Sample path LDP for traffic (in the long-timescale limit)

Large Deviations and Queues—Damon Wischik

1 Model

Let $\mathcal{X}$ be the set of discrete-time traffic processes

$$\mathcal{X} = \{x : \mathbb{Z}_+ \to \mathbb{R},\ x(0) = 0\}.$$

Let $\mathcal{C}$ be the set of continuous-time traffic processes

$$\mathcal{C} = \{x : \mathbb{R}_+ \to \mathbb{R},\ x(0) = 0\}.$$

Let $\mathcal{A}$ be the subset of $\mathcal{C}$ consisting of absolutely continuous traffic processes. Let $\mathcal{X}^T$ be and $\mathcal{C}^T$ be

$$\mathcal{X}^T = \{x : \{0, \dots, T\} \to \mathbb{R},\ x(0) = 0\},$$
$$\mathcal{C}^T = \{x : [0, T] \to \mathbb{R},\ x(0) = 0\},$$

and define $\mathcal{A}^T$ similarly. Interpret $x(t)$ as the amount of work arriving in the interval $(-t, 0]$. Say that a traffic process x has *mean rate* μ if $\lim_{t\to\infty} x(t)/t = \mu$. Write $\mathcal{X}_\mu$, $\mathcal{C}_\mu$ and $\mathcal{A}_\mu$ for the restrictions of $\mathcal{X}$, $\mathcal{C}$ and $\mathcal{A}$ to traffic processes with mean rate μ.

Define the *scaled uniform norm* $\|\cdot\|$ on these spaces by

$$\|x\| = \sup_{t\geq 0}\left|\frac{x(t)}{t+1}\right|$$

Also define π, the topology of uniform convergence on compact intervals.

Given $x \in \mathcal{X}$, define the *polygonalized version* $\tilde{x} \in \mathcal{A}$ to be

$$\tilde{x}(t) = \big(\lfloor t+1 \rfloor - t\big)x\big(\lfloor t \rfloor\big) + \big(t - \lfloor t \rfloor\big)x\big(\lfloor t+1 \rfloor\big).$$

Given $x \in \mathcal{C}$, define the *speeded-up version* $x^{\circlearrowleft L} \in \mathcal{C}$ to be

$$x^{\circlearrowleft L}(t) = x(Lt).$$

Use the following extended notation: write

$x(-t, 0]$	for	$x(t)$
$x\vert_{(-t,0]}$	for	the restriction of x to $[0, t]$
$\dot{x}_{-t}$	for	$dx(t)/dt$, where it is defined, for $x \in \mathcal{C}$
$\dot{x}_{-t}$	for	$x(t+1) - x(t)$, for $x \in \mathcal{X}$

2 Probabilistic setup

Let A be a random discrete-time traffic process, taking values in $\mathcal{X}$. Suppose that the $\dot{A}_{-t}$ are independent and identically distributed. Let

$$\Lambda(\theta) = \log \mathbb{E} \exp(\theta \dot{A}_0),$$

and assume that $\Lambda(\cdot)$ is finite in a neighbourhood of the origin.

3 Cramér's theorem

By Cramér's theorem, $L^{-1}A^{\circ L}(1)$ satisfies an LDP in $\mathbb{R}$ with good convex rate function

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}} \theta x - \Lambda(\theta).$$

4 Finite horizon SP-LDP

By Mogulskii's theorem, $L^{-1}\tilde{A}^{\circ L}|_{(-1,0]}$ satisfies a sample path LDP in $(\mathcal{C}^1, \pi)$ with rate function

$$I_1(x) = \begin{cases} \int_{-1}^{0} \Lambda^*(\dot{x}_s)\, ds & \text{if } x \in \mathcal{A}^1 \\ \infty & \text{otherwise.} \end{cases}$$

This can easily be extended to a sample path LDP for $L^{-1}\tilde{A}^{\circ L}|_{(-T,0]}$ with rate function

$$I_T(x) = \begin{cases} \int_{-T}^{0} \Lambda^*(\dot{x}_s)\, ds & \text{if } x \in \mathcal{A}^T \\ \infty & \text{otherwise.} \end{cases}$$

5 Infinite horizon SP-LDP

Write $\Pi_t : \mathcal{C} \to \mathcal{C}^T$ for the projection $x \mapsto x|_{(-t,0]}$. With these projections, $(\mathcal{C}, \pi)$ is the projective limit of the collection of spaces $(\mathcal{C}^T, \pi)$. By the Dawson-Gärtner theorem, $L^{-1}\tilde{A}^{\circ L}$ satisfies a sample path LDP in $(\mathcal{C}, \pi)$ with good rate function

$$I(x) = \sup_{T \geq 0} I_T(\Pi_t x).$$

By the non-negativity of Λ^*, the supremum is

$$I(x) = \begin{cases} \int_{-\infty}^{0} \Lambda^*(\dot{x}_s)\, ds & \text{if } x \in \mathcal{A} \\ \infty & \text{otherwise.} \end{cases}$$

6 Strengthening the topology

It can be shown that $L^{-1}\tilde{A}^{\circ L}$ is exponentially tight in $(\mathcal{C}, \|\cdot\|)$. Therefore, using the inverse contraction principle, the sample path LDP for $L^{-1}\tilde{A}^{\circ L}$ holds in $(\mathcal{C}, \|\cdot\|)$.

7 Restricting the space

Since $\Lambda(\cdot)$ is finite in a neighbourhood of the origin, it is differentiable at the origin. Let $\mu = \Lambda'(0)$. It can be shown that $\mathbb{P}(L^{-1}\tilde{A}^{\circ L} \in \mathcal{C}_\mu) = 1$, and that this space is closed. Therefore the sample path LDP for $L^{-1}\tilde{A}^{\circ L}$ holds in $(\mathcal{C}_\mu, \|\cdot\|)$.