Topology Notes

Large Deviations and Queues—Damon Wischik

Let $\mathcal{X}$ be a set.

Definition. A family τ of subsets of $\mathcal{X}$ is called a *topology* if
i. $\emptyset \in \tau$ and $\mathcal{X} \in \tau$
ii. The union of any family of sets in τ, is in τ
iii. The intersection of a finite number of sets in τ, is in τ
The elements of τ are called the *open sets*; their complements are the *closed sets*. The pair $(\mathcal{X}, \tau)$ is called a *topological space*.

Definition. A function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a *metric* if for all $x, y, z \in \mathcal{X}$,
i. $d(x, y) \geq 0$ with equality iff $x = y$
ii. $d(x, y) = d(y, x)$
iii. $d(x, z) \leq d(x, y) + d(y, z)$
This induces a natural topology, the smallest topology containing all the balls $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$.

Definition. A subset σ of τ is called *basis* of the topology τ if every set in τ is the union of sets in σ. A topological space is *separable* if it has a countable basis of open sets.

Definition. The topology *induced* on any subset $\mathcal{Y} \subset \mathcal{X}$ is $\{A \cap \mathcal{Y} : A \in \tau\}$.

Definition. A topology σ is finer/stronger than a topology τ on the same set $\mathcal{X}$, if σ contains all the sets in τ.

Definition. Let A be a subset of $\mathcal{X}$.
i. A is an *open neighbourhood* of x if A is open and $x \in A$
ii. A is a *neighbourhood* of x if it contains an open neighbourhood of x
iii. The *interior* of A, A°, is the union of all open subsets of A
iv. The *closure* of A, $\bar{A}$, is the intersection of all closed supersets of A

A set A is open iff for all $x \in A$ there exists an open neighbourhood B of x with $B \subset A$.

Definition. A set $A \subset \mathcal{X}$ is *dense* if its closure $\bar{A}$ is equal to $\mathcal{X}$.

A metric space is separable if it contains a countable dense set.

Definition. A topological space $\mathcal{X}$ is *Hausdorff* if for all $x, y \in \mathcal{X}$ there exist open neighbourhoods B_x and B_y of x and y that are disjoint. It is *regular* if in addition for every closed set $\bar{A} \subset \tau$ and point $y \in \tau$ there exist open neighbourhoods $B_{\bar{A}}$ and B_y of $\bar{A}$ and y that are disjoint.

Every metric space is regular.

Definition. Let A be a subset of a topological space. An *open cover* is a collection of open sets whose union contains A. A is *compact* if every open cover has a finite subcover.

i. A closed subset of a compact set is compact
ii. The intersection of a closed set and a compact set is compact
iii. In a Hausdorff space, every compact set is closed

Definition. A sequence $x_1, x_2, \dots$ converges to x if for all neighbourhoods B of x, $x_n \in B$ eventually.

In a metric space, $x_n \to x$ iff $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition. A metric space $\mathcal{X}$ is *complete* if every sequence for which $d(x_m, x_n) \to 0$ as $m \wedge n \to \infty$ (i.e. every *Cauchy sequence*) converges. A complete separable metric space is called *Polish*.

Definition. A subset A of a topological space is *sequentially compact* if every sequence of points in A contains a subsequence which converges to a point in A.

If A is compact then A is sequentially compact. If $\mathcal{X}$ is a metric space, and A is closed and sequentially compact, then A is compact.

Let A be a subset of a metric space. A is closed iff whenever x_n is a sequence in A and $x_n \to x$ then $x \in A$.

Definition. Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces, and let $f : \mathcal{X} \to \mathcal{Y}$. For $Y \subset \mathcal{Y}$, let $f^{-1}(Y) = \{x \in \mathcal{X} : f(x) \in Y\}$. We say f is *continuous* if whenever Y is open in $\mathcal{Y}$, $f^{-1}(Y)$ is open in $\mathcal{X}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are metric spaces, a function f is continuous iff whenever $x_n \to x$, $f(x_n) \to f(x)$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be continuous, and let $X \subset \mathcal{X}$ and $Y \subset \mathcal{Y}$.

i. If Y is open then $f^{-1}(Y)$ is open
ii. If Y is closed then $f^{-1}(Y)$ is closed
iii. If X is compact then $f(X)$ is compact

Definition. Let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$. The *level sets* of f are the sets of the form $\{x : f(x) \leq \alpha\}$ for $\alpha \in \mathbb{R}$. We say f is *lower-semicontinuous* if all level sets are closed.

Let $\mathcal{X}$ be a metric space and let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$. Then f is lower-semicontinous iff whenever $x_n \to x$, $\liminf_{n\to\infty} f(x_n) \geq f(x)$.

Let $\mathcal{X}$ be a Hausdorff space and let $f : \mathcal{X} \to \mathbb{R}$.

i. If f is lower-semicontinuous, then $\inf_{x\in A} f(x)$ is attained for any compact set A.
ii. If f has compact level sets, then $\inf_{x\in A} f(x)$ is attained for any closed set A.