Limitations of Affine CSP Algorithms

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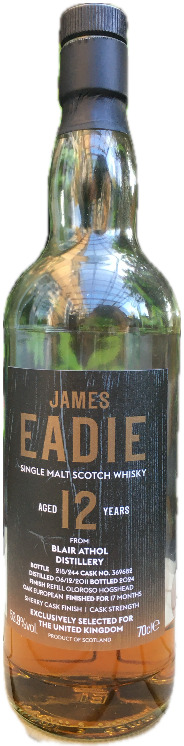
- For a relational structure **B** (the *template*), $\mathrm{CSP}(\mathbf{B})$ is the decision problem asking if an instance **A** admits a *homomorphism* to **B**.
- **CSP dichotomy:** For every finite **B**, $\mathrm{CSP}(\mathbf{B})$ is either in P or NP-complete.
- *No universal algorithm* known that uniformly solves all CSPs in P.

Conjecture (Dalmau & Opršal, LICS 2024):

"Every tractable finite-domain CSP is solved by a combination of k-consistency and a system of linear equations over $\mathbb{Z}$."

Prize: A bottle of whisky.



Our results

Theorem

There is a tractable (Maltsev) CSP template $\mathbf{B}$ *such that:*

- $\mathrm{CSP}(\mathbf{B})$ *is not solved by* $\mathbb{Z}$***-affine*** k***-consistency*** *for any sublinear* k.
- $\mathrm{CSP}(\mathbf{B})$ *is not solved by almost all other affine (P)CSP algorithms:* ***BLP+AIP*** *(Brakensiek, Guruswami, Wrochna, Živný 2020),* $\mathbf{BA}^k$ *(Ciardo, Živný 2023),* ***CLAP*** *(Ciardo, Živný 2023).*
- $\mathrm{CSP}(\mathbf{B})$ *is solved by the* ***cohomological*** k***-consistency*** *algorithm (Ó Conghaile 2022), for constant* k.

The basic LP relaxation for CSP

Let **B** be a template, **A** an instance, $k \in \mathbb{N}$ a width parameter.
The *width-k LP relaxation* $\mathrm{L}^{k,\mathbf{B}}_{\mathrm{CSP}}(\mathbf{A})$:

Variables: $\left\{ x_{X,f} \mid X \in \binom{A}{\leq k}, f \in \mathrm{Hom}(\mathbf{A}[X], \mathbf{B}) \right\}$.

Equations:

$$\sum_{f \in \mathrm{Hom}(\mathbf{A}[X],\mathbf{B})} x_{X,f} = 1 \qquad \text{for all } X \in \binom{A}{\leq k}$$

$$\sum_{f \in \mathrm{Hom}(\mathbf{A}[X],\mathbf{B}), f|_Y = g} x_{X,f} = x_{Y,g} \qquad \text{for all } Y \subset X \in \binom{A}{\leq k} \text{ and } g \in \mathrm{Hom}(\mathbf{A}[Y], \mathbf{B})$$

Fact: Let $k \geq \mathrm{ar}(\mathbf{A})$. $\mathrm{L}^{k,\mathbf{B}}_{\mathrm{CSP}}(\mathbf{A})$ has a $\{0,1\}$-solution iff $\mathbf{A} \to \mathbf{B}$.

Ensures consistency on overlapping subinstances.

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Ensures consistency on overlapping subinstances.

Result: $\mathrm{L}^{k,\mathbf{B}}_{\mathrm{CSP}}(\mathbf{A})$ can have a *$\mathbb{Z}$-solution* even if $\mathbf{A} \not\to \mathbf{B}$.

Let $k \in \mathbb{N}$ and a template **B** be fixed.

$\mathbb{Z}$-affine k-consistency

1: **Input:** Instance **A**.

2: Run the *k-consistency algorithm* and remove all partial k-solutions that are not consistent.
3: Find a $\mathbb{Z}$-solution Φ of $L_{\mathrm{CSP}}^{k,\mathbf{A}}(\mathbf{B})$ that only sets k-consistent partial homomorphisms to non-zero.
4: **Output** "YES" iff Φ exists.

Constructing the counterexample

Lemma

There exists an operation $\mathbf{OR}[\cdot,\cdot]$ *on finite structures such that:*
If $\mathbf{A}_1$ *is an instance of* $\mathrm{CSP}(\mathbf{B}_1)$ *and* $\mathbf{A}_2$ *is an instance of* $\mathrm{CSP}(\mathbf{B}_2)$*, then*

1. $\mathbf{OR}[\mathbf{A}_1, \mathbf{A}_2]$ *is an instance of* $\mathrm{CSP}(\mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2])$.
2. $\mathbf{OR}[\mathbf{A}_1, \mathbf{A}_2] \to \mathbf{OR}[\mathbf{B}_1, \mathbf{B}_2]$ *if and only if* $(\mathbf{A}_1 \to \mathbf{B}_1$ *or* $\mathbf{A}_2 \to \mathbf{B}_2)$.

The hard template for affine algorithms is $\mathbf{OR}[\mathbb{Z}_2, \mathbb{Z}_3]$.

Hardness for hierarchies like $\mathbb{Z}$*-affine k-consistency* or BA^k for $k \in o(n)$ is shown via Tseitin instances based on expander graphs (akin to Berkholz-Grohe, 2017).

Cohomological k-consistency (Ó Conghaile 2022) is a version of $\mathbb{Z}$-affine k-consistency that *fixes partial solutions* in the LP relaxation.

Theorem

There is a constant k such that cohomological k-consistency solves our counterexample **OR**$[\mathbb{Z}_2, \mathbb{Z}_3]$.
But there exists an NP-complete template that is not solved by cohomological k-consistency.

Exciting question: Is cohomological k-consistency a universal PTIME-algorithm for tractable CSPs?