Balanced Allocations with Incomplete Information: The Power of Two Queries

Dimitrios Los, Thomas Sauerwald

University of Cambridge, UK
Balanced allocations: Background
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

Goal: minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$. 
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$. 
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$. 
Balanced allocations setting

Allocate \( m \) tasks (balls) sequentially into \( n \) machines (bins).

**Goal:** minimise the maximum load \( \max_{i \in [n]} x_i^m \), where \( x^t \) is the load vector after ball \( t \).

\( \iff \) minimise the gap, where \( \text{Gap}(m) = \max_{i \in [n]} (x_i^m - m/n) \).
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$.

$\Leftrightarrow$ minimise the gap, where $\text{Gap}(m) = \max_{i \in [n]} (x_i^m - m/n)$. 
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$.

$\iff$ minimise the gap, where $\text{Gap}(m) = \max_{i \in [n]} (x_i^m - m/n)$.

Applications in hashing, load balancing and routing.
**One-Choice and Two-Choice processes**

**One-Choice Process:**

**Iteration:** For each $t \geq 0$, sample **one** bin uniformly at random (u.a.r.) and place the ball there.

**Two-Choice Process:**

**Iteration:** For each $t \geq 0$, sample **two** bins independently u.a.r. and place the ball in the least loaded of the two.

- **In the lightly-loaded case ($m = n$), w.h.p.** $\text{Gap}(n) = \Theta(\log n \log \log n)$ [Gon81].

- **In the heavily-loaded case ($m \gg n$), w.h.p.** $\text{Gap}(m) = \Theta(\sqrt{mn} \log n)$ (e.g. [RS98]).
**One-Choice and Two-Choice processes**

**One-Choice Process:**

Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].

- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta(\sqrt{m \cdot \log n})$ (e.g. [RS98]).

**Two-Choice Process:**

Iteration: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].

- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log_2 \log n + \Theta(1)$ [BCSV06].
**One-Choice and Two-Choice processes**

**One-Choice Process:**
Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].

**Two-Choice Process:**
Iteration: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].

- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log_2 \log n + \Theta(1)$ [BCSV06].

Meaning with probability at least $1 - n^{-c}$ for constant $c > 0$. 
**One-Choice and Two-Choice processes**

**One-Choice Process:**
Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**
Iteration: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log \frac{n}{2} + \Theta(1)$ [KLMadH96, ABKU99].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log \frac{n}{2} + \Theta(1)$ [BCSV06].
One-Choice and Two-Choice processes

One-Choice Process:
Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).

Two-Choice Process:
Iteration: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.
**One-Choice and Two-Choice processes**

**One-Choice Process:**
*Iteration:* For each $t \geq 0$, sample **one** bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**
*Iteration:* For each $t \geq 0$, sample **two** bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
**One-Choice and Two-Choice processes**

**One-Choice Process:**
Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**
Iteration: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
**One-Choice and Two-Choice processes**

**One-Choice Process:**

**Iteration:** For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n} \cdot \log n}\right)$ (e.g. [RS98]).

**Two-Choice Process:**

**Iteration:** For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log_2 \log n + \Theta(1)$ [BCSV06].
**One-Choice and Two-Choice processes**

**One-Choice Process:**

*Iteration:* For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**

*Iteration:* For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log_2 \log n + \Theta(1)$ [BCSV06].
(1 + $\beta$) process: Definition

(1 + $\beta$) Process:
- **Parameter:** A probability $\beta \in (0, 1]$.
- **Iteration:** For each $t \geq 0$, with probability $\beta$ allocate one ball via the TWO-CORE process, otherwise allocate one ball via the ONE-CORE process.
(1 + \( \beta \)) process: Definition

(1 + \( \beta \)) Process:
Parameter: A probability \( \beta \in (0, 1] \).
Iteration: For each \( t \geq 0 \), with probability \( \beta \) allocate one ball via the TWO-CHOICE process, otherwise allocate one ball via the ONE-CHOICE process.

\[ \text{[Mit99]} \] interpreted \( (1 - \beta)/2 \) as the probability of making an erroneous comparison.
(1 + $\beta$) process: Definition

(1 + $\beta$) Process:

Parameter: A probability $\beta \in (0, 1]$.

Iteration: For each $t \geq 0$, with probability $\beta$ allocate one ball via the TWO-CHOICE process, otherwise allocate one ball via the ONE-CHOICE process.

[Mit99] interpreted $(1 - \beta)/2$ as the probability of making an erroneous comparison.

In the heavily-loaded case, [PTW15] proved that the gap is w.h.p. $\Theta(\log n/\beta)$ for $1/n \leq \beta < 1 - \epsilon$ for constant $\epsilon > 0$. 
$k$-Threshold and $k$-Quantile
Adaptive 1-THRESHOLD

Adaptive THRESHOLD$(f)$ Process:

Parameter: A threshold function $f(x^t)$.

Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

\[
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 & \text{if } x_{i_1}^t \leq f(x^t), \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 & \text{otherwise.}
\end{align*}
\]
Adaptive 1-THRESHOLD

Adaptive THRESHOLD($f$) Process:

Parameter: A threshold function $f(x^t)$.

Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

\[
\begin{align*}
 x_{i_1}^{t+1} &= x_{i_1}^t + 1 & \text{if } x_{i_1}^t \leq f(x^t), \\
 x_{i_2}^{t+1} &= x_{i_2}^t + 1 & \text{otherwise}.
\end{align*}
\]
Adaptive 1-THRESHOLD

Adaptive THRESHOLD($f$) Process:
Parameter: A threshold function $f(x^t)$.
Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

$$
\begin{align*}
  x_{i_1}^{t+1} &= x_{i_1}^t + 1 & \text{if } x_{i_1}^t \leq f(x^t), \\
  x_{i_2}^{t+1} &= x_{i_2}^t + 1 & \text{otherwise}.
\end{align*}
$$

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]

\[\]
Adaptive 1-THRESHOLD

Adaptive THRESHOLD($f$) Process:
Parameter: A threshold function $f(x^t)$.
Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

$$
\begin{cases}
    x_{i_1}^{t+1} = x_{i_1}^t + 1 & \text{if } x_{i_1}^t \leq f(x^t), \\
    x_{i_2}^{t+1} = x_{i_2}^t + 1 & \text{otherwise}.
\end{cases}
$$
Adaptive 1-Threshold

Adaptive Threshold ($f$) Process:
Parameter: A threshold function $f(x^t)$.
Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:
\[
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } x_{i_1}^t \leq f(x^t), \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise.}
\end{align*}
\]
Adaptive 1-Threshold

Adaptive Threshold\(f\) Process:
Parameter: A threshold function \(f(x^t)\).
Iteration: For \(t \geq 0\), sample two uniform bins \(i_1\) and \(i_2\) independently, and update:
\[
\begin{align*}
  x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } x_{i_1}^t \leq f(x^t), \\
  x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise}.
\end{align*}
\]
Adaptive 1-THRESHOLD

Adaptive THRESHOLD($f$) Process:

Parameter: A threshold function $f(x^t)$.

Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

$$\begin{cases} 
    x_{i_1}^{t+1} = x_{i_1}^t + 1 & \text{if } x_{i_1}^t \leq f(x^t), \\
    x_{i_2}^{t+1} = x_{i_2}^t + 1 & \text{otherwise.}
\end{cases}$$
Adaptive 1-Threshold

Adaptive Threshold($f$) Process:

Parameter: A threshold function $f(x^t)$.

Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

$$\begin{cases} 
x_i^{t+1}_1 = x_i^t + 1 & \text{if } x_i^t \leq f(x^t), \\
x_i^{t+1}_2 = x_i^t + 1 & \text{otherwise.}
\end{cases}$$

For the lightly-loaded case, [FGG21] determined the optimal threshold, achieving w.h.p. $\text{Gap}(n) = \mathcal{O}\left(\sqrt{\frac{\log n}{\log \log n}}\right)$. 
Adaptive 1-THRESHOLD

Adaptive THRESHOLD($f$) Process:

Parameter: A threshold function $f(x^t)$.

Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

$$\begin{cases} 
    x_{i_1}^{t+1} = x_{i_1}^t + 1 & \text{if } x_{i_1}^t \leq f(x^t), \\
    x_{i_2}^{t+1} = x_{i_2}^t + 1 & \text{otherwise}.
\end{cases}$$

For the lightly-loaded case, [FGG21] determined the optimal threshold, achieving w.h.p. $\text{Gap}(n) = O\left(\sqrt{\frac{\log n}{\log \log n}}\right)$.

In the heavily-loaded case, [LSS21] proved for $f(x^t) = t/n$ that w.h.p. $\text{Gap}(m) = O(\log n)$. 
Adaptive 1-QUANTILE

Adaptive QUANTILE($\delta$) Process:

Parameter: A quantile function $\delta(x^t)$.

Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:

\[
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if} \quad \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise}.
\end{align*}
\]
Adaptive 1-QUANTILE

Adaptive QUANTILE(\(\delta\)) Process:

Parameter: A quantile function \(\delta(x^t)\).

Iteration: For \(t \geq 0\), sample two uniform bins \(i_1\) and \(i_2\) independently, and update:

\[
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise.}
\end{align*}
\]

\(\delta^t\)
Adaptive 1-QUANTILE

Adaptive QUANTILE(δ) Process:

Parameter: A quantile function δ(x^t).

Iteration: For t ≥ 0, sample two uniform bins i_1 and i_2 independently, and update:

\[
\begin{align*}
\text{if } \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
& x^t_{i_1} = x^t_{i_1} + 1
\end{align*}
\]

\[
\begin{align*}
\text{otherwise, } \\
& x^t_{i_2} = x^t_{i_2} + 1
\end{align*}
\]
Adaptive 1-QUANTILE

Adaptive QUANTILE(δ) Process:

Parameter: A quantile function δ(x^t).

Iteration: For t ≥ 0, sample two uniform bins i_1 and i_2 independently, and update:

\[
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if} \quad \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise}.
\end{align*}
\]
Adaptive 1-QUANTILE

Adaptive QUANTILE(δ) Process:

Parameter: A quantile function δ(x^t).

Iteration: For t ≥ 0, sample two uniform bins i₁ and i₂ independently, and update:

\[ \begin{align*} 
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise.} 
\end{align*} \]
Adaptive 1-QUANTILE

Adaptive QUANTILE($\delta$) Process:
Parameter: A quantile function $\delta(x^t)$.
Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:
\[
\begin{cases}
  x_{i_1}^{t+1} = x_{i_1}^t + 1 & \text{if } \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
  x_{i_2}^{t+1} = x_{i_2}^t + 1 & \text{otherwise}.
\end{cases}
\]

Adaptive QUANTILE($\delta$) processes can simulate any adaptive THRESHOLD($f$).
Adaptive 1-QUANTILE

Adaptive QUANTILE($\delta$) Process:
Parameter: A quantile function $\delta(x^t)$.
Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:
\[
\begin{align*}
x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise}.
\end{align*}
\]

- Adaptive QUANTILE($\delta$) processes can simulate any adaptive THRESHOLD($f$).
- Also, adaptive THRESHOLD($f$) process can simulate any adaptive QUANTILE($\delta$).
Adaptive 1-QUANTILE

<table>
<thead>
<tr>
<th>Adaptive QUANTILE(δ) Process:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parameter:</strong> A quantile function $\delta(x^t)$.</td>
</tr>
<tr>
<td><strong>Iteration:</strong> For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:</td>
</tr>
</tbody>
</table>
| $\begin{align*}
  x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } \text{Rank}(x^t, i_1) > \delta(x^t) \cdot n, \\
  x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{otherwise}.
\end{align*}$ |

- Adaptive QUANTILE(δ) processes can simulate any adaptive THRESHOLD($f$).
- Also, adaptive THRESHOLD($f$) process can simulate any adaptive QUANTILE(δ).
- Both are special cases of 2-THINNING [FGG21].
Adaptive 1-QUANTILE

Adaptive QUANTILE($\delta$) Process:
Parameter: A quantile function $\delta(x^t)$.
Iteration: For $t \geq 0$, sample two uniform bins $i_1$ and $i_2$ independently, and update:
\[
\begin{cases}
    x_{i_1}^{t+1} = x_{i_1}^{t} + 1 & \text{if } \text{Rank}(x^t, i_1) > \delta(x^{t}) \cdot n, \\
    x_{i_2}^{t+1} = x_{i_2}^{t} + 1 & \text{otherwise}.
\end{cases}
\]

- Adaptive QUANTILE($\delta$) processes can simulate any adaptive THRESHOLD($f$).
- Also, adaptive THRESHOLD($f$) process can simulate any adaptive QUANTILE($\delta$).
- Both are special cases of 2-THINNING [FGG21].
- [IK05, FL20] analyse $d$-THINNING in the lightly-loaded case.
1-Threshold as Two-Choice with incomplete information

We can interpret 1-Threshold as an instance of the Two-Choice process, where we are only able to compare the loads of the two sampled bins if one is above the threshold and one is below.
1-Threshold as Two-Choice with incomplete information

We can interpret 1-Threshold as an instance of the Two-Choice process, where we are only able to compare the loads of the two sampled bins if one is above the threshold and one is below.
1-Threshold as Two-Choice with incomplete information

We can interpret 1-Threshold as an instance of the Two-Choice process, where we are only able to compare the loads of the two sampled bins if one is above the threshold and one is below.
1-Threshold as Two-Choice with incomplete information

We can interpret 1-Threshold as an instance of the Two-Choice process, where we are only able to compare the loads of the two sampled bins if one is above the threshold and one is below.
1-QUANTILE as TWO-CHOICE with incomplete information

Similarly, 1-QUANTILE is as TWO-CHOICE but we can compare two bins only if these are on different sides of the quantile $\delta^t$. 
1-	extbf{QUANTILE as TWO-CHOICE with incomplete information}

Similarly, 1-QUANTILE is as TWO-CHOICE but we can compare two bins only if these are on different sides of the quantile $\delta^t$. 
$k$-Threshold process

Under this interpretation, we can extend the 1-Threshold process to $k$ thresholds.
Under this interpretation, we can extend the 1-THRESHOLD process to $k$ thresholds.

We can only distinguish two bins if they are in different regions.
$k$-Threshold process

- Under this interpretation, we can extend the 1-Threshold process to $k$ thresholds.
- We can only distinguish two bins if they are in different regions.
\textbf{$k$-Threshold process}

- Under this interpretation, we can extend the 1-\textsc{Threshold} process to $k$ thresholds.
- We can only distinguish two bins if they are in different regions.

\[\text{Diagram showing the process with different regions and thresholds.}\]
$k$-Threshold process

- Under this interpretation, we can extend the 1-Threshold process to $k$ thresholds.
- We can only distinguish two bins if they are in different regions.
- [IK05] analysed the lightly-loaded case for equidistant thresholds.
$k$-QUANTILE process

Similarly, we can extend $1$-QUANTILE to obtain the $k$-QUANTILE process.
Our results

Any adaptive $1$-Quantile / $1$-Threshold process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

A $k$-Quantile process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).
Our results

- Any adaptive $1$-QUANTILE/$1$-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).
- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $\mathcal{O}(k \cdot (\log n)^{1/k})$ gap.

![Graph showing the gap at $m = 1000 \cdot n$]
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $\mathcal{O}(k \cdot (\log n)^{1/k})$ gap.

![Graph showing the gap at $m = 1000 \cdot n$ for different quantile processes.](image-url)
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).
- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).
- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $\mathcal{O}(k \cdot (\log n)^{1/k})$ gap.
- Implications:
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.

- Implications:
  - For $k = \Theta(\log \log n)$, we recover the Two-Choice Gap($m$) = $O(\log \log n)$. 
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $\mathcal{O}(k \cdot (\log n)^{1/k})$ gap.

- Implications:
  - For $k = \Theta(\log \log n)$, we recover the Two-Choice Gap($m$) = $\mathcal{O}(\log \log n)$.
  - For $(1 + \beta)$ with $\beta = 1 - 2^{-0.5(\log n)^{(k-1)/k}}$, w.h.p. Gap($m$) = $\mathcal{O}(k \cdot (\log n)^{1/k})$. 
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.

- Implications:
  - For $k = \Theta(\log \log n)$, we recover the Two-Choice Gap($m$) = $O(\log \log n)$.
  - For $(1 + \beta)$ with $\beta = 1 - 2^{-0.5(\log n)^{(k-1)/k}}$, w.h.p. Gap($m$) = $O(k \cdot (\log n)^{1/k})$.
  - For $d$-THINNING, w.h.p. Gap($m$) = $O(d \cdot (\log n)^{2/d})$. 

Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.

Implications:

- For $k = \Theta(\log \log n)$, we recover the Two-Choice Gap($m$) = $O(\log \log n)$.
- For $(1 + \beta)$ with $\beta = 1 - 2^{-0.5(\log n)^{(k-1)/k}}$, w.h.p. Gap($m$) = $O(k \cdot (\log n)^{1/k})$.
- For $d$-THINNING, w.h.p. Gap($m$) = $O(d \cdot (\log n)^{2/d})$.
- For graphical allocations in dense expanders, w.h.p. Gap($m$) = $O(\log \log n)$ (progress in [PTW15, Open Question 2]).
Our results

- Any adaptive 1-QUANTILE/1-THRESHOLD process has w.h.p. an $\Omega(\log n / \log \log n)$ gap (disproves [FGG21, Problem 1.3]).

- A $k$-QUANTILE process with uniform quantiles that achieves w.h.p. an $O(k \cdot (\log n)^{1/k})$ gap.

Implications:

- For $k = \Theta(\log \log n)$, we recover the Two-Choice Gap($m$) = $O(\log \log n)$.
- For $(1 + \beta)$ with $\beta = 1 - 2^{-0.5(\log n)^{(k-1)/k}}$, w.h.p. Gap($m$) = $O(k \cdot (\log n)^{1/k})$.
- For $d$-THINNING, w.h.p. Gap($m$) = $O(d \cdot (\log n)^{2/d})$.
- For graphical allocations in dense expanders, w.h.p. Gap($m$) = $O(\log \log n)$ (progress in [PTW15, Open Question 2]).

- Use layered induction over super-exponential potential functions.
Lower bound: Proof Outline
Lower bound proof (I)

Theorem

For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process $P$, 

$$ \Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}). $$
Lower bound proof (I)

Theorem

For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process $P$,

$$
\Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
$$

Proof. We consider two cases:

Case A: $P$ uses at most $n$ quantiles with $\delta_t \geq \frac{1}{8 \log \log n}$.

Small quantile means that the first sample is used often.

$P$ disagrees with One-Choice w.h.p. in at most $n + O(m/\log n) = O(n)$ allocations.

Using Poissonisation w.h.p. there are $\Omega(n)$ balls above $\frac{m}{n} + \Omega(\log n)$.

Hence, $\text{Gap}(m) = \Omega(\log n)$.

15
Lower bound proof (I)

**Theorem**
For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process \( P \),

\[
\Pr \left[ \max_{t \in [0,n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

**Proof.** We consider two cases:

**Case A:** \( P \) uses at most \( n \) quantiles with \( \delta^t \geq \frac{1}{\log^2 n} \).
Lower bound proof (I)

**Theorem**

For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process $P$,

$$\Pr \left[ \max_{t \in [0,n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).$$

**Proof.** We consider two cases:

- **Case A:** $P$ uses at most $n$ quantiles with $\delta^t \geq \frac{1}{\log^2 n}$.
  - Small quantile means that the first sample is used often.
Lower bound proof (I)

**Theorem**

For any adaptive QUANTILE($\delta$) (or THRESHOLD($f$)) process $P$,

$$\Pr \left[ \max_{t \in [0,n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).$$

**Proof.** We consider two cases:

**Case A:** $P$ uses at most $n$ quantiles with $\delta^t \geq \frac{1}{\log^2 n}$.

- Small quantile means that the first sample is used often.
- $P$ disagrees with ONE-CHOICE w.h.p. in at most $n + O(m/\log^2 n) = O(n)$ allocations.
Lower bound proof (I)

**Theorem**

For any adaptive QUANTILE\((\delta)\) (or THRESHOLD\((f)\)) process \(P\),

\[
\Pr \left[ \max_{t \in [0, n\log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

**Proof.** We consider two cases:

Case A: \(P\) uses at most \(n\) quantiles with \(\delta^t \geq \frac{1}{\log^2 n}\).

- Small quantile means that the first sample is used often.
- \(P\) disagrees with ONE-CHOICE w.h.p. in at most \(n + \mathcal{O}(m/\log^2 n) = \mathcal{O}(n)\) allocations.
- Using Poissonisation w.h.p. there are \(\Omega(n)\) balls above \(\frac{m}{n} + \Omega(\log n)\).
Lower bound proof (I)

Theorem

For any adaptive QUANTILE(\(\delta\)) (or THRESHOLD(\(f\))) process \(P\),

\[
\Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

Proof. We consider two cases:

Case A: \(P\) uses at most \(n\) quantiles with \(\delta^t \geq \frac{1}{\log^2 n}\).

- Small quantile means that the first sample is used often.
- \(P\) disagrees with ONE-CHOICE w.h.p. in at most \(n + \mathcal{O}(m/\log^2 n) = \mathcal{O}(n)\) allocations.
- Using Poissonisation w.h.p. there are \(\Omega(n)\) balls above \(\frac{m}{n} + \Omega(\log n)\).
- Hence, \(\text{Gap}(m) = \Omega(\log n)\).
Lower bound proof (II)

**Theorem**

For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process $P$, 

$$\Pr \left[ \max_{t \in [0,n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).$$

**Proof (continued).** We consider two cases:
Lower bound proof (II)

**Theorem**

For any adaptive \textsc{Quantile}(\delta) (or \textsc{Threshold}(f)) process \( P \),

\[
\Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

**Proof (continued).** We consider two cases:

**Case B:** \( P \) uses at least \( n \) quantiles with \( \delta^t \geq \frac{1}{\log^2 n} \).
Lower bound proof (II)

**Theorem**
For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process $P$,

$$\Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).$$

**Proof (continued).** We consider two cases:

**Case B:** $P$ uses at least $n$ quantiles with $\delta^t \geq \frac{1}{\log^2 n}$.

- Break $m$ into intervals of $n$ allocations:
  
  $$\begin{array}{ccccccc}
  n & n & \ldots & n & n & n \\
  \log^2 n & \log^2 n & \ldots & \log^2 n & \log^2 n & \log^2 n
  \end{array}$$

Lower bound proof (II)

Theorem
For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process P,
\[
\Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

Proof (continued). We consider two cases:

Case B: P uses at least \( n \) quantiles with \( \delta^t \geq \frac{1}{\log^2 n} \).

- Break \( m \) into intervals of \( n \) allocations:

\[
\begin{array}{cccccc}
\text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\
\text{n} & \text{n} & \dots & \text{n} & \text{n} \\
\text{log}^2 n & \text{log}^2 n & \dots & \text{log}^2 n & \text{log}^2 n \\
\end{array}
\]

- One interval has \( \geq n/\log^2 n \) balls allocated with \( \delta^t \geq \frac{1}{\log^2 n} \).
Lower bound proof (II)

**Theorem**
For any adaptive QUANTILE(\(\delta\)) (or THRESHOLD(\(f\))) process \(P\),
\[
\Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

**Proof (continued).** We consider two cases:
- **Case B:** \(P\) uses at least \(n\) quantiles with \(\delta^t \geq \frac{1}{\log^2 n}\).

  □ Break \(m\) into intervals of \(n\) allocations:

  ![Diagram of intervals](image)

  □ One interval has \(\geq n/\log^2 n\) balls allocated with \(\delta^t \geq \frac{1}{\log^2 n}\).
Lower bound proof (II)

Theorem
For any adaptive QUANTILE(δ) (or THRESHOLD(f)) process \( P \),
\[
\Pr \left[ \max_{t \in [0,n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

Proof (continued). We consider two cases:

Case B: \( P \) uses at least \( n \) quantiles with \( \delta^t \geq \frac{1}{\log^2 n} \).

- Break \( m \) into intervals of \( n \) allocations:

- One interval has \( \geq n/\log^2 n \) balls allocated with \( \delta^t \geq \frac{1}{\log^2 n} \).

- In this interval, w.h.p. \( \Omega(n/\log^4 n) \) balls allocated using ONE-CHOICE.
Theorem
For any adaptive QUANTILE(\( \delta \)) (or THRESHOLD(\( f \))) process \( P \),
\[
\Pr \left[ \max_{t \in [0, n \log^2 n]} \text{Gap}(t) \geq \frac{1}{8} \cdot \frac{\log n}{\log \log n} \right] \geq 1 - o(n^{-2}).
\]

Proof (continued). We consider two cases:

Case B: \( P \) uses at least \( n \) quantiles with \( \delta^t \geq \frac{1}{\log^2 n} \).

- Break \( m \) into intervals of \( n \) allocations:

- One interval has \( \geq n/\log^2 n \) balls allocated with \( \delta^t \geq \frac{1}{\log^2 n} \).
- In this interval, w.h.p. \( \Omega(n/\log^4 n) \) balls allocated using ONE-CHOICE.
- Leads w.h.p. to an \( \Omega(\log n/\log \log n) \) gap.
Upper bound: Proof outline
Consider the QUANTILE($\delta_1, \delta_2, \ldots, \delta_k$) process with

$$\delta_j := \begin{cases} 
2^{-0.5}(\log n)^{(k-j)/k} & \text{if } j < k \\
\frac{1}{2} & \text{if } i = k.
\end{cases}$$

For any $t \geq 0$, $\Pr \left[ \text{Gap}(t) = O(k \cdot (\log n)^{1/k}) \right] \geq 1 - o(n^{-2})$. 

\[ \delta_{k-2} \delta_{k-1} \delta_k \]
The exponential potential function

[PTW15] used the two-sided **exponential potential**

\[
\Gamma^t(x^t) := \sum_{i=1}^{n} e^{\alpha(x^t_i - t/n)} + \sum_{i=1}^{n} e^{-\alpha(x^t_i - t/n)}.
\]

Overload potential: \( \Phi^t_0 \)  Underload potential
The exponential potential function

- [PTW15] used the two-sided **exponential potential**

\[
\Gamma^t(x^t) := \sum_{i=1}^{n} e^{\alpha(x^t_i - t/n)} + \sum_{i=1}^{n} e^{-\alpha(x^t_i - t/n)} .
\]

Overload potential: \(\Phi^t_0\)  
Underload potential

- For the \((1 + \beta)\) process, \(\alpha = \Theta(\beta)\).
The exponential potential function

- [PTW15] used the two-sided **exponential potential**

$$
\Gamma_t(x^t) := \sum_{i=1}^{n} e^{\alpha(x^t_i - t/n)} + \sum_{i=1}^{n} e^{-\alpha(x^t_i - t/n)}.
$$

Overload potential: $\Phi^t_0$  
Underload potential

- For the $(1 + \beta)$ process, $\alpha = \Theta(\beta)$.
- [PTW15] show that $E[\Gamma^{t+1} | \mathcal{F}^t] \leq \Gamma^t \cdot \left(1 - \frac{c_1}{n}\right) + c_2$.
The exponential potential function

- [PTW15] used the two-sided **exponential potential**

\[ \Gamma_t(x^t) := \sum_{i=1}^{n} e^{\alpha(x^t_i - t/n)} + \sum_{i=1}^{n} e^{-\alpha(x^t_i - t/n)} \]

Overload potential: \( \Phi^t_0 \)  
Underload potential

- For the \((1 + \beta)\) process, \( \alpha = \Theta(\beta) \).
- [PTW15] show that \( \mathbb{E} [\Gamma^{t+1} | \mathcal{F}^t] \leq \Gamma^t \cdot \left(1 - \frac{c_1}{n}\right) + c_2 \).
- This implies \( \mathbb{E} [\Gamma^t] \leq c \cdot n \) for any \( t \geq 0 \).
The exponential potential function

- [PTW15] used the two-sided **exponential potential**

\[
\Gamma^t(x^t) := \sum_{i=1}^{n} e^{\alpha(x^t_i - t/n)} + \sum_{i=1}^{n} e^{-\alpha(x^t_i - t/n)}.
\]

Overload potential: \(\Phi^t_0\)  
Underload potential

- For the \((1 + \beta)\) process, \(\alpha = \Theta(\beta)\).
- [PTW15] show that \(\mathbb{E} [\Gamma^{t+1} | \mathcal{F}^t] \leq \Gamma^t \cdot \left(1 - \frac{c_1}{n}\right) + c_2\).
- This implies \(\mathbb{E} [\Gamma^t] \leq c \cdot n\) for any \(t \geq 0\).
- By Markov’s inequality, we get \(\text{Pr} \left[ \Gamma^m \leq cn^3 \right] \geq 1 - n^{-2}\) which implies

\[
\text{Pr} \left[ \text{Gap}(m) \leq \frac{1}{\alpha} \left(3 \cdot \log n + \log c\right) \right] \geq 1 - n^{-2}.
\]
The exponential potential function

- [PTW15] used the two-sided \textbf{exponential potential}

\[ \Gamma^t(x^t) := \sum_{i=1}^{n} e^{\alpha(x^t_i - t/n)} + \sum_{i=1}^{n} e^{-\alpha(x^t_i - t/n)} . \]

\text{Overload potential: } \Phi^t_0 \quad \text{Underload potential}

- For the \((1 + \beta)\) process, \(\alpha = \Theta(\beta)\).

- [PTW15] show that \(\mathbb{E} [\Gamma^{t+1} \mid \mathcal{F}^t] \leq \Gamma^t \cdot \left(1 - \frac{c_1}{n}\right) + c_2\).

- This implies \(\mathbb{E} [\Gamma^t] \leq c \cdot n\) for any \(t \geq 0\).

- By Markov’s inequality, we get \(\Pr \left[ \Gamma^m \leq cn^3 \right] \geq 1 - n^{-2}\) which implies

\[ \Pr \left[ \text{Gap}(m) \leq \frac{1}{\alpha} (3 \cdot \log n + \log c) \right] \geq 1 - n^{-2}. \]

- In [PTW15], \(a = \mathcal{O}(1)\) so the tightest gaps proved were \(\mathcal{O}(\log n)\).
The exponential potential function

- [PTW15] used the two-sided exponential potential

\[ \Gamma_t(x^t) := \sum_{i=1}^n e^{\alpha(x^t_i - t/n)} + \sum_{i=1}^n e^{-\alpha(x^t_i - t/n)} . \]

- Overload potential: \( \Phi^t_0 \)
- Underload potential

- For the \((1 + \beta)\) process, \( \alpha = \Theta(\beta) \).
- [PTW15] show that \( \mathbb{E}[\Gamma^{t+1} | 3^t] \leq \Gamma^t \cdot \left(1 - \frac{c_1}{n}\right) + c_2 \).
- This implies \( \mathbb{E}[\Gamma^t] \leq c \cdot n \) for any \( t \geq 0 \).
- By Markov’s inequality, we get \( \mathbb{Pr}[\Gamma^m \leq cn^3] \geq 1 - n^{-2} \) which implies

\[ \mathbb{Pr} \left[ \text{Gap}(m) \leq \frac{1}{\alpha} (3 \cdot \log n + \log c) \right] \geq 1 - n^{-2}. \]

- In [PTW15], \( a = \mathcal{O}(1) \) so the tightest gaps proved were \( \mathcal{O}(\log n) \).
- [TW14] used this as a base case for TWO-CHOICE in the heavily-loaded case.
Technique 1: Super-exponential potential functions

We define the following super-exponential potential functions for $0 \leq j < k$ and $t \geq 0$:

$$\Phi_t^j := \sum_{i=1}^{n} \exp(\alpha \cdot (\log n)^{j/k} \cdot (x_t^n - t_n^{-2 \alpha j (\log n)^{1/k}})),$$

We prove that when $y_t \delta_k - j \cdot n < 2\alpha j (\log n)^{1/k}$ (good step $G_t^j$), then

$$E[\Phi_{t+1}^j | G_t^j] \leq \Phi_t^j \cdot (1 - 1/n) + 2.$$

So, after $s = n \cdot \text{polylog}(n)$ steps we get $E[\Phi_{t+s}^j | \Phi_0^j = O(n)$, $\cap \tau \in [t, t + s) G_\tau^j = O(n)$.

Observe that when $\Phi_0^j = O(n)$ then at most $O(n \cdot e^{-\alpha z})$ bins have load $\geq z$.

Similarly, when $\Phi_t^j = O(n)$, then $y_\delta k - j - 1 \cdot n < 2\alpha (j + 1)(\log n)^{1/k}$.
Technique 1: Super-exponential potential functions

- We define the following super-exponential potential functions for $0 \leq j < k$ and $t \geq 0$:

$$\Phi^t_j := \sum_{i=1}^{n} \exp \left( \alpha \cdot (\log n)^{j/k} \cdot \left( x_i^t - \frac{t}{n} - \frac{2}{\alpha j (\log n)^{1/k}} \right)^+ \right),$$
Technique 1: Super-exponential potential functions

We define the following super-exponential potential functions for $0 \leq j < k$ and $t \geq 0$:

$$
\Phi^t_j := \sum_{i=1}^{n} \exp \left( \alpha \cdot (\log n)^{j/k} \cdot \left( \frac{x^t_i}{n} - \frac{2}{\alpha} j (\log n)^{1/k} \right)^+ \right),
$$

We prove that when $y^{t}_{\delta_{k-j}n} < \frac{2}{\alpha} j (\log n)^{1/k}$ (good step $G^t_j$), then

$$
\mathbf{E} \left[ \Phi_j^{t+1} \mid G^t_j \right] \leq \Phi_j^t \cdot \left( 1 - \frac{1}{n} \right) + 2.
$$
Technique 1: Super-exponential potential functions

- We define the following \textbf{super-exponential potential functions} for $0 \leq j < k$ and $t \geq 0$:

$$
\Phi^j_t := \sum_{i=1}^{n} \exp \left( \alpha \cdot \frac{\log n}{k} \cdot \left( x_{i}^t - \frac{t}{n} - \frac{2}{\alpha} j \left( \log n \right)^{1/k} \right) \right),
$$

- We prove that when $y_{j,k-\cdot \cdot \cdot n} \leq \frac{2}{\alpha} j \left( \log n \right)^{1/k}$ (good step $G^t_j$), then

$$
\mathbb{E} \left[ \Phi^t_{j+1} \mid G^t_j \right] \leq \Phi^t_{j} \cdot \left( 1 - \frac{1}{n} \right) + 2.
$$

- So, after $s = n \cdot \text{polylog}(n)$ steps we get $\mathbb{E} \left[ \Phi^t_{j+s} \mid \Phi^t_0 = \mathcal{O}(n), \cap_{\tau \in [t,t+s]} G^\tau_j \right] = \mathcal{O}(n)$. 
Technique 1: Super-exponential potential functions

We define the following **super-exponential potential functions** for $0 \leq j < k$ and $t \geq 0$:

$$
\Phi^t_j := \sum_{i=1}^{n} \exp \left( \alpha \cdot \left( \log n \right)^{j/k} \cdot \left( x_t^i - \frac{t}{n} - \frac{2}{\alpha} j \left( \log n \right)^{1/k} \right) \right),
$$

We prove that when $y^t_{\delta k-j} \cdot n < \frac{2}{\alpha} j \left( \log n \right)^{1/k}$ (good step $G^t_j$), then

$$
E \left[ \Phi^{t+1}_j \mid G^t_j \right] \leq \Phi^t_j \cdot \left( 1 - \frac{1}{n} \right) + 2.
$$

So, after $s = n \cdot \text{polylog}(n)$ steps we get $E \left[ \Phi^{t+s}_j \mid \Phi^t_0 = O(n), \cap_{\tau \in [t,t+s]} G^\tau_j \right] = O(n)$.

Observe that when $\Phi^t_0 = O(n)$ then at most $O(n \cdot e^{-\alpha z})$ bins have load $\geq z$. 


Technique 1: Super-exponential potential functions

- We define the following super-exponential potential functions for $0 \leq j < k$ and $t \geq 0$:

\[
\Phi^t_j := \sum_{i=1}^{n} \exp \left( \alpha \cdot (\log n)^{j/k} \cdot \left( x^t_i - \frac{t}{n} - \frac{2}{\alpha} j (\log n)^{1/k} \right)^+ \right),
\]

- We prove that when $y^t_{\delta_{k-j} \cdot n} < \frac{2}{\alpha} j (\log n)^{1/k}$ (good step $G^t_j$), then

\[
\mathbb{E} \left[ \Phi^{t+1}_j \mid G^t_j \right] \leq \Phi^t_j \cdot \left( 1 - \frac{1}{n} \right) + 2.
\]

- So, after $s = n \cdot \text{polylog}(n)$ steps we get $\mathbb{E} \left[ \Phi^{t+s}_j \mid \Phi^t_0 = O(n), \cap_{\tau \in [t,t+s]} G^\tau_j \right] = O(n)$.
- Observe that when $\Phi^t_0 = O(n)$ then at most $O(n \cdot e^{-\alpha z})$ bins have load $\geq z$.
- Similarly, when $\Phi^t_j = O(n)$, then $y^t_{\delta_{k-j-1} \cdot n} < \frac{2}{\alpha} (j + 1) (\log n)^{1/k}$.
Proving \( \text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k}) \)
Proving \( \text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k}) \)
Proving $\text{Gap}(m) = O(k \cdot (\log n)^{1/k})$
Proving $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$
Proving $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$
Proving $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$
Proving $\text{Gap}(m) = O(k \cdot (\log n)^{1/k})$
Technique 2: Proving $\Phi^t_j$ is linear w.h.p.

Assume that $\mathbb{E}[\Phi^\tau_j] = O(n)$ and $G^\tau_j$ for all $\tau \in [t, t + n \cdot \text{polylog}(n))$. 

Using Markov's inequality we get that w.h.p. $\Phi^\tau_j = \text{poly}(n)$.

We define $\Psi^t_j$ as $\Phi^t_j$ with sufficiently smaller $\alpha$.

When $\Phi^\tau_j = \text{poly}(n)$, then $|\Psi^{\tau + 1}_j - \Psi^\tau_j| < n^{1/3}$.

Hence, we apply a bounded difference inequality to get that w.h.p. $\Psi^\tau_j = O(n)$. 


Technique 2: Proving $\Phi_j^t$ is linear w.h.p.

- Assume that $E[\Phi_j^\tau] = O(n)$ and $G_j^\tau$ for all $\tau \in [t, t + n \cdot \text{polylog}(n))$.
- Using Markov’s inequality we get that w.h.p. $\Phi_j^\tau = \text{poly}(n)$. 

We define $\Psi_j^t$ as $\Phi_j^t$ with sufficiently smaller $\alpha$. 

When $\Phi_j^\tau = \text{poly}(n)$, then $|\Psi_j^{\tau+1} - \Psi_j^\tau| < n^{1/3}$.

Hence, we apply a bounded difference inequality to get that w.h.p. $\Psi_j^\tau = O(n)$. 

22
Technique 2: Proving $\Phi^t_j$ is linear w.h.p.

- Assume that $\mathbb{E}[\Phi^\tau_j] = \mathcal{O}(n)$ and $\mathcal{G}^\tau_j$ for all $\tau \in [t, t + n \cdot \text{polylog}(n))$.
- Using Markov’s inequality we get that w.h.p. $\Phi^\tau_j = \text{poly}(n)$.
- We define $\Psi^t_j$ as $\Phi^t_j$ with sufficiently smaller $\alpha$. 
Technique 2: Proving $\Phi^t_j$ is linear w.h.p.

- Assume that $\mathbb{E}[\Phi^\tau_j] = \mathcal{O}(n)$ and $G^\tau_j$ for all $\tau \in [t, t + n \cdot \text{polylog}(n))$.
- Using Markov's inequality we get that w.h.p. $\Phi^\tau_j = \text{poly}(n)$.
- We define $\Psi^t_j$ as $\Phi^t_j$ with sufficiently smaller $\alpha$.
- When $\Phi^\tau_j = \text{poly}(n)$, then $|\Psi^{\tau+1}_j - \Psi^\tau_j| < n^{1/3}$. 
Technique 2: Proving $\Phi^t_j$ is linear w.h.p.

- Assume that $\mathbb{E} \left[ \Phi^\tau_j \right] = O(n)$ and $G^\tau_j$ for all $\tau \in [t, t + n \cdot \text{polylog}(n))$.
- Using Markov’s inequality we get that w.h.p. $\Phi^\tau_j = \text{poly}(n)$.
- We define $\Psi^t_j$ as $\Phi^t_j$ with sufficiently smaller $\alpha$.
- When $\Phi^\tau_j = \text{poly}(n)$, then $|\Psi^{\tau+1}_j - \Psi^\tau_j| < n^{1/3}$.
- Hence, we apply a bounded difference inequality to get that w.h.p. $\Psi^\tau_j = O(n)$. 

Conclusion

Summary of results:

- Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.

Future work:

- Prove lower bounds for adaptive $k$-QUANTILE for $k \geq 2$.
- Prove similar upper bounds for $k$-Threshold.
- Analyse Two-Choice with noise.
Conclusion

Summary of results:

- Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = O(k \cdot (\log n)^{1/k})$.
- Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive $1$-THRESHOLD and $1$-QUANTILE process (power of two queries).

Future work:

- Prove lower bounds for adaptive $k$-QUANTILE for $k \geq 2$.
- Prove similar upper bounds for $k$-THRESHOLD.
- Analyse Two-Choice with noise.
Conclusion

Summary of results:

- Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.
- Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive 1-THRESHOLD and 1-QUANTILE process (power of two queries).
- Implications:
  - For $k = \Theta(\log \log n)$, we get for Two-Choice $\text{Gap}(m) = \mathcal{O}(\log \log n)$ (power of two choices).
  - Tighter upper bounds for $d$-Thinning and $(1 + \beta)$ for $\beta$ close to 1.
  - Graphical allocations on dense expander graphs achieves $\text{Gap}(m) = \mathcal{O}(\log \log n)$.

Future work:

- Prove lower bounds for adaptive $k$-QUANTILE for $k \geq 2$.
- Prove similar upper bounds for $k$-THRESHOLD.
- Analyse Two-Choice with noise.
Conclusion

Summary of results:

■ Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.

■ Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive 1-THRESHOLD and 1-QUANTILE process (power of two queries).

■ Implications:
  
  ▶ For $k = \Theta(\log \log n)$, we get for TWO-CHOICE $\text{Gap}(m) = \mathcal{O}(\log \log n)$ (power of two choices).
Conclusion

Summary of results:

- Introduced a \( k \)-QUANTILE process which achieves w.h.p. \( \text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k}) \).
- Proved a lower bound of \( \Omega(\log n / \log \log n) \) for any adaptive 1-THRESHOLD and 1-QUANTILE process (\textit{power of two queries}).
- Implications:
  - For \( k = \Theta(\log \log n) \), we get for TWO-CHOICE \( \text{Gap}(m) = \mathcal{O}(\log \log n) \) (\textit{power of two choices}).
  - Tighter upper bounds for \( d \)-THINNING.

Future work:

- Prove lower bounds for adaptive \( k \)-QUANTILE for \( k \geq 2 \).
- Prove similar upper bounds for \( k \)-THRESHOLD.
- Analyse TWO-CHOICE with noise.
Conclusion

Summary of results:

- Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.
- Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive 1-THRESHOLD and 1-QUANTILE process (power of two queries).

Implications:

- For $k = \Theta(\log \log n)$, we get for TWO-CHOICE $\text{Gap}(m) = \mathcal{O}(\log \log n)$ (power of two choices).
- Tighter upper bounds for $d$-THINNING and $(1 + \beta)$ for $\beta$ close to 1.
Conclusion

Summary of results:

■ Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.
■ Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive $1$-THRESHOLD and $1$-QUANTILE process (power of two queries).

Implications:

▶ For $k = \Theta(\log \log n)$, we get for TWO-CHOICE $\text{Gap}(m) = \mathcal{O}(\log \log n)$ (power of two choices).
▶ Tighter upper bounds for $d$-THINNING and $(1 + \beta)$ for $\beta$ close to 1.
▶ Graphical allocations on dense expander graphs achieves $\text{Gap}(m) = \mathcal{O}(\log \log n)$. 

Future work:

■ Prove lower bounds for adaptive $k$-QUANTILE for $k \geq 2$.
■ Prove similar upper bounds for $k$-THRESHOLD.
■ Analyse TWO-CHOICE with noise.
Conclusion

Summary of results:

■ Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.

■ Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive 1-THRESHOLD and 1-QUANTILE process (power of two queries).

■ Implications:
  ▶ For $k = \Theta(\log \log n)$, we get for TWO-CHOICE $\text{Gap}(m) = \mathcal{O}(\log \log n)$ (power of two choices).
  ▶ Tighter upper bounds for $d$-THINNING and $(1 + \beta)$ for $\beta$ close to 1.
  ▶ Graphical allocations on dense expander graphs achieves $\text{Gap}(m) = \mathcal{O}(\log \log n)$.

Future work:
Conclusion

Summary of results:

■ Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.

■ Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive 1-THRESHOLD and 1-QUANTILE process (power of two queries).

■ Implications:
  ▶ For $k = \Theta(\log \log n)$, we get for TWO-CHOICE $\text{Gap}(m) = \mathcal{O}(\log \log n)$ (power of two choices).
  ▶ Tighter upper bounds for $d$-THINNING and $(1 + \beta)$ for $\beta$ close to 1.
  ▶ Graphical allocations on dense expander graphs achieves $\text{Gap}(m) = \mathcal{O}(\log \log n)$.

Future work:

■ Prove lower bounds for adaptive $k$-QUANTILE for $k \geq 2$. 
Conclusion

Summary of results:

- Introduced a $k$-QUANTILE process which achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k})$.
- Proved a lower bound of $\Omega(\log n / \log \log n)$ for any adaptive $1$-THRESHOLD and $1$-QUANTILE process (power of two queries).

Implications:

- For $k = \Theta(\log \log n)$, we get for TWO-CHOICE $\text{Gap}(m) = \mathcal{O}(\log \log n)$ (power of two choices).
- Tighter upper bounds for $d$-THINNING and $(1 + \beta)$ for $\beta$ close to 1.
- Graphical allocations on dense expander graphs achieves $\text{Gap}(m) = \mathcal{O}(\log \log n)$.

Future work:

- Prove lower bounds for adaptive $k$-QUANTILE for $k \geq 2$.
- Prove similar upper bounds for $k$-THRESHOLD.
Conclusion

Summary of results:

- Introduced a \( k \)-QUANTILE process which achieves w.h.p. \( \text{Gap}(m) = \mathcal{O}(k \cdot (\log n)^{1/k}) \).
- Proved a lower bound of \( \Omega(\log n / \log \log n) \) for any adaptive 1-THRESHOLD and 1-QUANTILE process (power of two queries).

Implications:

- For \( k = \Theta(\log \log n) \), we get for TWO-CHOICE \( \text{Gap}(m) = \mathcal{O}(\log \log n) \) (power of two choices).
- Tighter upper bounds for \( d \)-THINNING and \((1 + \beta)\) for \( \beta \) close to 1.
- Graphical allocations on dense expander graphs achieves \( \text{Gap}(m) = \mathcal{O}(\log \log n) \).

Future work:

- Prove lower bounds for adaptive \( k \)-QUANTILE for \( k \geq 2 \).
- Prove similar upper bounds for \( k \)-THRESHOLD.
- Analyse TWO-CHOICE with noise.
Questions?

More visualisations: tinyurl.com/ls21-visualisations
Questions?

More visualisations: tinyurl.com/ls21-visualisations
Appendix
Appendix A: Detailed experimental results

<table>
<thead>
<tr>
<th>$n$</th>
<th>$(1 + \beta)$, for $\beta = 0.5$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
<th>$k = 4$</th>
<th>Two-Choice</th>
</tr>
</thead>
</table>

Table: Summary of our Experimental Results ($m = 1000 \cdot n$).
Appendix B: Random $d$-regular graphs

Figure: Average Gap vs. $n \in \{10^3, 10^4, 5 \cdot 10^4\}$ for $d$-regular graphs generated using [SW99].


Bibliography II


