Limitations of affine CSP algorithms

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- $\mathbb{Z}$-affine k-consistency (Dalmau, Opršal 2024).
- BLP+AIP (Brakensiek, Guruswami, Wrochna, Živný 2020).
- BA^k (Ciardo, Živný 2023).
- CLAP and variants (Ciardo, Živný 2023).
- k-cohomological algorithm (Ó Conghaile 2022).

Question: All these algorithms run in PTIME. Which tractable finite-domain CSPs do they solve?
Conjecture: (Dalmau, Opršal 2024): $\mathbb{Z}$-affine k-consistency *solves all* tractable CSPs.

Results

Theorem

*The following algorithms **fail** to solve all finite domain CSPs with **Mal'tsev** templates:*

- $\mathbb{Z}$*-affine* k*-consistency for every fixed* $k \in \mathbb{N}$.
- BLP+AIP.
- BA^k for every fixed $k \in \mathbb{N}$.
- CLAP

The question remains open for the k**-cohomological algorithm** and a variant of CLAP called **C(BLP+AIP)**.

The basic LP relaxation for CSP

Let **A** be a template, **B** an instance, $k \in \mathbb{N}$ a width parameter.
The *width-k LP relaxation* $\mathrm{L}^{k,\mathbf{A}}_{\mathrm{CSP}}(\mathbf{B})$:

Variables: $\left\{x_{X,f} \mid X \in \binom{B}{\leq k}, f \in \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})\right\}$.

Equations:

$$\sum_{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A})} x_{X,f} = 1 \qquad \text{for all } X \in \binom{B}{\leq k}$$

$$\sum_{f \in \mathrm{Hom}(\mathbf{B}[X],\mathbf{A}), f|_Y = g} x_{X,f} = x_{Y,g} \qquad \text{for all } Y \subset X \in \binom{B}{\leq k} \text{ and } g \in \mathrm{Hom}(\mathbf{B}[Y], \mathbf{A})$$

Ensures consistency on overlapping subinstances.

Fact: Let $k \geq \mathrm{ar}(\mathbf{B})$. $\mathrm{L}^{k,\mathbf{A}}_{\mathrm{CSP}}(\mathbf{B})$ has a $\{0,1\}$-solution iff $\mathbf{B} \to \mathbf{A}$.

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Result: $\mathrm{L}^{k,\mathbf{A}}_{\mathrm{CSP}}(\mathbf{B})$ can have a *$\mathbb{Z}$-solution* even if $\mathbf{B} \not\to \mathbf{A}$.

Does not ensure consistency over $\mathbb{Z}$.

Summary of the algorithms

- **$\mathbb{Z}$-affine k-consistency:**
  1. Remove partial homomorphisms that are not k-consistent.
  2. Solve $L^{k,\mathbf{A}}_{\text{CSP}}(\mathbf{B})$ over $\mathbb{Z}$ for the k-consistent partial homomorphisms.
- **BLP+AIP:** Let k be the arity of $\mathbf{B}$. Refine $L^{k,\mathbf{A}}_{\text{CSP}}(\mathbf{B})$ by forcing every variable to 0 that is 0 in every *non-negative rational* solution. Solve the refined LP over $\mathbb{Z}$.
- **BA^k:** Like BLP+AIP, but here, k is a *parameter* of the algorithm.
- **CLAP:** Similar to BLP+AIP. Every variable that cannot receive value 1 in a non-negative rational solution is forced to 0.

Outline of the construction

1. The template $\mathbf{A}$: Coset CSPs.
2. Encoding the graph isomorphism problem as a coset CSP.
3. Constructing the instances $\mathbf{B}_n \not\to \mathbf{A}$: A disjunction of Cai-Fürer-Immerman graphs.
4. A $\mathbb{Z}$-solution for $\mathrm{L}^{k,\mathbf{A}}_{\mathrm{CSP}}(\mathbf{B}_n)$.

Coset CSPs

The template **A** of an r-ary *coset CSP* consists of:

- **Universe:** A group Γ.
- **Relations:** Cosets $\Delta\gamma$, where $\Delta \leq \Gamma^r, \gamma \in \Gamma^r$.

Example (Linear equations over Abelian groups)

Ternary linear equations over $\mathbb{Z}_p$:
Let $\Delta = \{(a_1, a_2, a_3) \in \mathbb{Z}_p^3 \mid a_1 + a_2 + a_3 = 0 \mod p\} \leq \mathbb{Z}_p^3$.
Equation $x_1 + x_2 + x_3 = b$ corresponds to the coset $\Delta(b, 0, 0)$.

Fact: Every coset CSP has the *Mal'tsev* polymorphism $f(x, y, z) = xy^{-1}z$.

Bounded colour class graph isomorphism

Problem (Bounded colour class graph isomorphism)

Let $d \in \mathbb{N}$ a constant.

Input: *Vertex-coloured graphs G, H where every colour is assigned to at most d vertices.*

Problem: *Is there a colour-preserving isomorphism from G to H?*

The problem is in PTIME via a group-theoretic algorithm due to Luks.

Given: Graphs G, H with colours $c_1, \dots, c_m$ and colour class size d.
Let $C_i \subseteq V(G)$ denote the vertices with colour c_i.

CSP formulation:

Template **A**: Symmetric group $\mathbf{Sym}_d$ with all binary cosets.

Instance **B**:

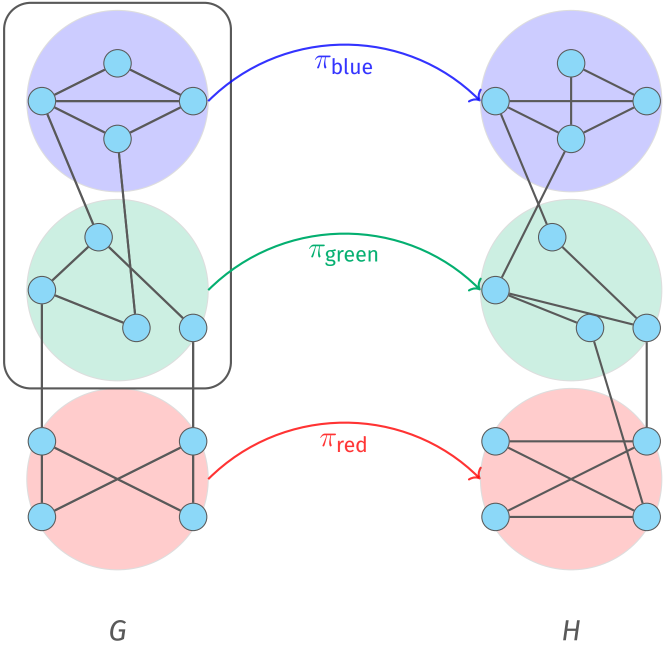
Variables: $\{x_{c_i} \mid i \in [m]\}$.

Constraints: $(x_{c_i}, x_{c_j}) \in \mathbf{Aut}(G[C_i \cup C_j])\gamma_{ij}$,
where γ_{ij} encodes a fixed local isomorphism from $G[C_i \cup C_j]$ to $H[C_i \cup C_j]$.

All isomorphisms from $G[C_i \cup C_j]$ to $H[C_i \cup C_j]$.

Bounded colour class graph isomorphism as a coset CSP

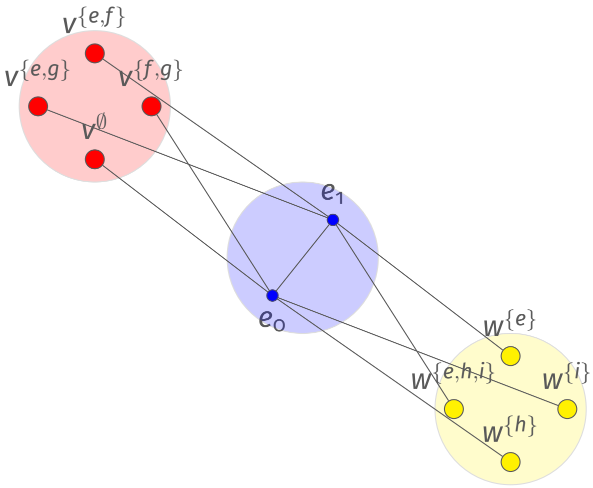
$(\pi_{\text{blue}}, \pi_{\text{green}}) \in \mathbf{Aut}(G[\text{Blue} \cup \text{Green}])\gamma$



Recap

- Graph isomorphism with colour class size d can be seen as a coset CSP over $\mathbf{Sym}_d$ (tractable).
- The template $\mathbf{A}$ is fixed, the instance $\mathbf{B}$ depends on the graphs.
- **Next step:** For all $n \in \mathbb{N}$, construct a pair of coloured graphs $G_n \not\cong H_n$ such that, for every $k \in \mathbb{N}$, $\mathrm{L}^{k,\mathbf{A}}_{\mathrm{CSP}}(\mathbf{B}_n)$ has a $\mathbb{Z}$-solution for almost all $n \in \mathbb{N}$.

- Fix a sequence $(G_n)_{n\in\mathbb{N}}$ of 3-regular expander graphs.
- For any prime p, and $t \in \mathbb{Z}_p$, $\mathrm{CFI}_{\mathbb{Z}_p}(G_n, t)$ is the CFI graph over $\mathbb{Z}_p$ with *twist* t.
- If $t \neq t'$, then $\mathrm{CFI}_{\mathbb{Z}_p}(G_n, t) \not\cong \mathrm{CFI}_{\mathbb{Z}_p}(G_n, t')$, but they look isomorphic for "*local consistency methods*".
- CFI graphs have constant colour class size.

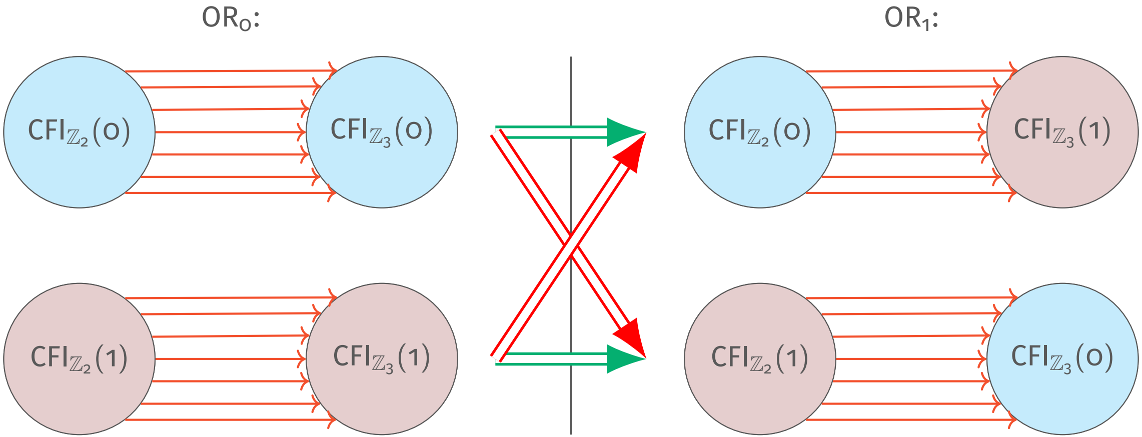


CFI gadget over $\mathbb{Z}_2$

Graph isomorphism disjunction construction

Goal: Given G_0, G_1, H_0, H_1, define graphs $\mathrm{OR}_0, \mathrm{OR}_1$ such that:

$$\mathrm{OR}_0 \cong \mathrm{OR}_1 \text{ if and only if } (G_0 \cong G_1) \text{ or } (H_0 \cong H_1).$$



Recap

- For every $n \in \mathbb{N}$, we have graphs $G_n \ncong H_n$ with bounded colour class size.
- The corresponding coset CSP over $\mathbf{Sym}_d$ has no solution.
- **Next step:** Construct a $\mathbb{Z}$-solution for $\mathrm{L}^{k,\mathbf{A}}_{\mathrm{CSP}}(\mathbf{B}_n)$.

Constructing integral solutions

Definition

A *p-solution* to a system of equations is a rational solution in which every variable has value p^z for some $z \in \mathbb{Z}$.

Lemma (Berkholz, Grohe 2017)

Let $p, q \in \mathbb{Z}$ be co-prime. If a system of linear equations over $\mathbb{Z}$ has both a p- and a q-solution, then it has an integral solution.

Lemma

The LP for "$\mathrm{CFI}_{\mathbb{Z}_2}(0) \cong \mathrm{CFI}_{\mathbb{Z}_2}(1)$?" has a 2-solution, and the LP for "$\mathrm{CFI}_{\mathbb{Z}_3}(0) \cong \mathrm{CFI}_{\mathbb{Z}_3}(1)$?" has a 3-solution.

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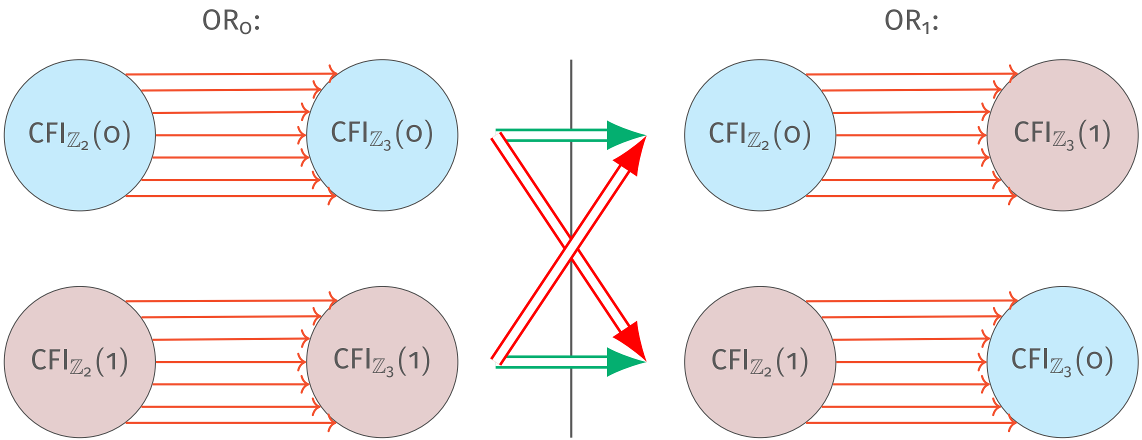
The cohomological algorithm

Algorithm 1 k-cohomology

Input: Instance $\mathbf{B}$.
Let $\mathcal{H}_0(X) := \mathrm{Hom}(\mathbf{B}[X], \mathbf{A})$ for every $X \in \binom{B}{\leq k}$.
repeat
    Let $\mathcal{H}'_i(X) \subseteq \mathcal{H}_i(X)$ be the partial homomorphisms
        that are not removed by the k-consistency procedure.
    Let $\mathcal{H}_{i+1}(X) \subseteq \mathcal{H}'_i(X)$ be the partial homomorphisms $f : X \to A$
        such that $\mathsf{L}^{k,\mathbf{A}}_{\mathsf{CSP}}(\mathbf{B})$, *augmented with the equation* $x_{X,f} = 1$, has a $\mathbb{Z}$-solution.
until $\mathcal{H}_{i+1} = \mathcal{H}_i$
If $\mathcal{H}_i(X) = \emptyset$ for some $X \in \binom{B}{\leq k}$, then return $\mathbf{B} \not\to \mathbf{A}$.

$L^{k,\mathbf{A}}_{CSP}(\mathbf{B})$, augmented with the equation $x_{X,f} = 1$, has only a 2- or a 3-solution, but not both. Hence *no $\mathbb{Z}$-solution*.



Open problems

- Find a counterexample (tractable or NP-complete) that cannot be solved by the *k-cohomological algorithm* for any fixed k.
- Simplify the counterexample. The current template is $\mathbf{Sym}_{72}$. Shouldn't $\mathbf{Sym}_3$ already work?
- Is there a *dichotomy* for coset CSPs: Do affine algorithms solve precisely the Abelian ones?