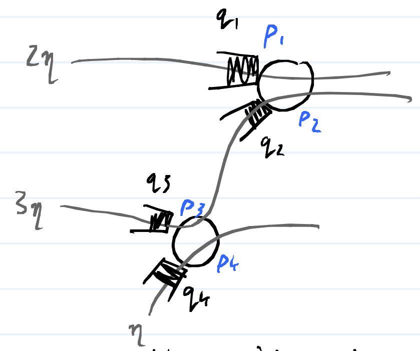
Worked example: revenue model



$$S = \{ (10,0,10,0),\ (10,0,0,10),\ (0,10,10,0),\ (0,10,0,10),\ (10,0,0,0),\ (0,10,0,0),\ (0,0,10,0),\ (0,0,0,10) \}$$

Under max-weight, with weights $w = (6,5,5,4)$ and weight function $f(q) = q^\alpha$, $\alpha > 0$, the queue growth rates solve the optimization problem

minimize $\frac{1}{1+\alpha}\left(6q_1^{1+\alpha} + 5(q_2^{1+\alpha} + q_3^{1+\alpha}) + 4q_4^{1+\alpha}\right)$

over $q \geq 0$

such that $(q_1, q_2, q_3, q_4) \geq (2\eta, 0, 3\eta, \eta) - (p_1, p_2 - p_3, p_3, p_4)$

for some $p \in \langle S \rangle$, ie

$p \geq 0$, $p_1 + p_2 \leq 10$, $p_3 + p_4 \leq 10$.

The first step is to rewrite the constraint. It can be checked explicitly that the constraint is equivalent to

$$\vec{q} \cdot \xi \geq \vec{\lambda} \cdot \xi - 10 \quad \text{for all } \xi \in \{ (1100), (0011), (1000), (0100), (0010), (0001) \}$$

where $\vec{q} = (I - R^T)^{-1} q = (q_1, q_2 + q_3, q_3, q_4)$

$\vec{\lambda} = (I - R^T)^{-1} \lambda = (2\eta, 3\eta, 3\eta, \eta)$.

This is true for a general switched network: the constraint can be rewritten

$\vec{q} \cdot \xi \geq \vec{\lambda} \cdot \xi - 1$ for all $\xi \in \Xi$

where Ξ is the set of extreme points of $\{ \xi \geq 0 : \xi \cdot \pi \leq 1 \text{ for all } \pi \in S \}$.

To solve the optimization, we used Mathematica. In order to get a numerically stable answer, it helps to use the objective function

$$\left(6q_1^{1+\alpha} + 5(q_2^{1+\alpha} + q_3^{1+\alpha}) + 4q_4^{1+\alpha}\right)^{\frac{1}{1+\alpha}}$$

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(* Version with queues only at nodes D and E; and A, B and C forward stuff immediately *)
rev[η_, α_] := Module[{res},
  res = FindMinimum[
    {(6 q1^(1+α) + 5 (q2^(1+α) + q3^(1+α)) + 4 q4^(1+α))^(1/(1+α)),
     {q1 ≥ 0, q2 ≥ 0, q3 ≥ 0, q4 ≥ 0,
      q1 + q2 + q3 ≥ 5 η - 10,
      q3 + q4 ≥ 4 η - 10,
      q1 ≥ 2 η - 10,
      q3 ≥ 3 η - 10,
      q4 ≥ η - 10}},
    {q1, q2, q3, q4}];
  {{q1, q2, q3, q4}, 31 η - (6 q1 + 5 (q2 + q3) + 4 q4)} /. res[[2]]
 ]
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total revenue rate

= total rate of arriving revenue − lost revenue

$= 6 \times 2\eta + 5 \times 3\eta + 4 \times \eta - (6q_1 + 5(q_2 + q_3) + 4q_4)$

$= 31\eta - (6q_1 + 5(q_2 + q_3) + 4q_4)$