Balanced Allocations: Relaxing Two-Choice

Dimitrios Los\textsuperscript{1}, Thomas Sauerwald\textsuperscript{1}, John Sylvester\textsuperscript{2}

\textsuperscript{1}University of Cambridge, UK
\textsuperscript{2}University of Glasgow, UK
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$. 
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$. 
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$. 
Balanced allocations setting

Allocate \( m \) tasks (balls) sequentially into \( n \) machines (bins).

**Goal:** minimise the **maximum load** \( \max_{i \in [n]} x_i^m \), where \( x^t \) is the load vector after ball \( t \).

\[ \Leftrightarrow \text{minimise the gap, where } \text{Gap}(m) = \max_{i \in [n]} (x_i^m - m/n). \]
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$.

$\iff$ minimise the gap, where $\text{Gap}(m) = \max_{i \in [n]} (x_i^m - m/n)$. 

$\text{Gap}$
Balanced allocations setting

Allocate $m$ tasks (balls) sequentially into $n$ machines (bins).

**Goal:** minimise the maximum load $\max_{i \in [n]} x_i^m$, where $x^t$ is the load vector after ball $t$.

$\iff$ minimise the gap, where $\text{Gap}(m) = \max_{i \in [n]} (x_i^m - m/n)$.

- Applications in hashing, load balancing and routing.
One-Choice and Two-Choice processes

One-Choice Process:
Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

Two-Choice Process:
Iteration: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta(\log n \log \log n)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta(\sqrt{mn} \log n)$ (e.g. [RS98]).

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log \frac{2}{\log n} + \Theta(1)$ [KLMadH96, ABKU99].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log \frac{2}{\log n} + \Theta(1)$ [BCSV06].
**One-Choice and Two-Choice processes**

**One-Choice Process:**

**Iteration:** For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].

- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{mn \cdot \log n}\right)$ (e.g. [RS98]).

**Two-Choice Process:**

**Iteration:** For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log 2^{\log n} + \Theta(1)$ [KLMadH96, ABKU99].

- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log 2^{\log n} + \Theta(1)$ [BCSV06].
### One-Choice and Two-Choice processes

**One-Choice Process:**

**Iteration**: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].

**Two-Choice Process:**

**Iteration**: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].

- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log_2 \log n + \Theta(1)$ [BCSV06].

Meaning with probability at least $1 - n^{-c}$ for constant $c > 0$. 

---

3
**One-Choice and Two-Choice processes**

**One-Choice Process:**
Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].

- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).
One-Choice and Two-Choice processes

**One-Choice Process:**

Iteration: For each $t \geq 0$, sample **one** bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**

Iteration: For each $t \geq 0$, sample **two** bins independently u.a.r. and place the ball in the least loaded of the two.
**One-Choice and Two-Choice processes**

**One-Choice Process:**
*Iteration:* For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**
*Iteration:* For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
**One-Choice and Two-Choice processes**

**One-Choice Process:**

*Iteration:* For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right) [\text{Gon81}].$
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**

*Iteration:* For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
**One-Choice and Two-Choice processes**

**One-Choice Process:**
Iteration: For each $t \geq 0$, sample **one** bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta\left(\frac{\log n}{\log \log n}\right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta\left(\sqrt{\frac{m}{n}} \cdot \log n\right)$ (e.g. [RS98]).

**Two-Choice Process:**
Iteration: For each $t \geq 0$, sample **two** bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log_2 \log n + \Theta(1)$ [BCSV06].
**One-Choice and Two-Choice processes**

**One-Choice Process:**

Iteration: For each $t \geq 0$, sample one bin uniformly at random (u.a.r.) and place the ball there.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \Theta \left( \frac{\log n}{\log \log n} \right)$ [Gon81].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \Theta \left( \sqrt{\frac{m}{n}} \cdot \log n \right)$ (e.g. [RS98]).

**Two-Choice Process:**

Iteration: For each $t \geq 0$, sample two bins independently u.a.r. and place the ball in the least loaded of the two.

- In the lightly-loaded case ($m = n$), w.h.p. $\text{Gap}(n) = \log_2 \log n + \Theta(1)$ [KLMadH96, ABKU99].
- In the heavily-loaded case ($m \gg n$), w.h.p. $\text{Gap}(m) = \log_2 \log n + \Theta(1)$ [BCSV06].
One-Choice and Two-Choice processes

Gap for $n = 10^4$

"Power of two choices"
One-Choice and Two-Choice processes

Gap for $n = 10^4$

“Power of two choices”
Relaxing with incomplete information

**Mean-Thinning Process:**
Iteration: For each $t \geq 0$, sample two bins $i_1$ and $i_2$ u.a.r., and update:

\[
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 & \text{if } x_{i_1}^t < \frac{t}{n}, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 & \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{align*}
\]
Relaxing with incomplete information

Mean-Thinning Process:
Iteration: For each $t \geq 0$, sample two bins $i_1$ and $i_2$ u.a.r., and update:

\[
\begin{align*}
   x_{i_1}^{t+1} &= x_{i_1}^t + 1 & \text{if } x_{i_1}^t < \frac{t}{n}, \\
   x_{i_2}^{t+1} &= x_{i_2}^t + 1 & \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{align*}
\]

Achieves w.h.p. $\operatorname{Gap}(m) = O(\log n)$ and uses $2 - \epsilon$ samples.
Relaxing with incomplete information

**Mean-Thinning Process:**

**Iteration:** For each \( t \geq 0 \), sample two bins \( i_1 \) and \( i_2 \) u.a.r., and update:

\[
\begin{align*}
  x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } x_{i_1}^t < \frac{t}{n}, \\
  x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{align*}
\]

Achieves w.h.p. \( \text{Gap}(m) = O(\log n) \) and uses \( 2 - \epsilon \) samples.
Relaxing with incomplete information

**Mean-Thinning Process:**

Iteration: For each $t \geq 0$, sample two bins $i_1$ and $i_2$ u.a.r., and update:

\[
\begin{cases}
    x_{i_1}^{t+1} = x_{i_1}^t + 1 & \text{if } x_{i_1}^t < \frac{t}{n}, \\
    x_{i_2}^{t+1} = x_{i_2}^t + 1 & \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{cases}
\]
Relaxing with incomplete information

**Mean-Thinning Process:**

Iteration: For each $t \geq 0$, sample two bins $i_1$ and $i_2$ u.a.r., and update:

$$
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if} \quad x_{i_1}^t < \frac{t}{n}, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{if} \quad x_{i_1}^t \geq \frac{t}{n}.
\end{align*}
$$

Achieves w.h.p. $\text{Gap}(m) = O(\log n)$ and uses $2 - \epsilon$ samples.
Relaxing with incomplete information

**Mean-Thinning Process:**

**Iteration:** For each $t \geq 0$, sample two bins $i_1$ and $i_2$ u.a.r., and update:

\[
\begin{cases}
    x_{i_1}^{t+1} = x_{i_1}^t + 1 & \text{if } x_{i_1}^t < \frac{t}{n}, \\
    x_{i_2}^{t+1} = x_{i_2}^t + 1 & \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{cases}
\]
Mean-Thinning Process:
Iteration: For each \( t \geq 0 \), sample two bins \( i_1 \) and \( i_2 \) u.a.r., and update:

\[
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 \quad \text{if } x_{i_1}^t < \frac{t}{n}, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 \quad \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{align*}
\]

\( \frac{w_t}{n} \) achieves w.h.p. \( \text{Gap}(m) = O(\log n) \) and uses \( 2 - \epsilon \) samples.
Relaxing with incomplete information

**Mean-Thinning Process:**

Iteration: For each $t \geq 0$, sample two bins $i_1$ and $i_2$ u.a.r., and update:

$$
\begin{align*}
    x_{i_1}^{t+1} &= x_{i_1}^t + 1 & \text{if } x_{i_1}^t < \frac{t}{n}, \\
    x_{i_2}^{t+1} &= x_{i_2}^t + 1 & \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{align*}
$$

Achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(\log n)$ and uses $2 - \epsilon$ samples.
Relaxing with incomplete information

**Mean-Thinning Process:**

**Iteration:** For each $t \geq 0$, sample two bins $i_1$ and $i_2$ u.a.r., and update:

$$
\begin{cases}
  x_{i_1}^{t+1} = x_{i_1}^t + 1 & \text{if } x_{i_1}^t < \frac{t}{n}, \\
  x_{i_2}^{t+1} = x_{i_2}^t + 1 & \text{if } x_{i_1}^t \geq \frac{t}{n}.
\end{cases}
$$

Achieves w.h.p. $\text{Gap}(m) = \mathcal{O}(\log n)$ and uses $2 - \epsilon$ samples.
Relaxing with outdated information

Allocate balls in batches of size $b$. For $b = n$, improved $\text{Gap}(m) = O(\log n)$ to $\text{Gap}(m) = \Theta(\log n / \log \log n)$. For $b \geq n \log n$, achieves $\text{Gap}(m) = \Theta(b/n)$. 

Open in Visualiser.
Relaxing with outdated information

- Allocate balls in batches of size $b$ [BCE$^+12$].
Relaxing with outdated information

- Allocate balls in batches of size $b$ [BCE$^+$12].
Relaxing with outdated information

- Allocate balls in batches of size $b$ [BCE+12].
Relaxing with outdated information

- Allocate balls in batches of size $b$ [BCE$^+$12].
Relaxing with outdated information

Allocate balls in batches of size $b$ [BCE+12].

For $b = n$, improved $\text{Gap}(m) = \mathcal{O}(\log n)$ to $\text{Gap}(m) = \Theta(\log n / \log \log n)$. 

For $b = n$, improved $\text{Gap}(m) = \mathcal{O}(\log n)$ to $\text{Gap}(m) = \Theta(\log n / \log \log n)$. 

Relaxing with outdated information

- Allocate balls in batches of size $b$ [BCE$^+$12].

- For $b = n$, improved $\text{Gap}(m) = \Theta(\log n)$ to $\text{Gap}(m) = \Theta(\log n / \log \log n)$.

- For $b \geq n \log n$, achieves $\text{Gap}(m) = \Theta(b/n)$. 
Our techniques

- Interplay between (i) linear, (ii) quadratic and (iii) exponential potentials.
Bibliography I


