Limitations of Game Comonads for Invertible-Map Equivalence via Homomorphism Indistinguishability

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The Game Comonad Programme

- Initiated 2017 by Abramsky, Dawar, Wang: ***The Pebbling Comonad in Finite Model Theory***.
- Aims at comonadic characterisations of concepts from finite model theory.

Game comonads for FO with all Lindström quantifiers

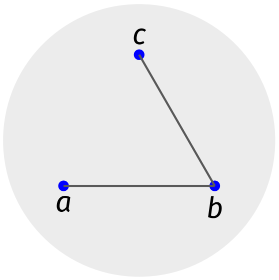
Question: Game comonads for Linear-Algebraic logic?
Answer: Do not exist.

Game comonads for $\mathcal{C}^k$

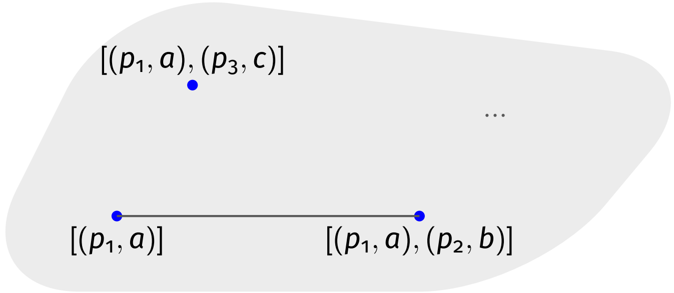
- Logics induce equivalence relations on finite structures.
- $\mathfrak{A} \equiv_{\mathcal{C}^k} \mathfrak{B}$ if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same $\mathcal{C}^k$-sentences.
- Logical equivalences are characterised by pebble games: $\mathfrak{A} \equiv_{\mathcal{C}^k} \mathfrak{B}$ iff Duplicator has a winning strategy in the *bijective k-pebble game*:
  1. Position: Pebbles $a_1, \dots, a_k$ on $\mathfrak{A}$; $b_1, \dots, b_k$ on $\mathfrak{B}$.
  2. Spoiler picks up a pebble-pair (a_i, b_i).
  3. Duplicator gives bijection $f : A \longrightarrow B$.
  4. Spoiler places a_i on an element of $\mathfrak{A}$ and b_i on $f(a_i)$.
  5. If the pebbles do not induce a local isomorphism, Spoiler wins.

For every τ-structure $\mathfrak{A}$, define $\mathbb{T}_k\mathfrak{A}$ as the structure of all Spoiler plays:

$\mathfrak{A}$:



$\mathbb{T}_k\mathfrak{A}$:



$\mathbb{T}_k$ is an endofunctor on the category of τ-structures and forms a *comonad* with a suitable counit and comultiplication.

Duplicator has a winning strategy iff there exist homomorphisms $f : \mathbb{T}_k\mathfrak{A} \longrightarrow \mathfrak{B}$ and $g : \mathbb{T}_k\mathfrak{B} \longrightarrow \mathfrak{A}$ ($\mathfrak{A}$ and $\mathfrak{B}$ are *co-Kleisli isomorphic*).

Highlights of Game Comonads

- Game comonad for *bijective k-pebble game* (Abramsky, Dawar, Wang).
- Game comonad for *k-round Ehrenfeucht-Fraïssé game* (Abramsky, Shah).
- Game comonad for FO with all n-ary *Lindström quantifiers* (Dawar, Ó Conghaile).
- *Coalgebras* of the respective comonads capture *graph decompositions* associated with treewidth, treedepth, pathwidth...
- Uniform framework for proving *Lovász-style theorems* on homomorphism indistinguishability.

Definition

Let $\mathcal{F}$ be a class of finite graphs. Let G and H be two graphs. Then $G \equiv_{\mathcal{F}} H$ iff for every $F \in \mathcal{F}$, the number of homomorphisms from F to G is the same as from F to H.

$\mathcal{F}$	$\equiv_{\mathcal{F}}$	
all graphs	isomorphism	Lovász, 1967
treewidth $\leq k$	$\equiv_{\mathcal{C}^k}$	Dvořák, 2009
treedepth $\leq k$	quantifier-rank k $\mathcal{C}$-equivalence	Grohe, 2020

Theorem

There is no graph class $\mathcal{F}$ such that $\equiv_{\mathcal{F}}$ is invertible-map equivalence (i.e. equivalence in Linear-Algebraic logic).

Theorem (Dawar, Jakl, Reggio; 2021)

Let $\mathbb{C}$ be a comonad on the category of graphs that sends finite graphs to finite graphs. Then there exists a graph class $\mathcal{F}$ such that co-Kleisli isomorphism for $\mathbb{C}$ is the same as $\equiv_{\mathcal{F}}$.

Corollary

There is no comonad that characterises invertible-map equivalence.

- *Linear-Algebraic logic* (LA) is infinitary FO augmented with all isomorphism-invariant linear-algebraic operators over finite fields.
- For every $k \in \mathbb{N}$ and every subset Q of prime numbers, the *invertible-map equivalence* $\equiv^{\mathsf{IM}}_{k,Q}$ is equivalence in LA with $\leq k$ variables and linear-algebraic operators over prime fields $\mathbb{F}_p$ for $p \in Q$.
- Each $\equiv^{\mathsf{IM}}_{k,Q}$ has a *pebble game* characterisation.

Proof of the main result

Theorem

For every k, Q, there exists no graph class $\mathcal{F}$ such that $\equiv_{\mathcal{F}}$ is $\equiv^{IM}_{k,Q}$.

Proof: Construct non-isomorphic $\equiv^{IM}_{k,Q}$-equivalent CFI-graphs over *planar* base graphs. **Why?**

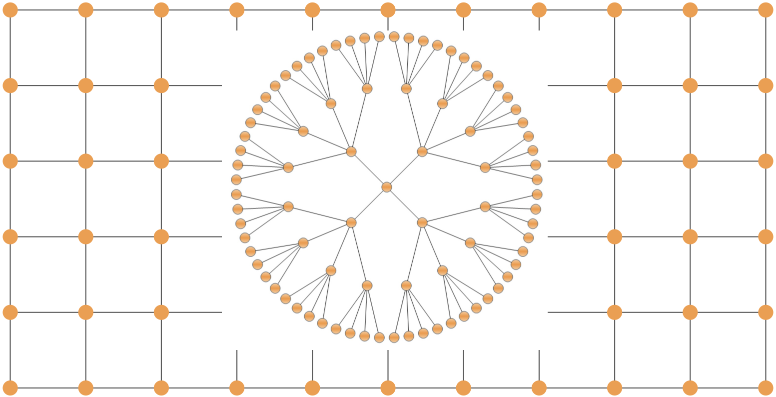
1. CFI-graphs over planar base graphs are **not $\equiv_{\mathcal{P}}$-equivalent**, for $\mathcal{P}$ the class of planar graphs (technique due to Roberson, 2022).
2. Any logical equivalence that strictly refines $\equiv_{\mathcal{C}^k}$ for every $k \in \mathbb{N}$ is **at least as fine as $\equiv_{\mathcal{P}}$** if it is characterised by a hom-ind-relation (Seppelt, 2023).

$\Rightarrow$ $\equiv^{IM}_{k,Q}$ cannot be a hom-ind-relation because it **does not refine $\equiv_{\mathcal{P}}$** (but it does refine all $\equiv_{\mathcal{C}^k}$).

Theorem (Dawar, Grädel, Lichter; 2021)

For every k and Q, there exist non-isomorphic CFI-graphs (over non-planar base graphs) which are $\equiv^{IM}_{k,Q}$-equivalent.

The construction can be adapted to planar base graphs:



Homomorphism counting over finite fields

Theorem

There is no graph class $\mathcal{F}$ and number $m \in \mathbb{N}$ such that $\equiv^{IM}_{k,Q}$ is captured by $\equiv_{\mathcal{F}}$ where homomorphisms are counted modulo m.

The 1-vertex graph and the edgeless graph on $m+1$ vertices are $\equiv_{\mathcal{F}}$-equivalent modulo m for any $\mathcal{F}$.

Summary

- There is *no homomorphism-indistinguishability relation* (neither in $\mathbb{N}$ nor with modular counting) characterising the IM-equivalences.

$\Rightarrow$ There is *no game comonad* for the IM-equivalences.

- This can likely be shown for every logic strictly stronger than $\mathcal{C}^k$ which cannot distinguish CFI graphs over planar base graphs.