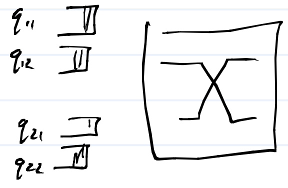
Worked example: 2x2 input-queued switch running max-weight



$$\mathcal{S} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$\lambda = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & \lambda_{22} \end{pmatrix}$$

The queue growth rates solve:

minimize $\frac{1}{1+\alpha} \sum q_{ij}^{1+\alpha}$ over $q \geq 0$ such that $q \geq \lambda - \begin{pmatrix} \rho & 1-\rho \\ 1-\rho & \rho \end{pmatrix}$ for some $\rho \in [0,1]$.

Minimum is obviously at $q = \begin{pmatrix} (0.7-\rho)^+ & (\rho-0.7)^+ \\ (\rho-0.7)^+ & (\lambda_{22}-\rho)^+ \end{pmatrix}$ for some $\rho \in [0,1]$.

If $\lambda_{22} \leq 0.7$: answer is clearly $q=0$.

If $\lambda_{22} > 0.7$ $\quad \frac{d}{d\rho} \frac{1}{1+\alpha} q^{1+\alpha} \cdot 1 \Big|_{\rho=0.7} = -(\lambda_{22}-0.7)^\alpha$, so minimum must be at $\rho > 0.7$,
Also, clearly, can restrict to $\rho \leq \lambda_{22}$.

so $q = \begin{pmatrix} 0 & \rho-0.7 \\ \rho-0.7 & \lambda_{22}-\rho \end{pmatrix}$.

To find ρ, $\quad \frac{d}{d\rho} \frac{1}{1+\alpha} q^{1+\alpha} \cdot 1 = 2(\rho-0.7)^\alpha - (\lambda_{22}-\rho)^\alpha$

This is <0 at $\rho=0.7$, >0 at $\rho=\lambda_{22}$, and increasing in ρ.
So there is only one zero, at

$$2^{1/\alpha}(\rho-0.7) = \lambda_{22}-\rho$$

$$\Rightarrow \rho(1+2^{1/\alpha}) = \lambda_{22} + 2^{1/\alpha}\, 0.7$$

$$\Rightarrow \rho = \frac{\lambda_{22} + 2^{1/\alpha}\, 0.7}{1+2^{1/\alpha}} < \lambda_{22}.$$

Since we're only allowed $\rho \in [0,1]$, the solution is

$$\rho = \begin{cases} 0.7 & \text{if } \lambda_{22} \leq 0.7 \\ \frac{\lambda_{22}+2^{1/\alpha}\, 0.7}{1+2^{1/\alpha}} & \text{if this is} \leq 1 \text{ ie if } \lambda_{22} \leq 1+0.3\times 2^{1/\alpha} \\ 1 & \text{if } \lambda_{22} > 1+0.3\times 2^{1/\alpha}. \end{cases}$$

The net departure rate is $1 \cdot \begin{pmatrix} 0.7 & 1-\rho \\ 1-\rho & \rho \end{pmatrix}$ when $\lambda_{22} \geq 0.7$ which, when simplified, gives

$$\text{dep rate} = \begin{cases} 1.3+\lambda_{22} & \text{if } \lambda_{22} \leq 0.7 \\ \frac{\lambda_{22}-0.7}{1+2^{1/\alpha}} & \text{else.} \\ 1.7 & \text{if } \lambda_{22} > 1+0.3\times 2^{1/\alpha} \end{cases}$$

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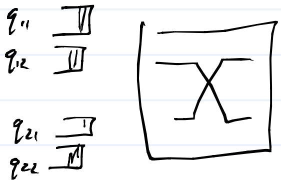

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$\lambda = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & \lambda_{22} \end{pmatrix}$$

The queue growth rates solve:

minimize $\sum \int_0^{q_{ij}} \left(\frac{\lambda_{ij}}{x} - 1\right)^{-\alpha} dx$ over $q \geq 0$ such that $q \geq \lambda - \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$ for some $p \in [0,1]$.

Minimum is obviously at $q = \begin{pmatrix} (0.7-p)^+ & (p-0.7)^+ \\ (p-0.7)^+ & (\lambda_{22}-p)^+ \end{pmatrix}$ for some $p \in [0,1]$.

If $\lambda_{22} \leq 0.7$: answer is clearly $q=0$.
If $\lambda_{22} > 0.7$:

$$\frac{d}{dp} \sum \int_0^{q_{ij}} \left(\frac{\lambda_{ij}}{x} - 1\right)^{-\alpha} dx \,\Bigg|_{p=0.7} = -\left(\frac{\lambda_{22}}{\lambda_{22}-0.7} - 1\right)^{-\alpha} < 0.$$

So minimum is at $p > 0.7$.
The proofs in the paper also tell us that $q < \lambda$ componentwise, so $p < 1$.

So $q = \begin{pmatrix} 0 & p-0.7 \\ p-0.7 & \lambda_{22}-p \end{pmatrix}$.

Differentiating the objective, and setting derivative equal to 0, gives

$$2\left(\frac{0.3}{p-0.7} - 1\right)^{-\alpha} - \left(\frac{\lambda_{22}}{\lambda_{22}-p} - 1\right)^{-\alpha} = 0$$

$$\Rightarrow \quad 2^{-1/\alpha}\left(\frac{1-p}{p-0.7}\right) = \frac{p}{\lambda_{22}-p} \quad \Rightarrow \quad 2^{1/\alpha}\,\frac{p-0.7}{1-p} = \frac{\lambda_{22}-p}{p}$$

$$\Rightarrow \quad 2^{1/\alpha} p(p-0.7) = (1-p)(\lambda_{22}-p)$$

$$\Rightarrow \quad p^2\left(2^{1/\alpha}-1\right) + p\left(1+\lambda_{22} - 0.7\times 2^{1/\alpha}\right) - \lambda_{22} = 0$$

$$\Rightarrow \quad p = \frac{-\left(1+\lambda_{22}-0.7\times 2^{1/\alpha}\right) \pm \sqrt{\left(1+\lambda_{22}-0.7\times 2^{1/\alpha}\right)^2 + 4\lambda_{22}\left(2^{1/\alpha}-1\right)}}{2\left(2^{1/\alpha}-1\right)}$$

In order to have $p \geq 0$ we always need the +ve root.

The net departure rate is $\begin{cases} 1.3 + \lambda_{22} & \text{if } \lambda_{22} \leq 0.7 \\ 0.7 + 2(1-p) + p & \text{if } \lambda_{22} > 0.7 \end{cases}$

The latter case is $2 - (p-0.7) = 2 - \dfrac{0.4 - \lambda_{22} - 0.7\times 2^{1/\alpha} + \sqrt{\cdots}}{2\left(2^{1/\alpha}-1\right)}$