A note on the Calculus of Variations

Large Deviations and Queues—Damon Wischik

Let Λ^* be a convex function $\Lambda^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. Let

$$I(f) = \int_0^1 \Lambda^*(\dot{f}_t)\,dt$$

for absolutely continuous functions $f \in \mathcal{C}[0,1]$.

Lemma 1 *For any absolutely continuous* $f \in \mathcal{C}[0,1]$,

$$I(f) \geq \Lambda^*\Big(f(1) - f(0)\Big).$$

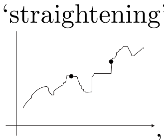
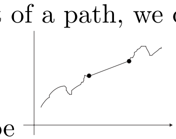
Proof. Let $U \sim \text{Uniform}[0,1]$. Then

$$\mathbb{E}\Lambda^*(\dot{f}_U) = \int_0^1 \Lambda^*(\dot{f}_t)\,dt = I(f).$$

However, by Jensen's inequality and by convexity of Λ^*,

$$\mathbb{E}\Lambda^*(\dot{f}_U) \geq \Lambda^*(\mathbb{E}\dot{f}_U) = \Lambda^*\Big(\int_0^1 \dot{f}_t\,dt\Big) = \Lambda^*\Big(f(1) - f(0)\Big). \qquad \square$$

Typical application

"By 'straightening' a segment of a path, we can reduce the rate function." Let f be , and let g be  like f but with the segment $[t_1, t_2]$ 'straightened'. Then

$$\begin{aligned}
I(f) &= \int_0^{t_1} \Lambda^*(\dot{f}_t)\,dt + \int_{t_1}^{t_2} \Lambda^*(\dot{f}_t)\,dt + \int_{t_2}^1 \Lambda^*(\dot{f}_t)\,dt \\
&\geq \int_0^{t_1} \Lambda^*(\dot{f}_t)\,dt + \Lambda^*\Big(\frac{f(t_2) - f(t_1)}{t_2 - t_1}\Big) + \int_{t_2}^1 \Lambda^*(\dot{f}_t)\,dt \\
&= \int_0^1 \Lambda^*(\dot{g}_t)\,dt = I(g).
\end{aligned}$$

Here we have used Lemma 1 extended in the obvious way to absolutely continuous functions $f \in \mathcal{C}[t_1, t_2]$.