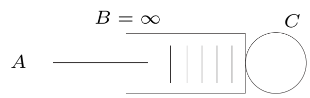
LDP for queue size

Large Deviations and Queues—Damon Wischik

Consider a queue with constant service rate C, with buffer size $B = \infty$, and with arrival process $A = (\dots, A_{-1}, A_0)$ where the A_t are independent and identically distributed. Recall that the queue size Q is given by $Q = q(A)$ where $q(a) = \sup_{t\geq 0} a(-t,0] - Ct$ and $a(-t,0] = a_{-t+1} + \cdots + a_0$.



Let $\Lambda(\theta) = \log \mathbb{E} e^{\theta A_0}$ and $\Lambda^*(x) = \sup_{\theta\in\mathbb{R}} \theta x - \Lambda(\theta)$. Assume that $\Lambda(\theta)$ is finite for all θ (and thus that $\mathbb{E}A_0$ is finite, and $\Lambda(\theta)$ is infinitely differentiable for all θ).

Theorem 1 *If $\mathbb{E}A_0 < C$ then, for $q > 0$,*

$$\lim_{l\to\infty} \frac{1}{l} \log \mathbb{P}\Big(\frac{Q}{l} > q\Big) = -q \sup\big\{\theta > 0 : \Lambda(\theta) < \theta C\big\} \tag{1}$$

$$= -\inf_{t>0} t\Lambda^*(C + q/t) \tag{2}$$

$$= -\inf_{t>0} \sup_{\theta\geq 0} \theta(q + Ct) - t\Lambda(\theta) \tag{3}$$

(The limit, infimum and supremum are taken over $l \in \mathbb{R}$, $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$.)

We will split the proof into three parts: the large deviations upper bound

$$\limsup l^{-1} \log \mathbb{P}(Q > lq) \leq (1), \tag{4}$$

the large deviations lower bound

$$\liminf l^{-1} \log \mathbb{P}(Q > lq) \geq (2), \tag{5}$$

and finally (1) = (2) = (3).

Proof of LD upper bound. Write out the probability we wish to estimate, and then use the Chernoff bound. For any $\theta > 0$ such that $\Lambda(\theta) < \theta C$,

$$\begin{aligned}
\mathbb{P}(Q > lq) &= \mathbb{P}\big(\sup_{t\geq 0} A(-t,0] - Ct > lq\big) \\
&= \mathbb{P}\big(A(-t,0] - Ct > lq \text{ for some } t \geq 0\big) \\
&\leq \sum_{t\geq 0} \mathbb{P}\big(A(-t,0] - Ct \geq lq\big) \\
&\leq \sum_{t\geq 0} e^{-\theta l q} e^{t\{\Lambda(\theta)-\theta C\}} \quad \text{by Chernoff's bound, since } \theta > 0 \\
&= e^{-\theta l q} \frac{e^{\Lambda(\theta)-\theta C}}{1 - e^{\Lambda(\theta)-\theta C}} \quad \text{the series is summable, since } \Lambda(\theta) < \theta C
\end{aligned}$$

and so $\limsup l^{-1} \log \mathbb{P}(Q > lq) \leq -\theta q$. Take the infimum over all such θ to prove the result (4).

Note that if no such θ existed then the supremum would be $-\infty$, by convention, and so the bound would be trivial. But such a θ does exist, because $\Lambda(\theta)$ is finite in a neighbourhood of $\theta = 0$, hence differentiable at $\theta = 0$, and we've assumed that $\Lambda'(0) = \mathbb{E}A_0 < C$; therefore $\Lambda(\theta) < \theta C$ for θ sufficiently small. □

Proof of LD lower bound. Pick any $u > 0$, $u \in \mathbb{R}$. We will find a lower bound for $\mathbb{P}(Q > lq)$ by estimating the probability that the queue reaches level lq in time lu using Cramér's theorem:

$$\liminf_{l\to\infty} \frac{1}{l} \log \mathbb{P}(Q > lq) \tag{6}$$

$$\begin{aligned}
&= \liminf_{l\to\infty} \frac{1}{l} \log \mathbb{P}\Big(\sup_v A(-v,0] - Cv > lq\Big) \\
&= \liminf_{l\to\infty} \frac{1}{l} \log \mathbb{P}(A(-v,0] - Cv > lq \text{ for some } v) \\
&\geq \liminf_{l\to\infty} \frac{1}{l} \log \mathbb{P}\Big(A\big(-\lceil lu\rceil, 0\big] > lq + C\lceil lu\rceil\Big) \quad \text{by choosing } v = \lceil lu\rceil \\
&\geq \liminf_{l\to\infty} \frac{u}{\lceil lu\rceil - 1} \log \mathbb{P}\Big(A\big(-\lceil lu\rceil, 0\big] > \frac{\lceil lu\rceil}{u} q + C\lceil lu\rceil\Big) \quad \text{by bounds}^1 \text{ for } \lceil lu\rceil \\
&= u \liminf_{n\to\infty} \frac{1}{n-1} \log \mathbb{P}\Big(\frac{1}{n} A(-n,0] > C + \frac{q}{u}\Big) \quad \text{where } n = \lceil lu\rceil \\
&\geq u \liminf_{n\to\infty} \frac{1+\varepsilon}{n} \log \mathbb{P}\Big(\frac{1}{n} A(-n,0] > C + \frac{q}{u}\Big) \quad \text{for any } \varepsilon > 0 \\
(6) &\geq -u \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\Big(\frac{1}{n} A(-n,0] > C + \frac{q}{u}\Big) \quad \text{since } \varepsilon > 0 \text{ arbitrary} \\
&\geq -u \inf_{x > C + q/u} \Lambda^*(x) \quad \text{by Cramér's theorem} \\
&= -u\Lambda^*(C + q/u+) \quad \text{since } \Lambda^*(x) \text{ is increasing for } x \geq \mathbb{E}A_0 \\
&\qquad\qquad \text{where by } f(x+) \text{ we mean } \lim_{y\downarrow x} f(y) \\
(6) &\geq -\inf_{u>0} u\Lambda^*(C + q/u+) \quad \text{since } u > 0 \text{ arbitrary} \\
&\geq -(t+\delta)\Lambda^*\big(C + q/(t+\delta)+\big) \quad \text{choosing } u = t + \delta,\ \delta > 0 \\
&\geq -(t+\delta)\Lambda^*(C + q/t) \quad \text{since } q/(t+\delta)+ < q/t \text{ and } \Lambda^* \text{ is increasing} \\
(6) &\geq -t\Lambda^*(C + q/t) \quad \text{since } \delta > 0 \text{ arbitrary} \\
(6) &\geq -\inf_{t>0} t\Lambda^*(C + q/t) \quad \text{since } t > 0 \text{ arbitrary}
\end{aligned}$$

This completes the proof. □

Equality of rate functions. First, (2)=(3)*:* Expand Λ^*, and use the fact that the supremum over θ in $\Lambda^*(x) = \sup_{\theta\in\mathbb{R}} \theta x - \Lambda(\theta)$ can be taken over $\theta \geq 0$ for $x \geq \mathbb{E}A_0$, as we saw in the proof of Cramér's theorem.

Second, (3)≥(1)*:* For any $\theta > 0$ with $\Lambda(\theta) < \theta C$,

$$\theta(q + Ct) - t\Lambda(\theta) = \theta q + t\big(\theta C - \Lambda(\theta)\big). \geq \theta q$$

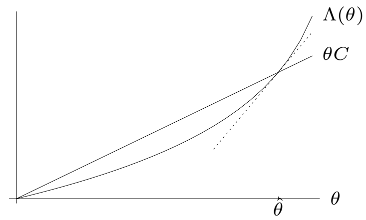
[1]Recall that $\lceil x\rceil - 1 < x \leq \lceil x\rceil$, so $l \leq \lceil lu\rceil/u$ and $1/l < u/(\lceil lu\rceil - 1)$.

Taking the supremum over such θ,

$$\begin{aligned}
&\sup_{\theta>0:\Lambda(\theta)<\theta C} \theta(q+Ct) - t\Lambda(\theta) \geq \sup_{\theta>0:\Lambda(\theta)<\theta C} \theta q \\
\implies &\sup_{\theta\geq 0} \theta(q+Ct) - t\Lambda(\theta) \geq q \sup\{\theta>0 : \Lambda(\theta)<\theta C\}.
\end{aligned}$$

Now take the infimum over $t>0$.

Finally, (3)$\leq$(1)*:* Let $\hat\theta = \sup\{\theta>0 : \Lambda(\theta)<\theta C\}$. (The set is non-empty, by our remark in the proof of the LD upper bound.) If $\hat\theta=\infty$, we are done. Otherwise, using the fact that Λ is convex and differentiable, it must be that $\Lambda(\hat\theta)=\hat\theta C$ and $\Lambda'(\hat\theta)>C$.



Now consider the supporting tangent to $\Lambda(\theta)$ at $\hat\theta$: by convexity, $\Lambda(\theta)\geq \hat\theta C + \Lambda'(\hat\theta)(\theta-\hat\theta)$, and so

$$\begin{aligned}
(3) &= \inf_{t>0}\sup_{\theta\geq 0} \theta(q+Ct) - t\Lambda(\theta) \\
&\leq \inf_{t>0}\sup_{\theta\geq 0} \theta(q+Ct) - t\Big(\hat\theta C + \Lambda'(\hat\theta)(\theta-\hat\theta)\Big) \quad \text{from supporting tangent at } \hat\theta \\
&= \inf_{t>0}\sup_{\theta\geq 0} \theta\Big(q - t\big(\Lambda'(\hat\theta)-C\big)\Big) + \hat\theta t\big(\Lambda'(\hat\theta)-C\big) \quad \text{gathering } \theta \text{ terms} \\
&= \inf_{t>0} \begin{cases} \infty & \text{if } t < q/\big(\Lambda'(\hat\theta)-C\big) \\ \hat\theta t\big(\Lambda'(\hat\theta)-C\big) & \text{else} \end{cases} \quad \text{performing the } \theta\text{-optimization} \\
&= \hat\theta q \quad \text{performing the } t\text{-optimization} \\
&= (1).
\end{aligned}$$

This completes the proof. □