III Large Deviations and Queues

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Attempt THREE questions. All questions carry equal weight. You may find helpful the reference material at the end of the paper.

Question 1. Let $A_1, A_2, \ldots$ be normal random variables with mean μ and variance σ^2. Let B be an exponential random variable with mean $1/\lambda$. Let C be a normal random variable with mean ν and variance ρ^2. Let all of these random variables be independent.

(a). State, without proof, a large deviations principle for $L^{-1}B$.
(b). Find a large deviations principle for $L^{-1}(A_1 + \cdots + A_L)$.
(c). Find a large deviations principle for $L^{-1}(B + A_1 + \cdots + A_L)$.
(d). Find a large deviations principle for $L^{-1}(C + A_1 + \cdots + A_L)$.
(e). Comment on your results.

State clearly any general results to which you appeal.

Question 2.

(a). Define these terms: rate function, good rate function, large deviations principle.

Recall that a sequence of random variables $(X_L, L \in \mathbb{N})$ is said to be *exponentially tight* if for all $\alpha \geq 0$ there exists a compact set K_α such that

$$\limsup_{L \to \infty} \frac{1}{L} \log P(X_L \notin K_\alpha) < -\alpha.$$

The sequence $(X_L, L \in \mathbb{N})$ is said to satisfy a *weak large deviations principle* if the large deviations upper bound is required to hold only for compact sets.

Suppose that the sequence $(X_L, L \in \mathbb{N})$ is exponentially tight, and satisfies a weak large deviations principle with rate function I.

(b). Show that I is a good rate function.
(c). Show that the large deviations upper bound holds for closed sets.

Conclude that $(X_L, L \in \mathbb{N})$ satisfies a large deviations principle with good rate function I.

Question 3.

(a). Consider a queue operating in slotted time, with infinite buffer and fixed service rate c, and receiving an amount of work a_t in timeslot $(t-1, t)$. What is the Lindley recursion for queue size? Writing a for $(a_t, t \in \mathbb{Z})$, define the queue size function $Q_0(a, c)$.
(b). Fix $\lambda > 0$ and consider the space of input process

$$\mathcal{X} = \left\{ a : \lim_{t \to \infty} \frac{a_{-t} + \cdots + a_{-1}}{t} = \lambda \right\}$$

equipped with the norm

$$\|a\| = \sup_{t \in \mathbb{N}} \left| \frac{a_{-t} + \cdots + a_{-1}}{t + 1} \right|$$

Show that, if $\lambda < c$, the queue size function $Q_0(\cdot, c)$ is continuous on $(\mathcal{X}, \|\cdot\|)$.

(c). Suppose that work from this queue is fed into another queue downstream: any work served by the first queue in timeslot $(t-1, t)$ reaches the downstream queue in the same timeslot, and may be served in the same timeslot. Let the downstream queue have service rate $d < c$. Write down a recursion for the downstream queue size $R_t(a, c, d)$, and show that $R_t(a, c, d)$ satisfies

$$Q_t(a, c) + R_t(a, c, d) = \left[Q_{t-1}(a, c) + R_{t-1}(a, c, d) + a_t - d\right]^+.$$

(d). Define the downstream queuesize function $R_0(a, c, d)$.

(e). Suppose that $\lambda < d < c$. Show that $R_0(\cdot, c, d)$ is continuous on $(\mathcal{X}, \|\cdot\|)$. Explain how one might use this in finding a large deviations principle for the downstream queuesize.

Question 4. Define the *effective bandwidth* of an arrival process. Write an essay on effective bandwidths and large deviations. In your essay you should describe a queueing model, explain the use of large deviations theory in analysing it, interpret the results in terms of effective bandwidth, and give examples, including an example of a queue fed by several independent arrival processes.

(You should prove a large deviations upper bound for the queue length distribution, but you need not prove a large deviations lower bound.)

Reference: Gärtner-Ellis theorem

A convex function $\Lambda : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is *essentially smooth* if
(a). the interior of its effective domain is non-empty
(b). $\Lambda(\cdot)$ is differentiable throughout the interior of its effective domain
(c). $\Lambda(\cdot)$ is steep, namely, $|\nabla\Lambda(\theta_n)| \to \infty$ whenever (θ_n) is a sequence in the interior of the effective domain converging to a point on the boundary of the effective domain.

Let $(X_L, \ L \in \mathbb{N})$ be a sequence of random vectors in $\mathbb{R}^d$, and let

$$\Lambda^L(\theta) = \frac{1}{L} \log E \exp(L\theta \cdot X_L)$$

for $\theta \in \mathbb{R}^d$. Assume that for each θ the limit

$$\Lambda(\theta) = \lim_{L\to\infty} \Lambda^L(\theta)$$

exists in $\mathbb{R} \cup \{\infty\}$. Assume further that 0 is in the interior of the effective domain of Λ, and that Λ is essentially smooth and lower-semicontinuous. Then $(X_L, \ L \in \mathbb{N})$ satisfies an LDP in $\mathbb{R}^d$ with good rate function

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}^t} \theta \cdot x - \Lambda(\theta).$$