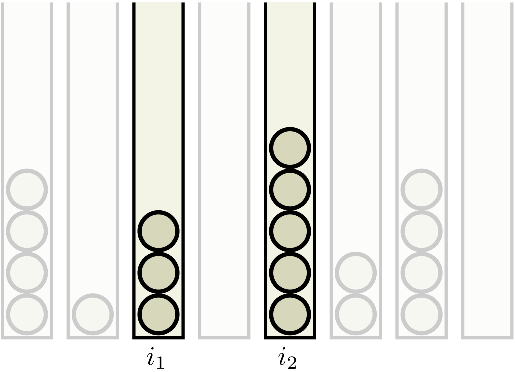

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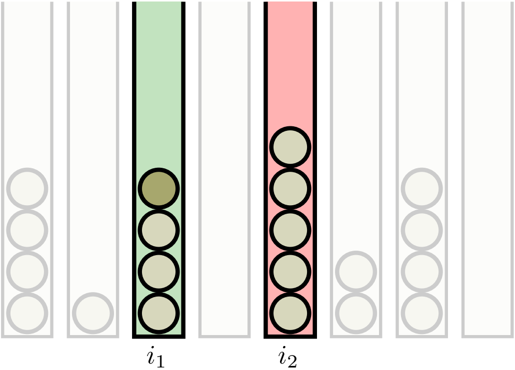

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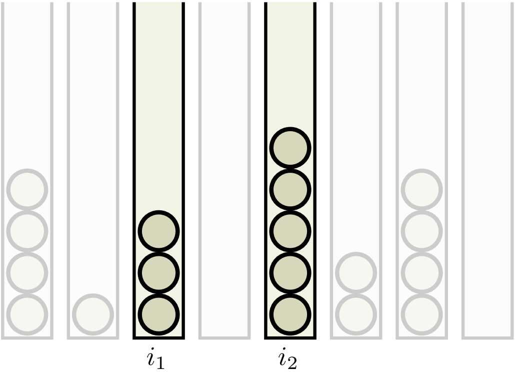

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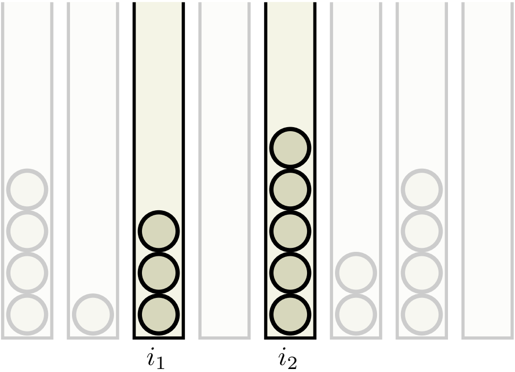

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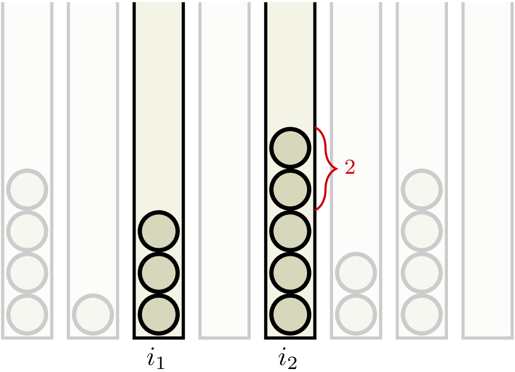

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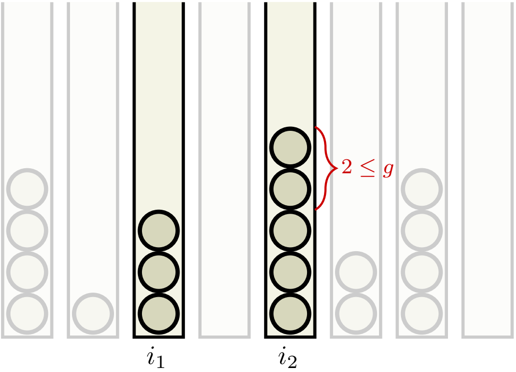

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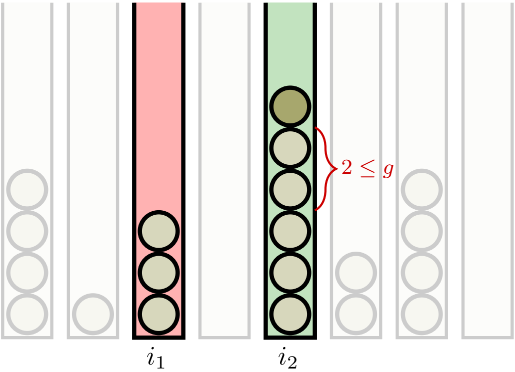

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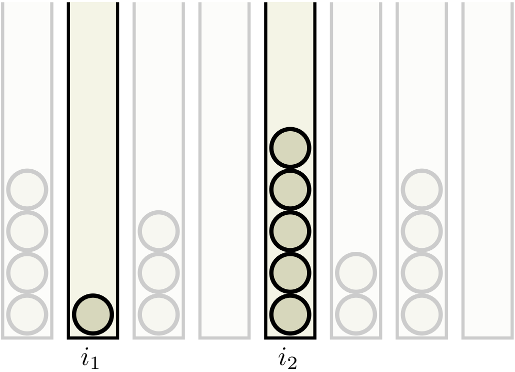

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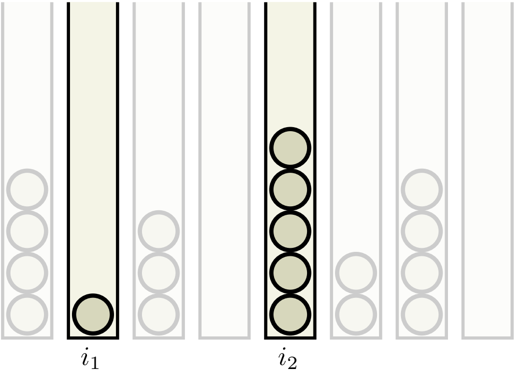

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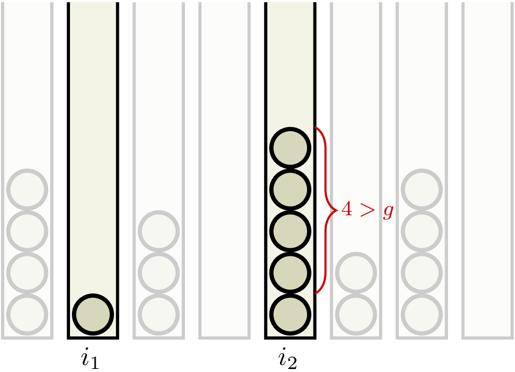

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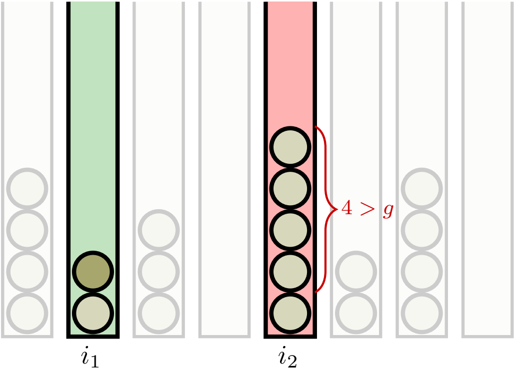

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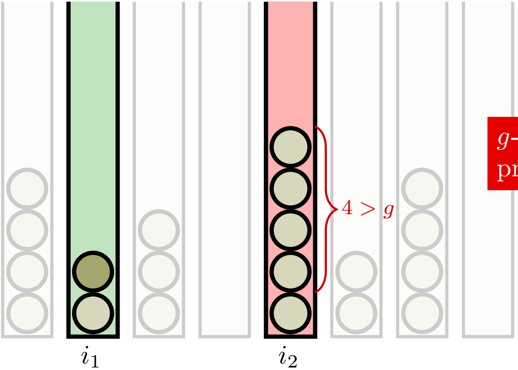

g-BOUNDED: Adversary prefers **heavier** bins.

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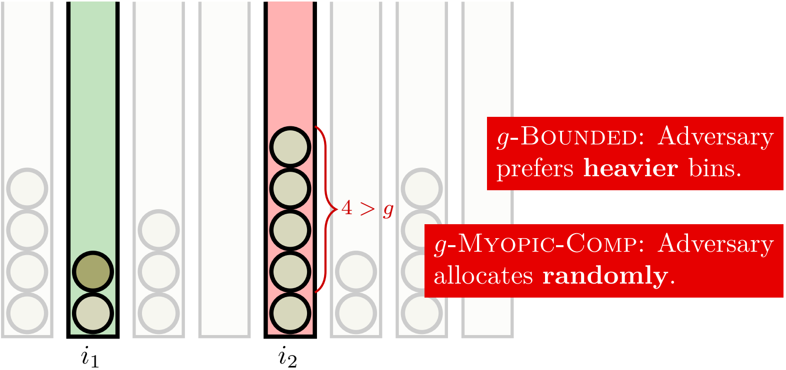

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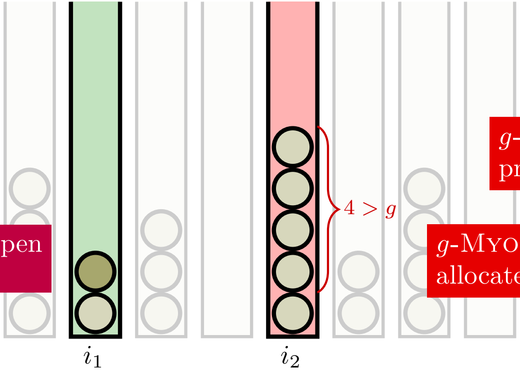

g-Bounded: Open in Visualiser.

g-Myopic-Comp: Open in Visualiser.

g-Bounded: Adversary prefers **heavier** bins.

g-Myopic-Comp: Adversary allocates **randomly**.

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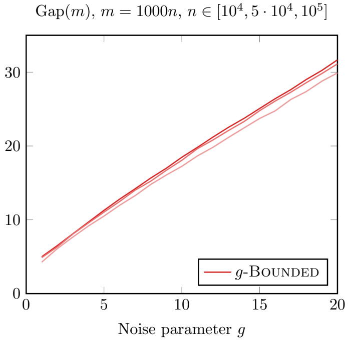

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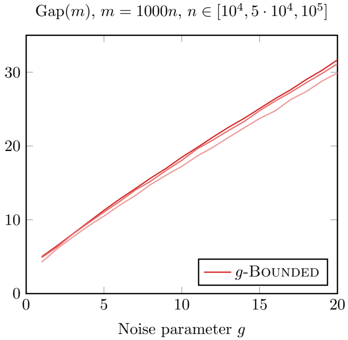

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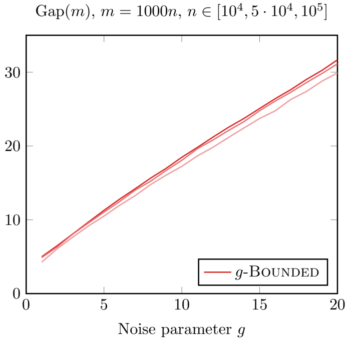

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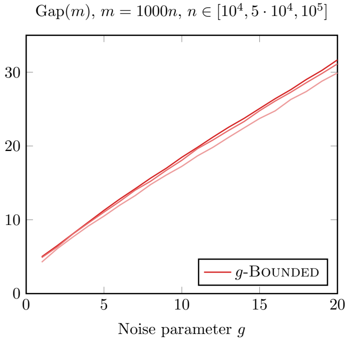

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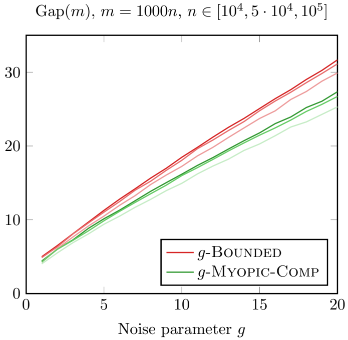
  - Otherwise, for any m, w.h.p. $\mathrm{Gap}(m) = \mathcal{O}(\frac{g}{\log g} \cdot \log \log n)$.
- For both cases, we prove a matching lower bound for g-MYOPIC-COMP.



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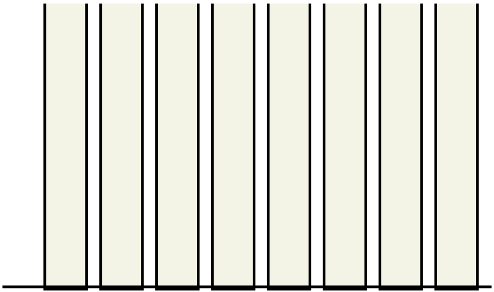

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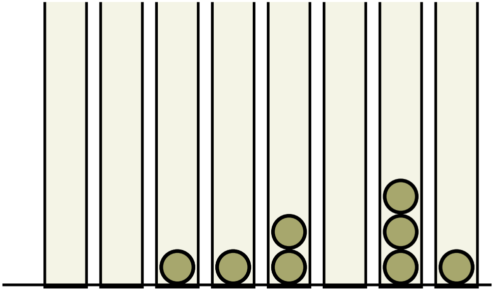

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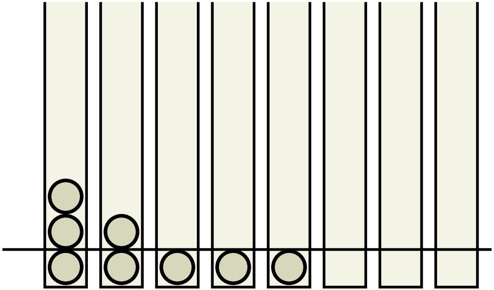

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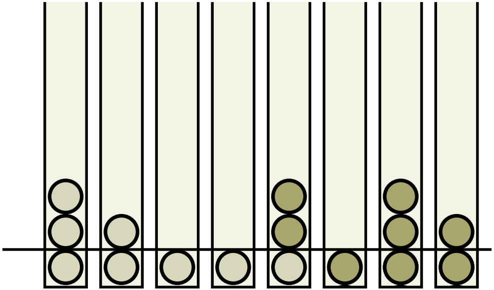

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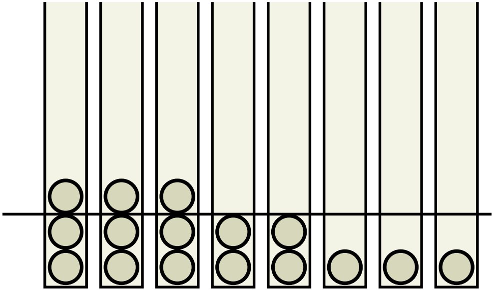

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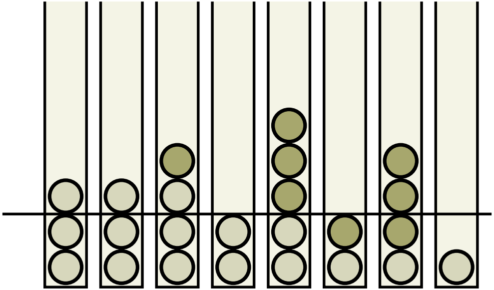

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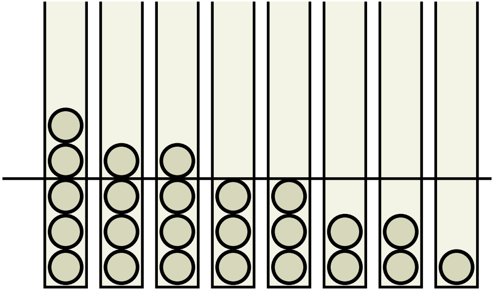

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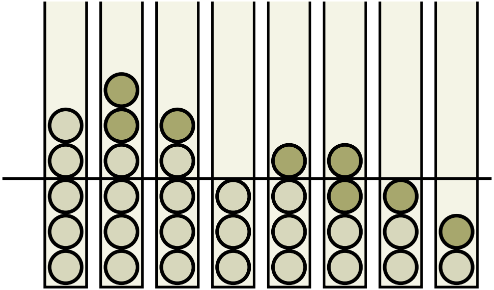

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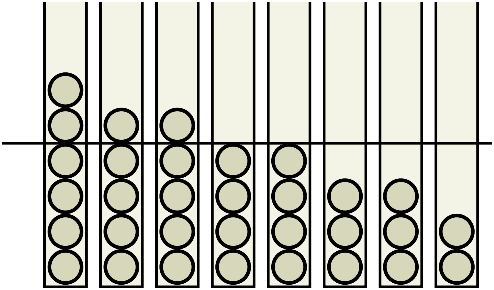

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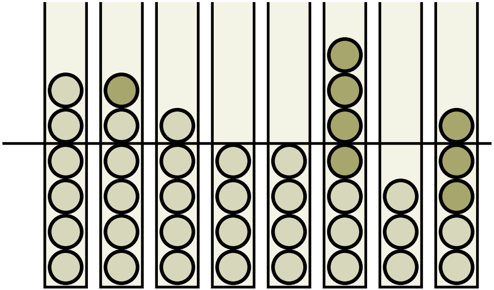

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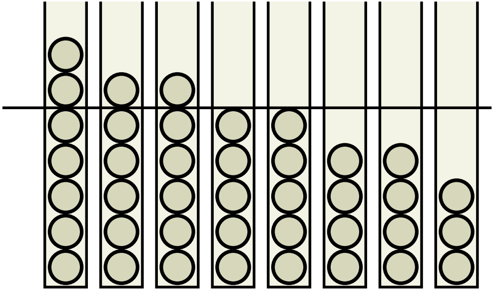

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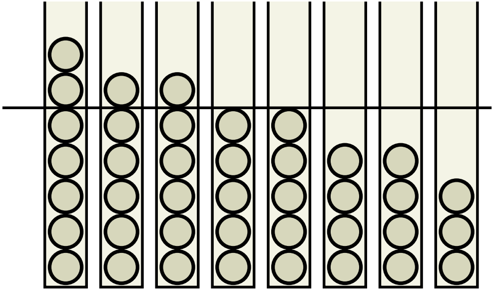

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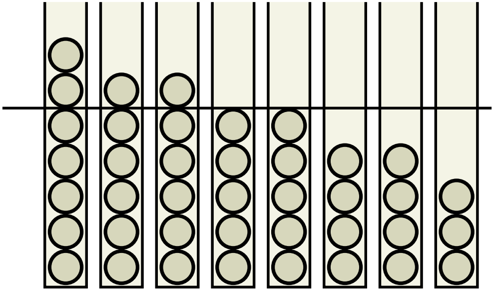

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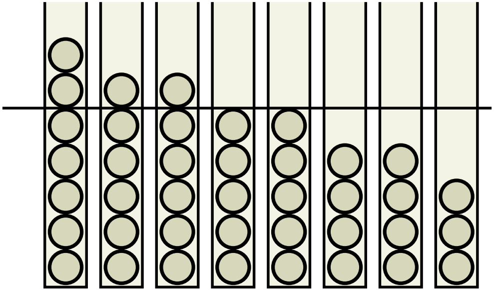

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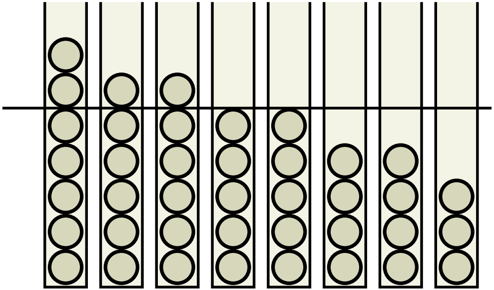

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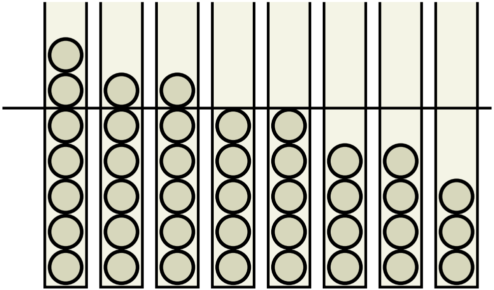

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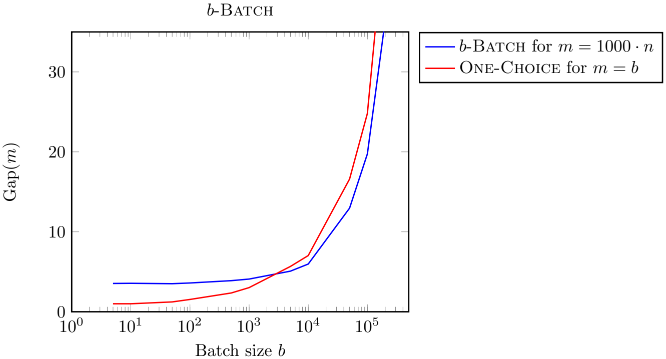

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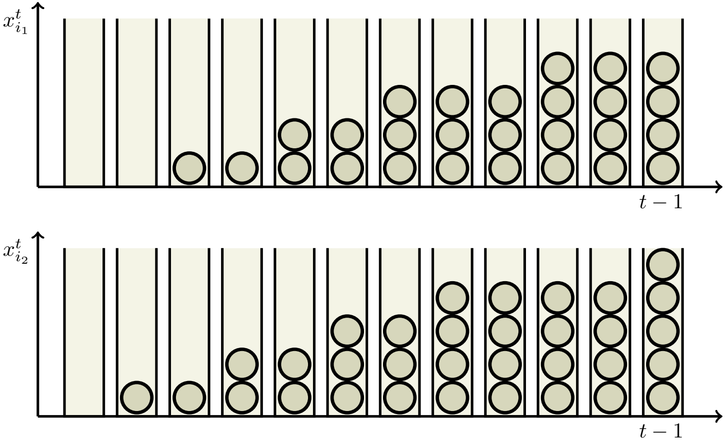

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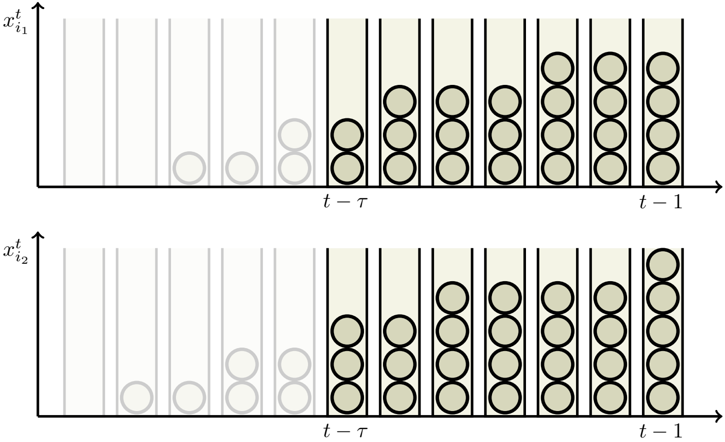

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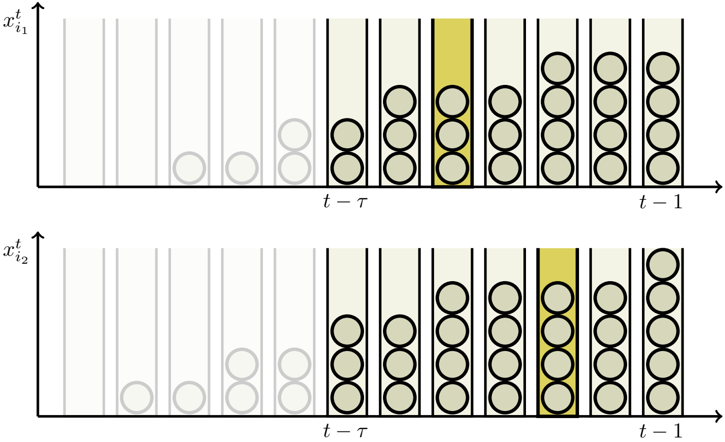

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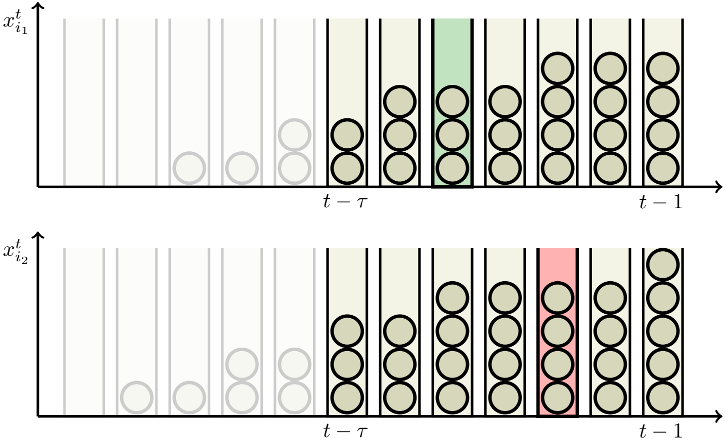

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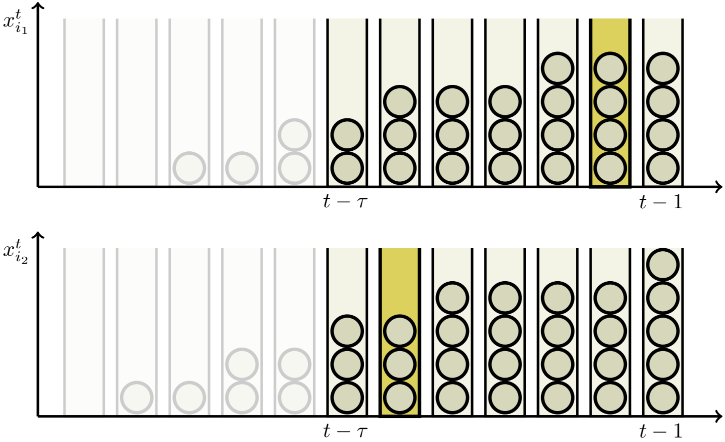

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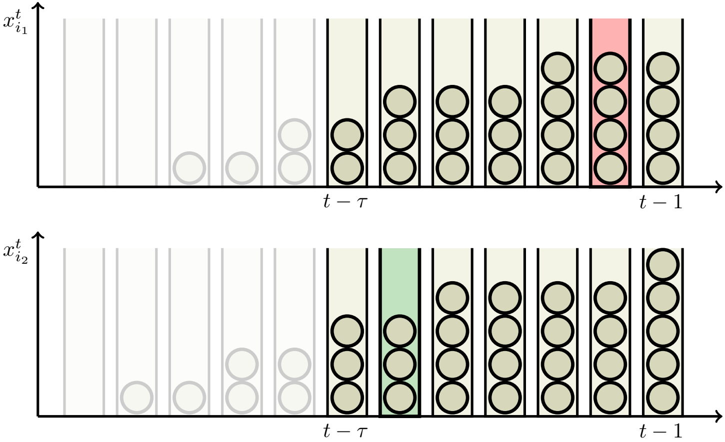

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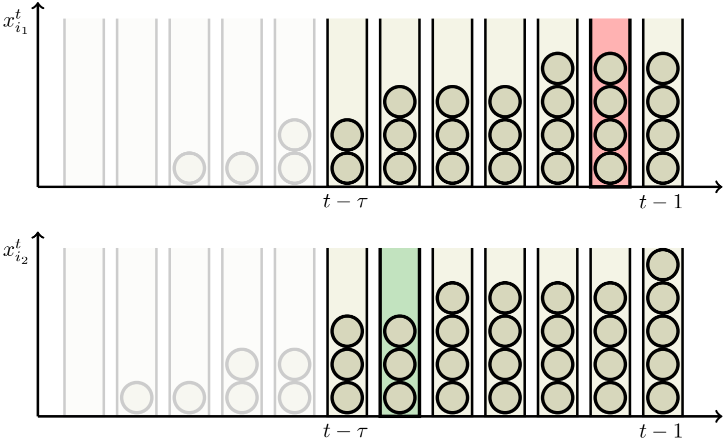

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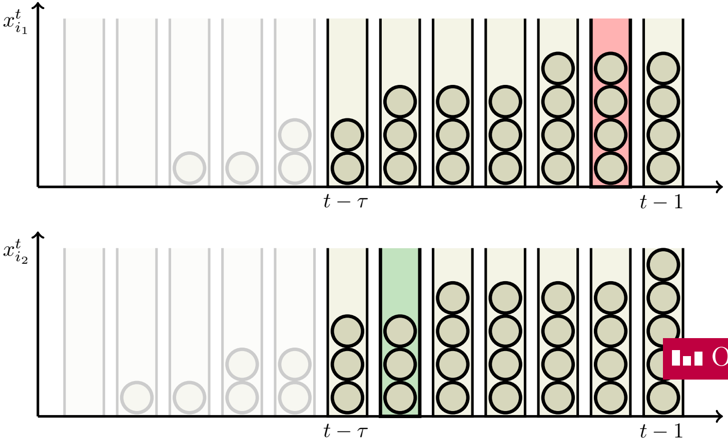

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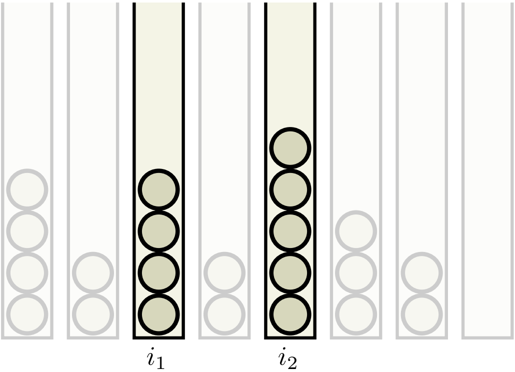

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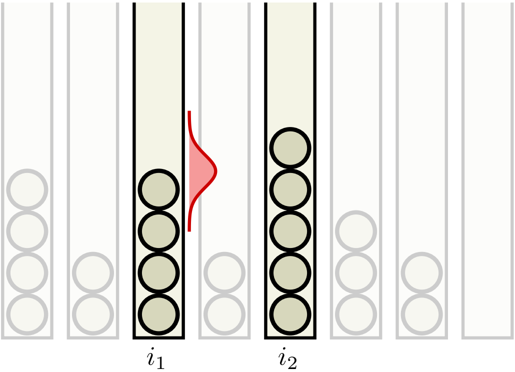

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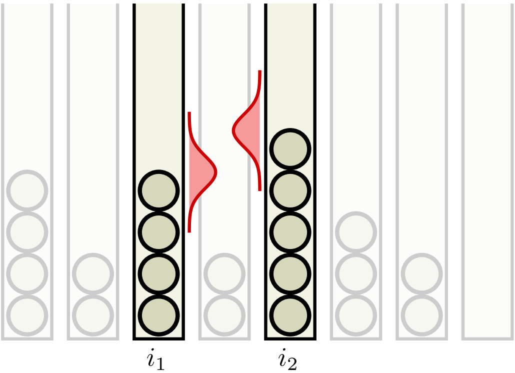

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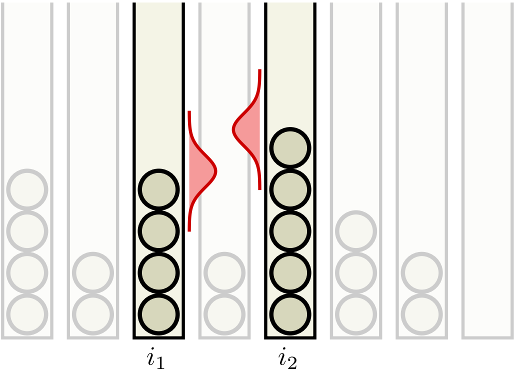

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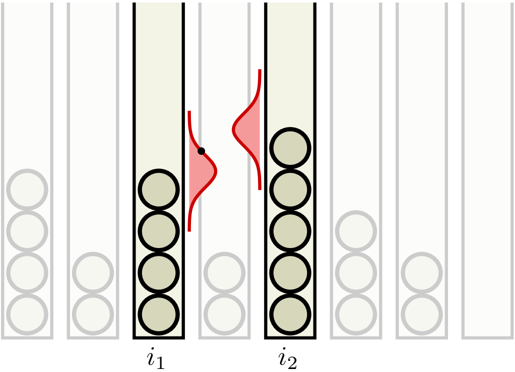

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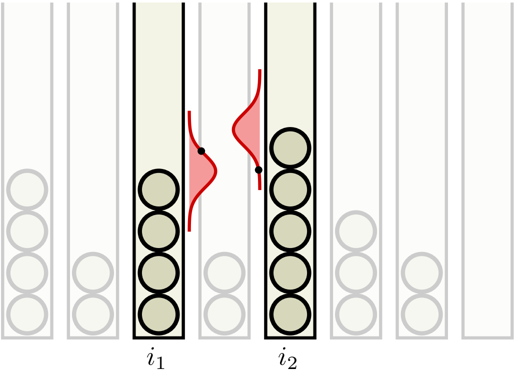

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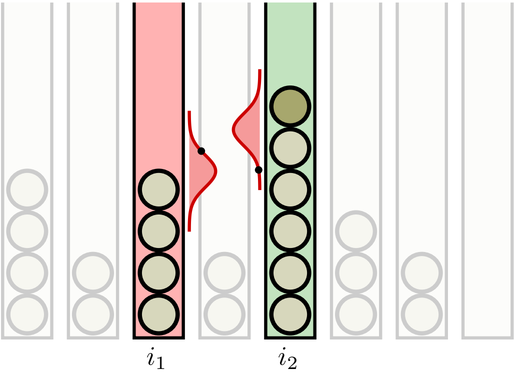

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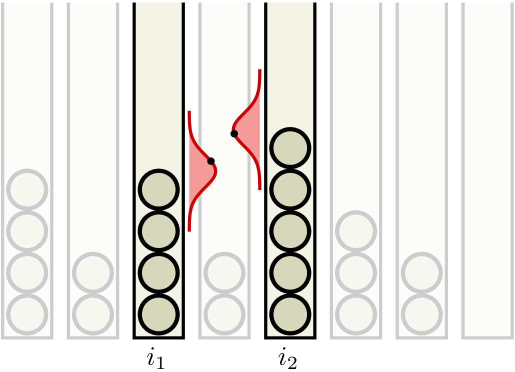

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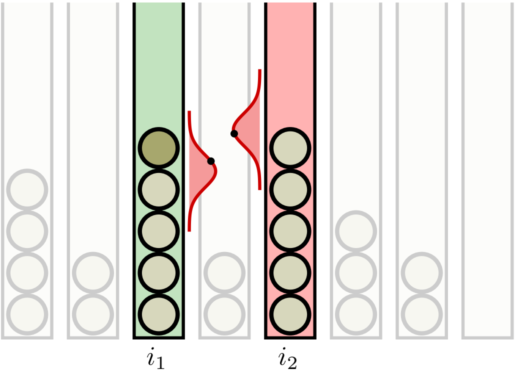

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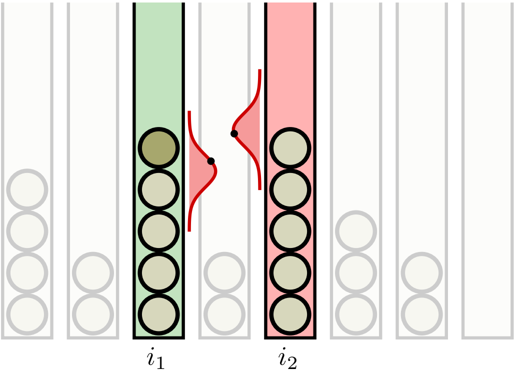

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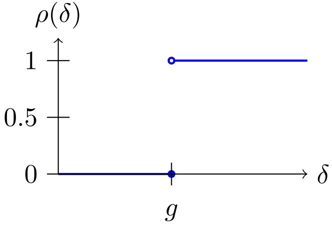

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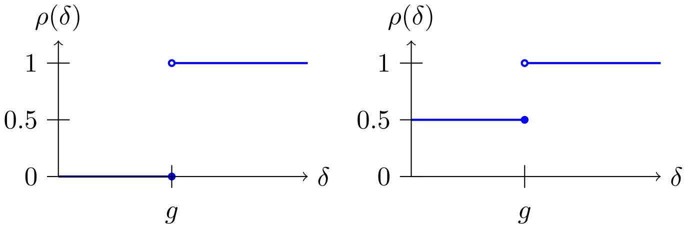

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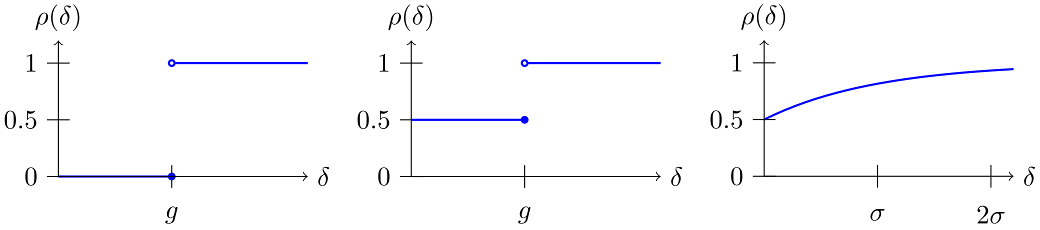

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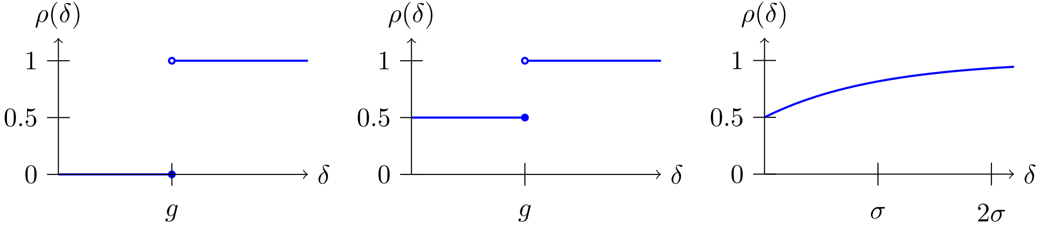

Techniques

Overview

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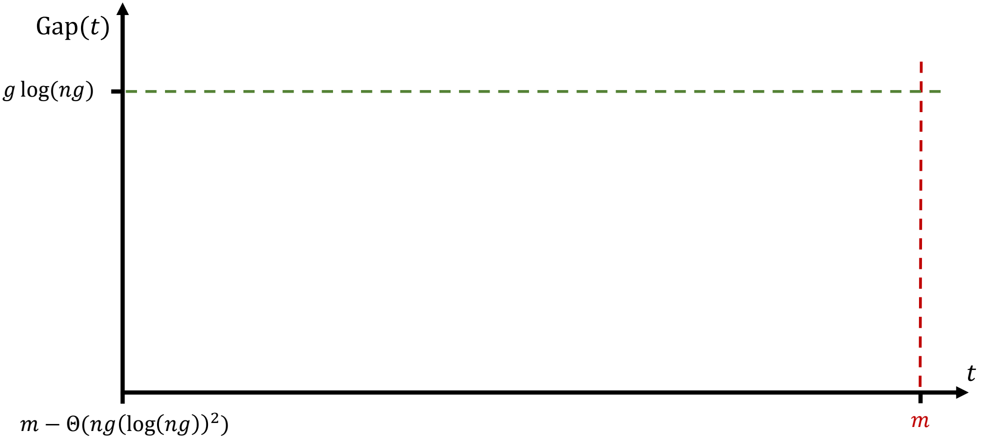

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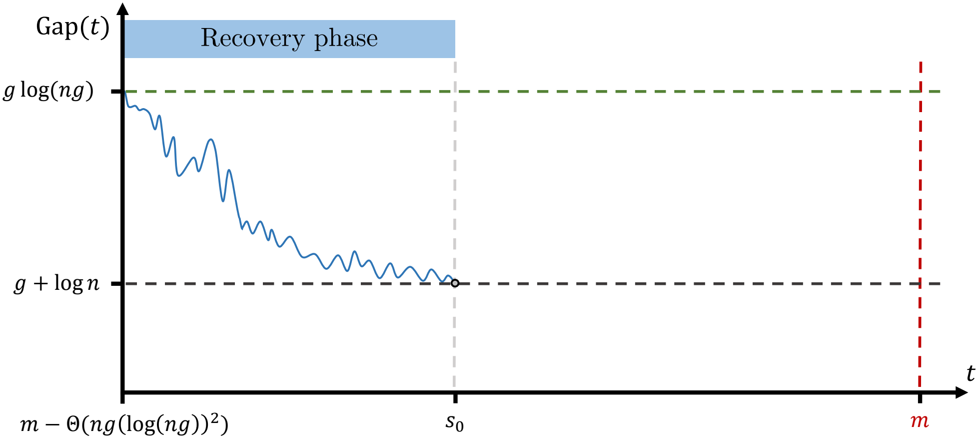

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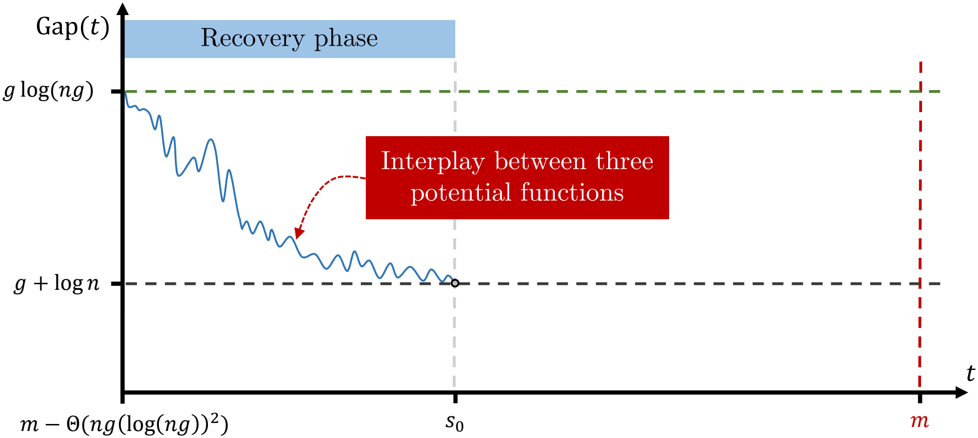
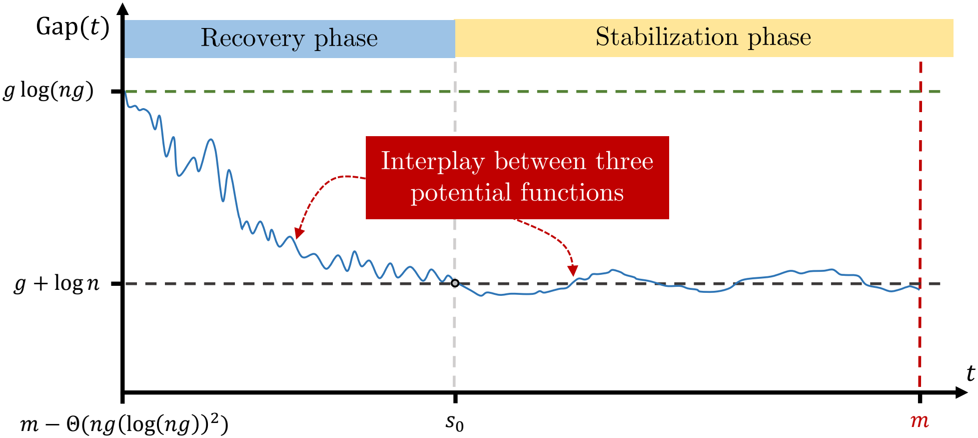


Gap(t)
Recovery phase
$g \log(ng)$
Interplay between three potential functions
$g + \log n$
t
$m - \Theta(ng(\log(ng))^2)$
s_0
m

Overview



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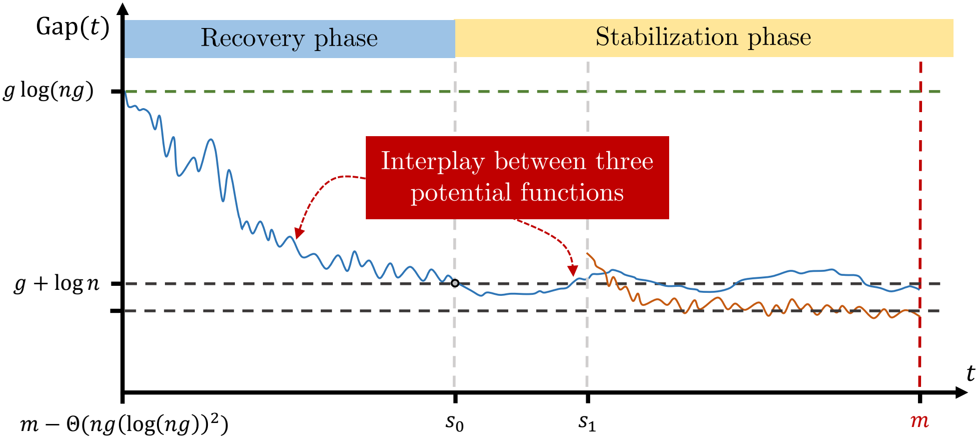

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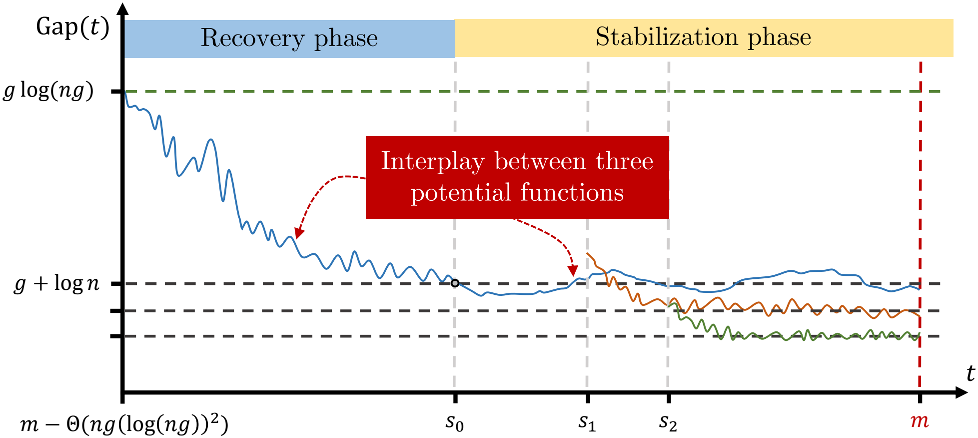

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Recovery phase
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s_2
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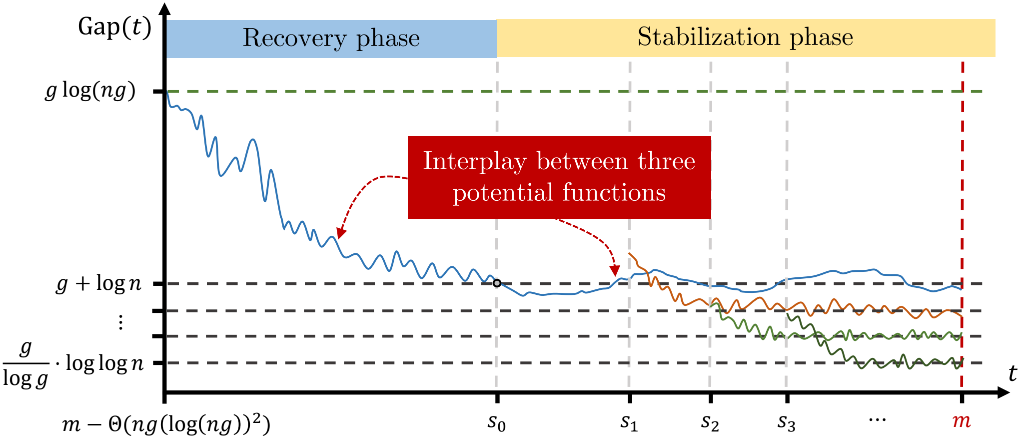

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Recovery phase
Stabilization phase
$g \log(ng)$
Interplay between three potential functions
$g + \log n$
⋮
$\frac{g}{\log g} \cdot \log\log n$
t
$m - \Theta(ng(\log(ng))^2)$
s_0
s_1
s_2
s_3
⋯
m

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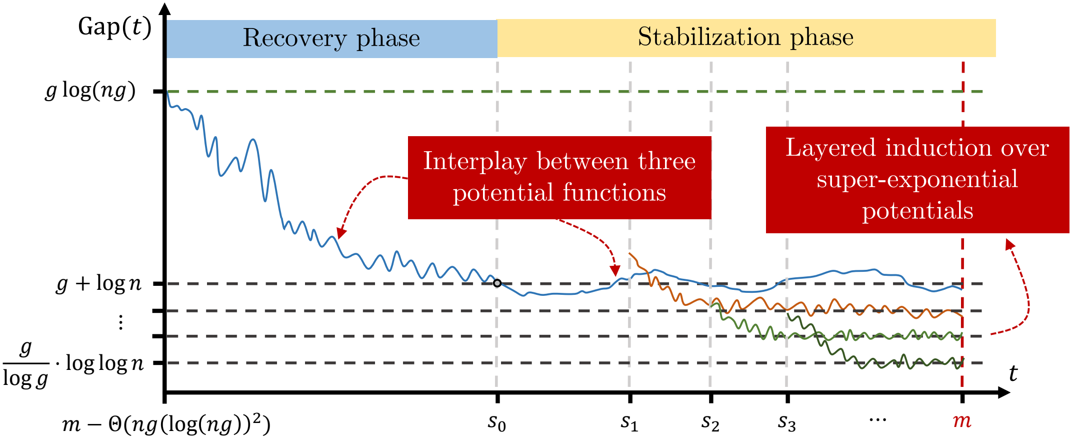

Gap(t)
Recovery phase
Stabilization phase
$g\log(ng)$
Interplay between three potential functions
Layered induction over super-exponential potentials
$g+\log n$
⋮
$\frac{g}{\log g}\cdot\log\log n$
t
$m-\Theta(ng(\log(ng))^2)$
s_0
s_1
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Probability allocation vectors

- **Probability allocation vector** p^t, where p_i^t is the prob. of allocating to i-th heaviest bin.

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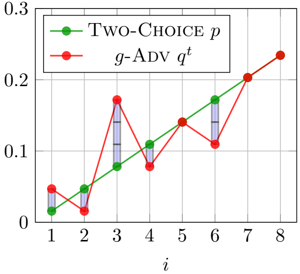

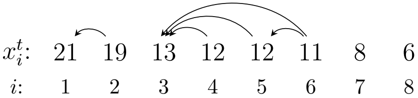

Upper Bound: $\mathcal{O}(g + \log n)$

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- We use the **hyperbolic cosine potential** [PTW15] with constant $\gamma > 0$:

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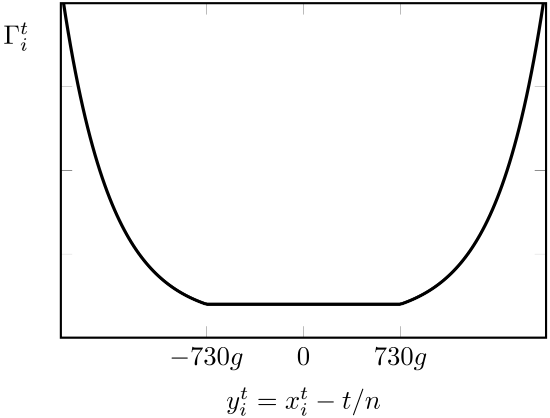

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How can we prove that the potential drops in expectation over multiple steps when large?

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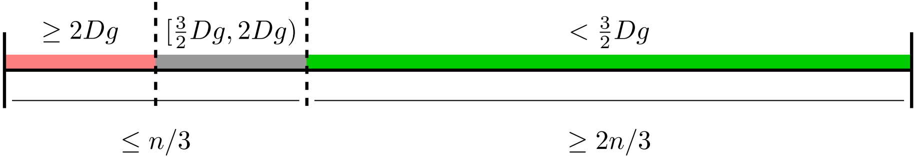

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- This concludes the $\mathcal{O}(g + \log n)$ bound.

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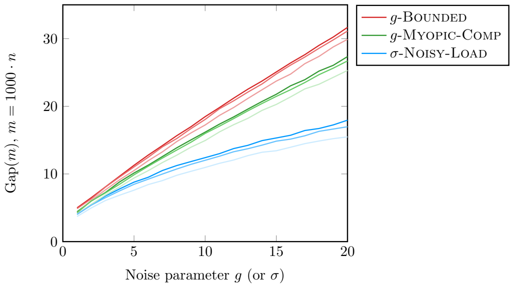
Future work:

- Improve the bounds for σ-Noisy-Load (or other distributions ρ).



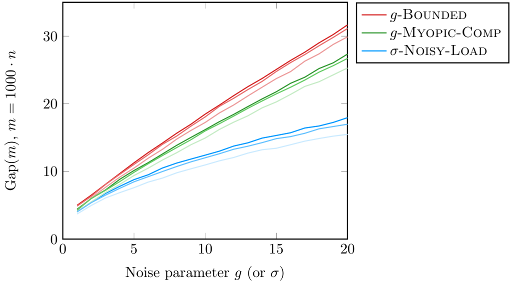
Summary & Future Work

Summary of results:

- For any g-Adv process,
  - If $g \geq \log n$, then for any m, w.h.p. $\mathrm{Gap}(m) = \mathcal{O}(g)$.
  - Otherwise, for any m, w.h.p. $\mathrm{Gap}(m) = \mathcal{O}\left(\frac{g}{\log g} \cdot \log\log n\right)$.
- Matching lower bound for the g-Myopic-Comp process.
- Tight bounds for Two-Choice with outdated information.

Future work:

- Improve the bounds for σ-Noisy-Load (or other distributions ρ).
- Analyze the noisy and outdated setting for other processes.



Summary & Future Work

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- For any g-ADV process,
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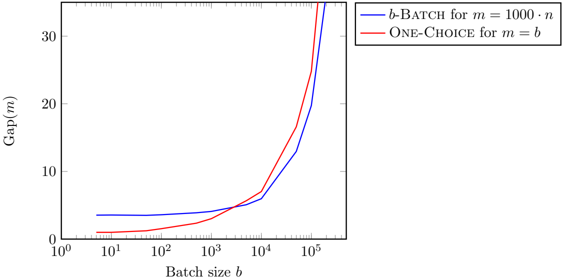
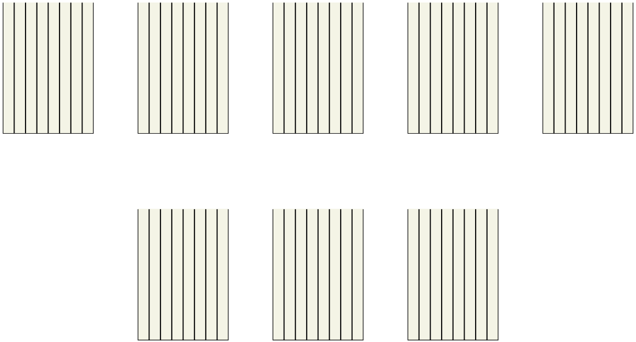


Questions?



Visualisations: `dimitrioslos.com/podc22`

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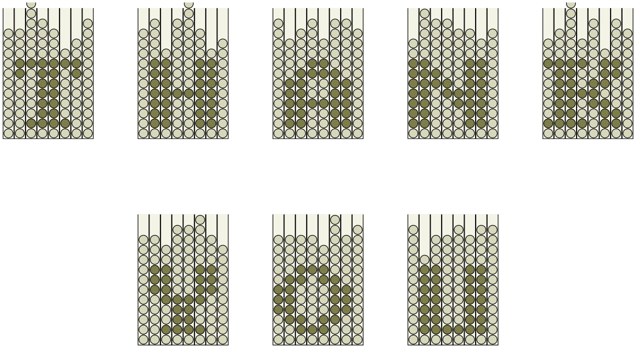

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Appendix A: Detailed results for noise models

Model	Range	Lower Bound	Upper Bound
g-Bounded	$1 \leq g$	–	$\mathcal{O}(g \cdot \log(ng))$
g-Adv	$1 \leq g$	–	$\mathcal{O}(g + \log n)$
g-Adv	$1 < g \leq \log n$	–	$\mathcal{O}\big(\frac{g}{\log g} \cdot \log \log n\big)$
g-Myopic-Comp	$\frac{\log n}{\log \log n} \leq g$	$\Omega(g)$	—
g-Myopic-Comp	$1 < g \leq \frac{\log n}{\log \log n}$	$\Omega\big(\frac{g}{\log g} \cdot \log \log n\big)$	–
σ-Noisy-Load	$1 \leq \sigma$	–	$\mathcal{O}(\sigma \sqrt{\log n} \cdot \log(n\sigma))$
σ-Noisy-Load	$2 \cdot (\log n)^{-1/3} \leq \sigma$	$\Omega(\min\{1, \sigma\} \cdot (\log n)^{1/3})$	–
σ-Noisy-Load	$32 \leq \sigma$	$\Omega(\min\{\sigma^{4/5}, \sigma^{2/5} \cdot \sqrt{\log n}\})$	–

Table: Overview of the lower and upper bounds for Two-Choice with noisy information derived in previous works (rows in Gray) and in this work (rows in Green). Upper bounds hold for all values of $m \geq n$, while lower bounds may only hold for a suitable value of m.

Appendix A: Detailed results for outdated information

Model	Range	Lower Bound	Upper Bound
b-Batch	$b = \Omega(n \log n)$	$\Omega(b/n)$	$\mathcal{O}(b/n)$
b-Batch	$b = n$	$\Omega(\frac{\log n}{\log \log n})$	$\mathcal{O}(\log n)$
τ-Delay	$\tau = n$	–	$\mathcal{O}(\frac{\log n}{\log \log n})$
τ-Delay	$\tau \in \left[n \cdot e^{-(\log n)^c}, n \log n\right]$	–	$\mathcal{O}\left(\frac{\log n}{\log((4n/\tau) \log n)}\right)$
τ-Delay	$\tau = n^{1-\epsilon}$	–	$\mathcal{O}(\log \log n)$
b-Batch	$b = n$	$\Omega(\frac{\log n}{\log \log n})$	–
b-Batch	$b \in \left[n \cdot e^{-(\log n)^c}, n \log n\right]$	$\Omega\left(\frac{\log n}{\log((4n/b) \log n)}\right)$	–
b-Batch	$b = n^{1-\epsilon}$	$\Omega(\log \log n)$	–

Table: Overview of the lower and upper bounds for Two-Choice with outdated information, derived in previous works (rows in Gray) and in this work (rows in Green). Upper bounds hold for all values of $m \geq n$, while lower bounds may only hold for a suitable value of m.

Appendix B: Analysis outline for outdated information

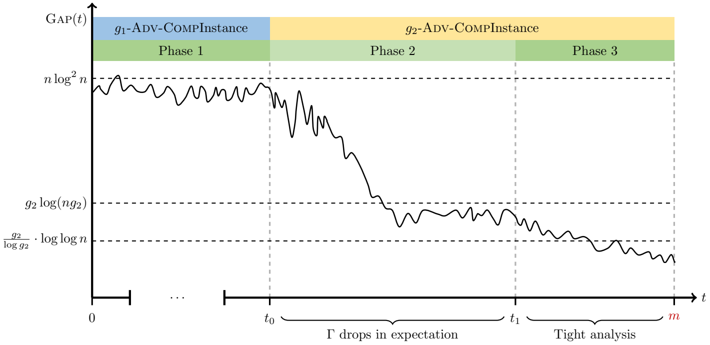

Figure: τ-Delay (and b-Batch) can be exactly simulated using a g_1-Adv-Comp process with $g_1 = \tau \leq n \log n$. This gives the $\mathcal{O}(n \log^2 n)$ gap (since $\tau \leq n \log n$). Then w.h.p. for n^3 steps it can be simulated using a g_2-Adv-Comp process where g_2 is the One-Choice gap for 2τ balls.

Appendix C: Upper bound of $\mathcal{O}(g \log(ng))$ (I)

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- [PTW15] used the **hyperbolic cosine potential** (with no offset)

$$\Gamma^t(x^t) := \underbrace{\sum_{i=1}^{n} e^{\gamma(x_i^t - t/n)}}_{\text{Overload potential}} + \underbrace{\sum_{i=1}^{n} e^{-\gamma(x_i^t - t/n)}}_{\text{Underload potential}} .$$

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- [PTW15] showed that for Two-Choice, for small enough γ,

$$\begin{aligned}\mathbf{E}\left[\, \Gamma^{t+1} \mid y^t \,\right] &\leq \Gamma^t + \sum_{i=1}^{n} p_i \cdot (\gamma + \gamma^2) \cdot e^{\gamma y_i^t} + p_i \cdot (-\gamma + \gamma^2) \cdot e^{-\gamma y_i^t} + \mathcal{O}\left(\frac{\gamma}{n}\Gamma^t\right) \\ &\leq \Gamma^t \cdot \left(1 - \frac{\gamma}{48n}\right) + c.\end{aligned}$$

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- This gives that $\mathrm{Gap}(m) = \mathcal{O}(\frac{\log(n/\gamma)}{\gamma})$.

Appendix C: Upper bound of $\mathcal{O}(g \log(ng))$ (II)

- For g-ADV, the adversary can "transfer" $2/n^2$ probability from i_1 to i_2 if $|x^t_{i_1} - x^t_{i_2}| \leq g$.

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by choosing $\gamma = \Theta(1/g)$.

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$$\mathbf{E}\left[\, \Gamma^{t+1} \mid y^t \,\right] \leq \Gamma^t + \sum_{i=1}^{n} p_i \cdot (\gamma + \gamma^2) \cdot e^{\gamma y_i^t} + p_i \cdot (-\gamma + \gamma^2) \cdot e^{-\gamma y_i^t} + \mathcal{O}\left(\frac{\gamma}{n}\Gamma^t\right) + \mathcal{O}\left(\frac{\gamma^2}{n}\Gamma^t\right)$$
$$\leq \Gamma^t \cdot \left(1 - \frac{\gamma}{96n}\right) + c.$$

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Appendix D: Upper bound of $\mathcal{O}(\frac{g}{\log g} \log \log n)$ for $g \leq \log n$

Appendix D: Upper bound of $\mathcal{O}(\frac{g}{\log g} \log\log n)$ for $g \leq \log n$

- We define the **super-exponential potentials**, for $1 \leq j \leq \frac{\log\log n}{\log g} := k$:

$$\Phi_j^t := \sum_{i=1}^{n} \exp\left(\gamma \cdot (\log n) \cdot g^{j-k} \cdot \left(x_i^t - \frac{t}{n} - z_j\right)^+\right),$$

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- When $\Phi_{j-1}^t = \mathcal{O}(n)$, then the number of bins i with $x_i^t \geq \frac{t}{n} + z_j$ is at most

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- And so, after $s = n \cdot \text{polylog}(n)$ steps, we get

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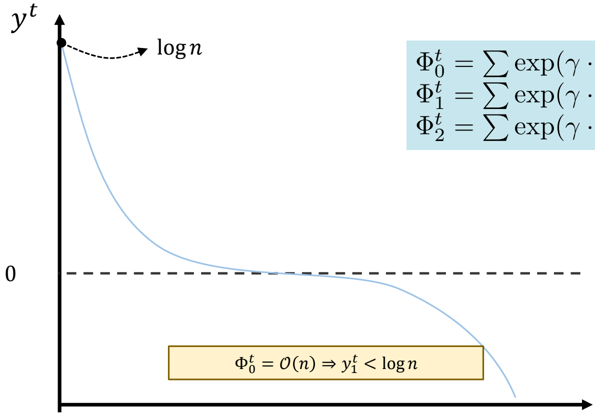
- Finally, when $\Phi_{k-1}^t = \mathcal{O}(n)$, we obtain that

$$\mathrm{Gap}(t) = \mathcal{O}(k \cdot g) = \mathcal{O}\left(\frac{g}{\log g} \log\log n\right).$$

Appendix E: Proving $\mathrm{Gap}(m) = \mathcal{O}(k \cdot g)$, for $g = (\log n)^{1/3}$



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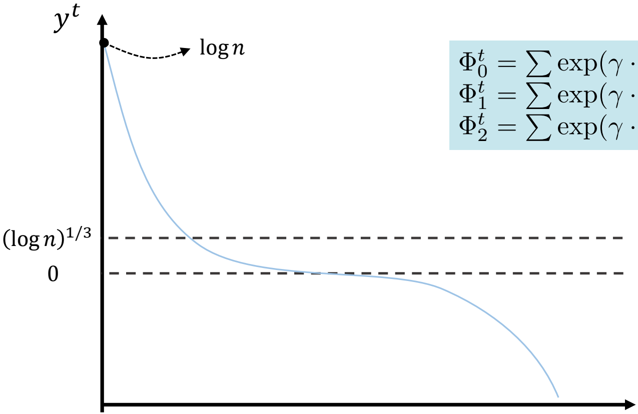

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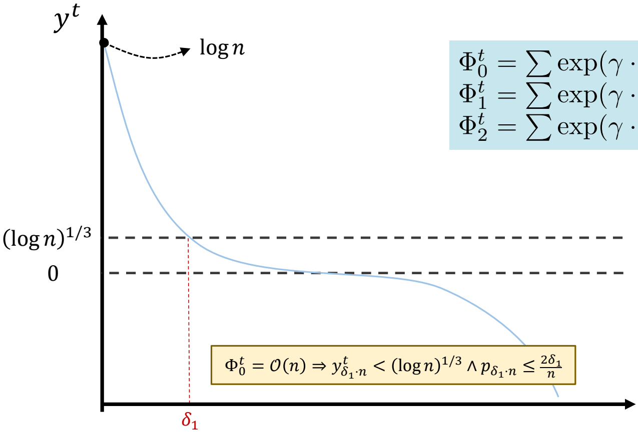

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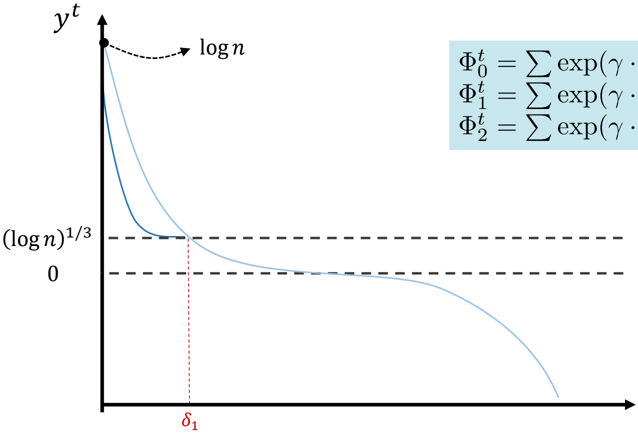
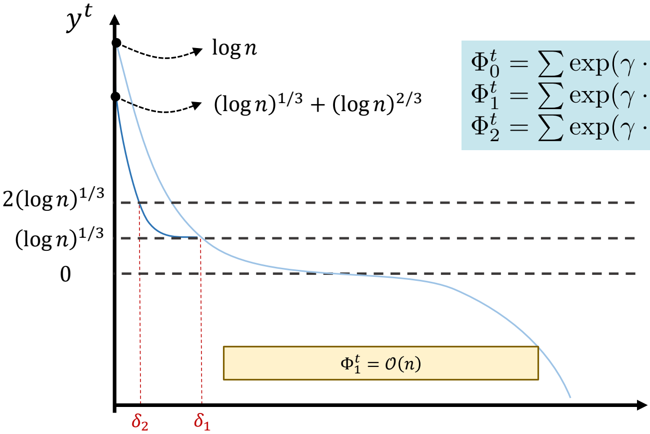
$\Phi_0^t = \mathcal{O}(n) \Rightarrow y_{\delta_1 \cdot n}^t < (\log n)^{1/3} \wedge p_{\delta_1 \cdot n} \leq \frac{2\delta_1}{n}$

Appendix E: Proving $\mathrm{Gap}(m) = \mathcal{O}(k \cdot g)$**, for** $g = (\log n)^{1/3}$



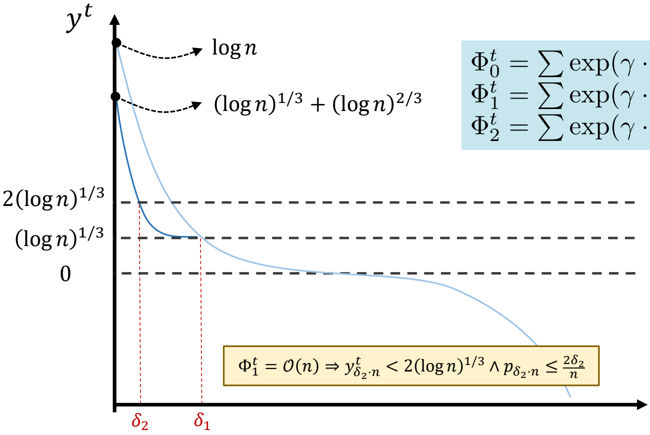
$$\Phi_0^t = \sum \exp(\gamma \cdot (\dots))$$
$$\Phi_1^t = \sum \exp(\gamma \cdot (\log n)^{1/3} \cdot (\dots))$$
$$\Phi_2^t = \sum \exp(\gamma \cdot (\log n)^{2/3} \cdot (\dots))$$

Appendix E: Proving $\mathrm{Gap}(m) = \mathcal{O}(k \cdot g)$, for $g = (\log n)^{1/3}$



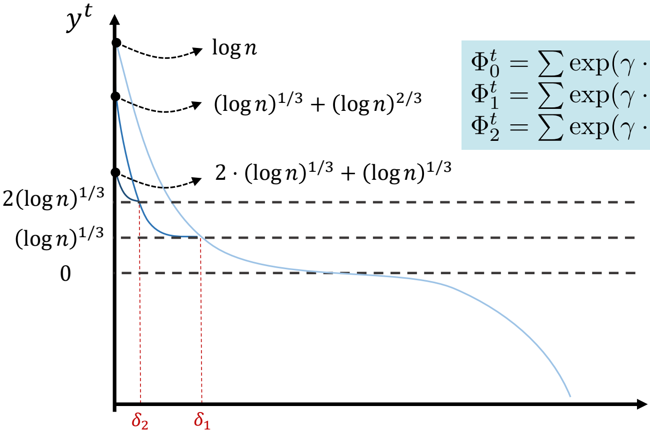
$$\Phi_0^t = \sum \exp(\gamma \cdot (\ldots))$$
$$\Phi_1^t = \sum \exp(\gamma \cdot (\log n)^{1/3} \cdot (\ldots))$$
$$\Phi_2^t = \sum \exp(\gamma \cdot (\log n)^{2/3} \cdot (\ldots))$$

Appendix E: Proving $\text{Gap}(m) = \mathcal{O}(k \cdot g)$, for $g = (\log n)^{1/3}$



$$\Phi_0^t = \sum \exp(\gamma \cdot (\ldots))$$
$$\Phi_1^t = \sum \exp(\gamma \cdot (\log n)^{1/3} \cdot (\ldots))$$
$$\Phi_2^t = \sum \exp(\gamma \cdot (\log n)^{2/3} \cdot (\ldots))$$

Appendix E: Proving $\text{Gap}(m) = \mathcal{O}(k \cdot g)$, for $g = (\log n)^{1/3}$



$$\Phi_0^t = \sum \exp(\gamma \cdot (\ldots))$$
$$\Phi_1^t = \sum \exp(\gamma \cdot (\log n)^{1/3} \cdot (\ldots))$$
$$\Phi_2^t = \sum \exp(\gamma \cdot (\log n)^{2/3} \cdot (\ldots))$$

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