Large Deviations and Queues

Example Sheet—Lent 2003—D.J. Wischik

The questions marked with (*) I intend to discuss; the rest I hope to discuss.

Q 1 (Principle of the largest term). Let $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ be sequences in $\mathbb{R}_+$. Prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log(a_n + b_n) \le \limsup_{n\to\infty} \frac{1}{n} \log(a_n) \vee \limsup_{n\to\infty} \frac{1}{n} \log(b_n)$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log(a_n + b_n) \ge \liminf_{n\to\infty} \frac{1}{n} \log(a_n) \vee \liminf_{n\to\infty} \frac{1}{n} \log(b_n).$$

[Need: elementary limits]

Q 2 (Broadcast problem). Let $W_N = T_1 + \cdots + T_N$, where the T_i are independent and $T_i \sim \text{Geometric}(\lambda_N(N-i+1)/N)$, where $\lambda_N = Np_N(1-p_N)^{N-1}$ and $p_N = 1/N$. Find a large deviations principle for $(W_N, N \in \mathbb{N})$. [Need: abstract large deviations]

Q 3 (*Useful LDPs). Let $(X_n, n \in \mathbb{N})$ and $(Y_n, n \in \mathbb{N})$ satisfy large deviations principles in Hausdorff spaces $\mathcal{X}$ and $\mathcal{Y}$ with good rate functions I and J.

i. Suppose that (for each n) X_n and Y_n are independent, and that $\mathcal{X}$ and $\mathcal{Y}$ have countable bases of open sets. Show that (X_n, Y_n) satisfies a large deviations principle in $\mathcal{X} \times \mathcal{Y}$ with good rate function $(x, y) \mapsto I(x) + J(y)$.

ii. Suppose $\mathcal{X} = \mathcal{Y}$, and let

$$Z_n = \begin{cases} X_n & \text{if } B_n = 0 \\ Y_n & \text{if } B_n = 1 \end{cases}$$

where $B_n \sim \text{Bin}(1, p)$, and is independent of X_n and Y_n. Show that Z_n satisfies an LDP in $\mathcal{X}$ with rate function $z \mapsto I(z) \wedge J(z)$.

[Need: abstract large deviations]

Q 4 (Dawson-Gärtner Theorem). Let $(X^L, L \in \mathbb{N})$ be a sequence of random variables taking values in $\mathcal{X} = \{x : \mathbb{N} \to \mathbb{R}\}$. For $x \in \mathcal{X}$, write $x|_{[1,j]}$ for $(x(1), \ldots, x(j))$. Suppose that for each j, $X^L|_{[1,j]}$ satisfies a large deviations principle in $\mathbb{R}^j$ with good rate function I_j. Show that X^L satisfies a large deviations principle in $\mathcal{X}$, under the topology of pointwise convergence, with good rate function $I(x) = \sup_j I_j(x|_{[1,j]})$. [Hint. The product of a family of compact sets is itself compact. Need: abstract large deviations]

Q 5 (*Empirical distributions). A discrete-time Markov chain (X_t) on the states $\{1, 2, 3, 4\}$ moves according to the transition matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-p & p & 0 & 0 \\ 1-q & 0 & q & 0 \\ 0 & r & 1-r & 0 \end{pmatrix}$$

and $X_0 = 4$. Given that the empirical distribution of $X_1, \dots, X_n$ on $\{1, 2, 3\}$ satisfies a large deviations principle as $n \to \infty$, write down (without proof) what you expect its rate function to be. For what choices of p, q and r is the rate function good? convex? [Need: abstract large deviations]

Q 6 (*LDP for minimum). Let $(X_n, n \in \mathbb{N})$ satisfy a large deviations principle in $\mathbb{R}$ with good convex rate function I, and suppose $I(\mu) = 0$. Let M_n be the minimum of k independent copies of X_n. Prove that M_n satisfies a large deviations principle in $\mathbb{R}$ with good rate function

$$J(m) = \begin{cases} kI(m) & \text{if } m \geq \mu \\ I(m) & \text{if } m < \mu \end{cases}$$

[Need: contraction principle]

Q 7 (*Restricted contraction principle). Suppose that X_n satisfies a large deviations principle in some Hausdorff space $\mathcal{X}$ with good rate function I, and let $f : \mathcal{X} \to \mathcal{Y}$ be a map to another Hausdorff space $\mathcal{Y}$. Suppose there exists an open neighbourhood $\mathcal{E}$ of the effective domain of I, such that f is continuous on $\bar{\mathcal{E}}$. Show that $f(X_n)$ satisfies a large deviations principle in $\mathcal{Y}$ with good rate function $J(y) = \inf_{x:f(x)=y} I(x)$. [Need: contraction principle]

Q 8 (*Moderate Deviations). Let X be a real-valued random variable, with log moment generating function $\Lambda(\theta) = \log \mathbb{E}e^{\theta X}$ finite in a neighbourhood of the origin. Let X_n be the average of n independent copies of X. Show that for any $\beta \in (0, 1)$,

$$\frac{1}{n^\beta} \log \mathbb{P}\big(n^{(1-\beta)/2}(X_n - \mu) \in B\big) \approx - \inf_{x \in B} \tfrac{1}{2}x^2/\sigma^2$$

where $\mu = \mathbb{E}X$ and $\sigma^2 = \operatorname{Var} X > 0$, and the approximation means that the appropriate large deviations upper and lower bounds apply. Interpret this result, in light of Cramér's Theorem and the Central Limit Theorem. [Need: Cramér's Theorem]

Q 9 (Continuity of queueing functions). Consider a queue operating in slotted time, with infinite buffer and constant service rate C. Define the space $\mathcal{X}_{[a,b]}$, and the queue size function Q.

i. Show that if $a \leq b < C$ then Q is continuous on $\mathcal{X}_{[a,b]}$.
ii. Define the departure map

$$\big(D(x)\big)(-t, 0] = Q_{-t+1}(x) + x(-t, 0] - Q_0(x).$$

Show that if $a < C$ then D is a continuous map $\mathcal{X}_{[a,a]} \to \mathcal{X}_{[a,a]}$.

iii. If the buffer is finite, is the departure map still continuous?
[Need: continuity of queue size function]

Q 10 (Extended LDP for simple queue).

i. Let A be a random stationary arrival process, and define

$$\Lambda_t(\theta) = \frac{1}{t} \log \mathbb{E} e^{\theta A(-t,0]}.$$

Suppose that the limit

$$\Lambda(\theta) = \lim_{t \to \infty} \Lambda_t(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$ for each $\theta \in \mathbb{R}$, and that it is essentially smooth, finite in a neighbourhood of $\theta = 0$, and lower-semicontinuous. State a large deviations principle for $(L^{-1}A(-L,0], L \in \mathbb{N})$.

ii. Consider a queue fed by A. Suppose the queue has infinite buffer, and constant service rate $C > \mathbb{E}X_1$. Let $Q(A)$ be the queue size at time 0. State and prove a large deviations principle for $(l^{-1}Q(A), l \in \mathbb{R}_+)$.

[Need: Gärtner-Ellis theorem, LDP for a simple queue]

Q 11 (Example arrival processes). In the setting of Question 10, verify the conditions and find the rate function for queue size, for the following arrival processes.

i. $(A_t, t \in \mathbb{Z})$ is a two-state Markov chain, representing a traffic source which produces an amount of work h in each timestep while in the on state, and no work while in the off state, and which flips from on to off with probability p, and from off to on with probability q.

ii. $(A_t, t \in \mathbb{Z})$ is a stationary autoregressive process of degree 1, that is, $A_t = \mu + X_t$ where

$$X_t = \alpha X_{t-1} + (1 - \alpha^2)\varepsilon_t$$

where $|\alpha| < 1$ and the ε_t are independent normal random variables with mean 0 and variance σ^2. [Hint. The marginal distribution of X_t is $\mathrm{N}(0, \sigma^2)$.]

[Need: question 10]

Q 12 (*Underflow in queues fed by many flows). Consider a queue, operating in slotted time, fed by the aggregate of L copies of some arrival process A, with an infinite buffer, and service rate LC strictly greater than the aggregate mean arrival rate. Find non-trivial large deviations bounds for the probability of the event that the queue is non-empty. [Need: many sources limit]

Q 13 (Traffic measurement). Listen to BBC Radio 4's "I'm sorry I haven't a clue" on your computer. What is the effective bandwidth of the traffic stream? What is its Hurst parameter? [Need: effective bandwidth, LRD]