Useful concrete LDPs

Large Deviations and Queues—Damon Wischik

1 A trivial LDP

Let $X \sim \text{Exp}(\lambda)$. Then $(X/l,\ l \in \mathbb{R}_+)$ satisfies an LDP in $\mathbb{R}_+$ with good rate function

$$I(x) = \lambda x.$$

2 Cramér's Theorem

Let X^L be the average of L independent copies of a real-valued random variable X. Then X^L satisfies an LDP in $\mathbb{R}$ with convex rate function

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}} \theta x - \Lambda(\theta)$$

where the log moment generating function $\Lambda(\cdot)$ is

$$\Lambda(\theta) = \log \mathbb{E} \exp(\theta X).$$

If $\Lambda(\cdot)$ is finite in a neighbourhood of the origin then Λ^* is a good rate function.

3 Gärtner-Ellis Theorem

The *effective domain* of a function $\Lambda(\cdot)$ taking values in $\mathbb{R} \cup \{\infty\}$ is $\mathcal{D}\Lambda = \{\theta : \Lambda(\theta) \in \mathbb{R}\}$.

A convex function $\Lambda : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is *essentially smooth* if

i. $(\mathcal{D}\Lambda)^\circ$ is non-empty
ii. $\Lambda(\cdot)$ is differentiable throughout $(\mathcal{D}\Lambda)^\circ$
iii. $\Lambda(\cdot)$ is steep, namely, $|\nabla\Lambda(\theta_n)| \to \infty$ whenever (θ_n) is a sequence in $(\mathcal{D}\Lambda)^\circ$ converging to a point on the boundary of $\mathcal{D}\Lambda$.

Let $(X^L,\ L \in \mathbb{N})$ be a sequence of random vectors in $\mathbb{R}^d$, and let

$$\Lambda^L(\theta) = \frac{1}{L} \log \mathbb{E} \exp(L\theta \cdot X^L)$$

for $\theta \in \mathbb{R}^d$. Assume that for each θ the limit

$$\Lambda(\theta) = \lim_{L \to \infty} \Lambda^L(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$.

Assume further that $0 \in (\mathcal{D}\Lambda)^\circ$ and that Λ is essentially smooth and lower-semicontinuous. Then $(X^L,\ L \in \mathbb{N})$ satisfies an LDP in $\mathbb{R}^d$ with good rate function

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}^t} \theta \cdot x - \Lambda(\theta).$$

Note that $\Lambda(\cdot)$, being the pointwise limit of convex functions, is itself convex. Note that $\Lambda^*(\cdot)$, being a convex conjugate, is convex and lower-semicontinuous. Since $\Lambda(\cdot)$ is convex, it satisfies the following: if $\Lambda(\cdot)$ is differentiable at θ and $\nabla\Lambda(\theta) = x$ then $\Lambda^*(x) = \theta \cdot x - \Lambda(\theta)$.

4 Schilder's Theorem

Let $\mathcal{C}[0,1]$ be the space of continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the supremum norm

$$\|f\| = \sup_{0 \le t \le 1} |f(t)|.$$

Say that a function $f : [0,1] \to \mathbb{R}$ is *absolutely continuous* (AC) if for all $\varepsilon > 0$ there exists a $\delta > 0$ such for every finite collection of non-overlapping intervals $\{[s_i, t_i], 1 \le i \le N\}$,

$$\sum_{1 \le i \le N} (t_i - s_i) < \delta \implies \sum_{1 \le i \le N} |f(t_i) - f(s_i)| < \varepsilon.$$

Absolutely continuous functions are dense in the space of continuous functions. An absolutely continuous function is differentiable almost everywhere.

Let $(B(t), t \in [0,1])$ be a standard Brownian motion, taking values in $\mathcal{C}[0,1]$. Let

$$B^l(t) = \frac{1}{\sqrt{l}} B(t).$$

Then $(B^l(\cdot),\ l \in \mathbb{R}_+)$ satisfies an LDP in $\mathcal{C}[0,1]$ with good rate function

$$I(f) = \begin{cases} \int_{t=0}^1 \frac{1}{2} \dot{f}(t)^2 dt & \text{if } f \text{ is AC and } f(0) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

5 Mogulskii's theorem

Let $X_1, X_2, \dots$ be independent copies of a real-valued random variable X with log moment generating function $\Lambda(\cdot)$. Define the partial sums process

$$S_n = X_1 + \cdots + X_n, \quad \text{for } n \in \mathbb{N};$$

and the polygonalized partial sums process

$$\tilde{S}_t = (\lceil t \rceil - t) S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor) S_{\lceil t \rceil}, \quad \text{for } t \in \mathbb{R}_+;$$

and the scaled polygonalized partial sums process

$$Y^L(t) = \frac{1}{L} \tilde{S}_{Lt}, \quad \text{for } 0 \le t \le 1.$$

Then $(Y^L(\cdot),\ L \in \mathbb{N})$ satsfies an LDP in $\mathcal{C}[0,1]$ with good rate function

$$I(f) = \begin{cases} \int_{t=0}^1 \Lambda^*\big(\dot{f}(t)\big) dt & \text{if } f \text{ is AC and } f(0) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

This has been extended, along the lines of the Gärtner-Ellis theorem, by Dembo and Zajic (1995), *From empirical mean and measure to partial sums processes*, Stochastic Processes and their Applications.