Can we relax 

\[
\text{Gap for } n = 10^4
\]

One-Choice: each ball is allocated in a bin sampled uniformly at random.

\[
\text{Gap}(m) = \Theta \left( \sqrt{\frac{m}{n} \log n} \right)
\]

Two-Choice: each ball is allocated in the lesser loaded of two bins sampled uniformly at random.

\[
\text{Gap}(m) = \Theta \left( \log \log n + \Theta(1) \right)
\]

Processes

Mean-Thinning (B, 2)

For each ball:
- Sample one bin; if load at most \( t/n \), then allocate.
- Otherwise, sample a second bin and allocate there.

Twinning (B, 2)

For each ball:
- Sample one bin; if load at most \( Wt/n \), then allocate 2 balls.
- Otherwise, allocate 1 ball.

Quantile (B)

For each ball:
- Sample two bins \( i_1 \) and \( i_2 \).
- Send 1 queries of the form: is load at median?
- Based on responses, allocate to smaller one, or randomly.

Outline of the Analysis for Mean-Thinning

Our analysis is based on an interaction between the following functions:

- The Exponential potential \( \Gamma^t := \sum_{i \in [m]} e^{t(x_i - 1/n)} + \sum_{i \in [m]} e^{-t(x_i - 1/n)} \).
- The Absolute potential: \( \Delta^t := \sum_{i \in [m]} (x_i - t/n)^2 \).
- The Quadratic potential: \( \Upsilon^t := \sum_{i \in [m]} (x_i - t/n)^2 \).
- The Quantile position: \( \delta^t = \{ r \in [n] : y_r \geq 0 \} / n \).

Rough idea (cf. [7]):

- Observe \( e^{\delta^t} \leq \Gamma^t \). Want: \( \Gamma^t \in \text{Poly}(n) \) w.h.p. Easy to get: \( \Gamma^t \leq e^{\log n} \) w.h.p.
- As long as \( \Delta^t = \Omega(n) \), then \( \Gamma^t \) drops in expectation.
- If \( \Delta^t = O(n) \), then \( \delta^t \in (\epsilon, 1 - \epsilon) \) w.h.p., for a constant fraction of the next \( \Theta(n) \) steps.
- If \( \delta^t \in (\epsilon, 1 - \epsilon) \) then, in expectation, \( \Gamma^t \) decreases in the next step.
- Once \( \Gamma^t > cn \), then, w.h.p., for the next \( n^2 \) steps, it must be \( \leq cn \) once every \( n \log n \) steps.
- Between these events, w.h.p., the gap cannot rise by more than \( O(\log n) \).

Batching (B)

Two-Choice where balls are allocated in batches of size \( b \).

Graphical Allocations

Two-Choice on graphs: Sample an edge and allocate in the lesser loaded of the two bins.

On expanders, we show \( \text{Gap}(m) = O(\log n) \) with weighted balls and batching [4].

On dense expanders, \( \text{Gap}(m) = O(\log \log n) \) [5].

References