Choiceless Polynomial Time

Benedikt Pago [1]

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[1]University of Cambridge

UNIVERSITY OF CAMBRIDGE
Department of Computer Science and Technology

Landscape of polynomial time logics

Ptime

$\rotatebox{-60}{$\lneqq$}$

Group order logic

$\rotatebox{90}{$\lneq$}$

Rank logic

$\rotatebox{60}{$\lneq$}$

FPC

$\rotatebox{90}{$\lneq$}$

IFP

$\rotatebox{90}{$\lneq$}$

First-order logic (FO)

PTIME

$\leq$ $\qquad\qquad$ $\geq$ **(=?)**

Group order logic

$\lneq$

Rank logic $\xleftrightarrow[\neq]{}$ **Choiceless Polynomial Time** (CPT)

$\gneq$ $\qquad\qquad$ $\lneq$

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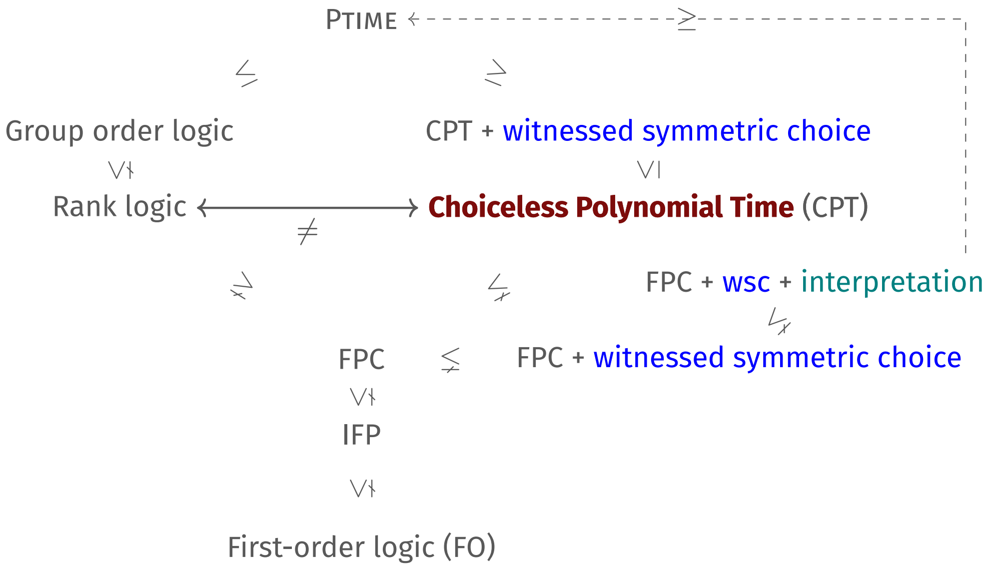
$\lneq$

IFP

$\lneq$

First-order logic (FO)

Landscape of polynomial time logics



Landscape of polynomial time logics

PTIME

Group order logic

Rank logic

CPT + witnessed symmetric choice

Choiceless Polynomial Time (CPT)

FPC + wsc + interpretation

FPC

FPC + witnessed symmetric choice

IFP

First-order logic (FO)

Candidates for capturing PTIME

Up to now, all logics were *extensions* of FO/fixed-point logics.

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CPT can be seen as a *restriction*:

It is obtained by enforcing **polynomial-time Turing machines** to be isomorphism-invariant [Blass, Gurevich, Shelah, 1999].

```c
void DFS(int graph[MAX_NODES][MAX_NODES], bool visited[MAX_NODES], int current_node, int num_nodes)
{
  visited[current_node] = true;

  for (int i = 0; i < num_nodes; i++) {
    if (graph[current_node][i] == 1 && !visited[i]) {
      DFS(graph, visited, i, num_nodes);
    }
  }
}
```

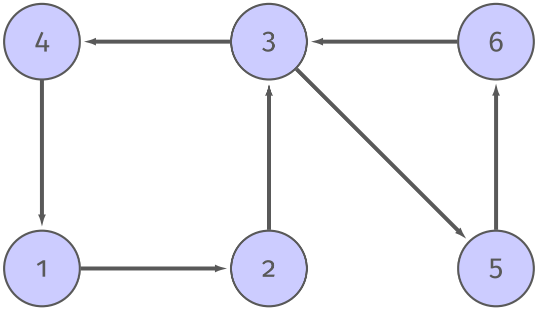
A C-program for depth-first search

```
void G = (V, E), current_node ∈ V DES])
{
  visited[current_node] = true;

  all i ∈ V do in parallel: i++) {
    if (graph[current_node][i] == 1 && !visited[i]) {
      G, i graph);
    }
  }
}
```

A choiceless program for DFS

DFS with choices



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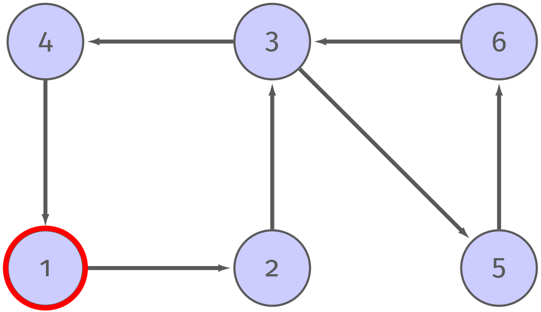

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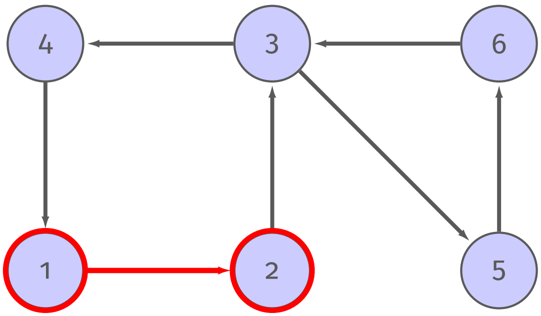

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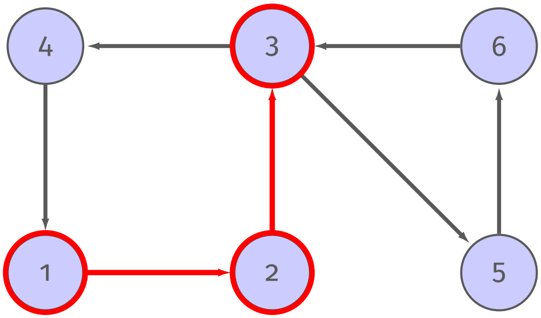

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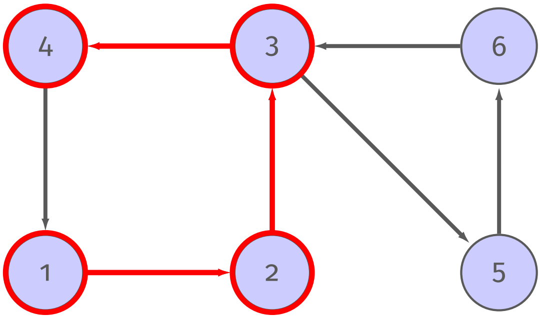

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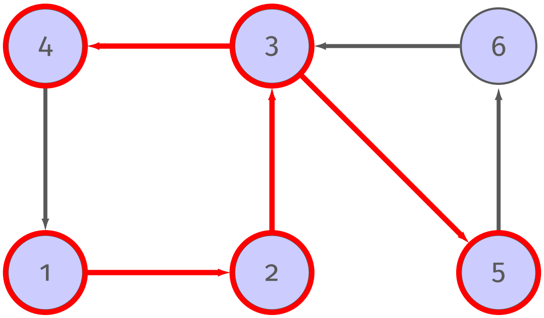

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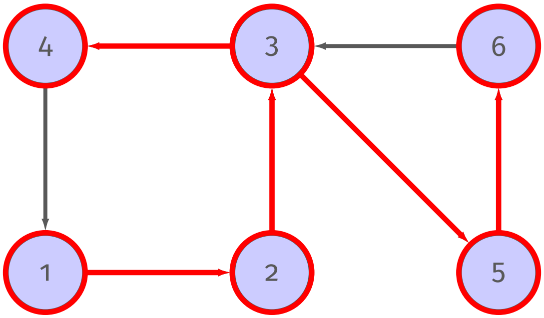

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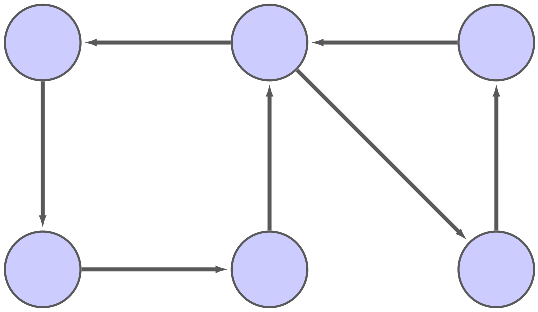

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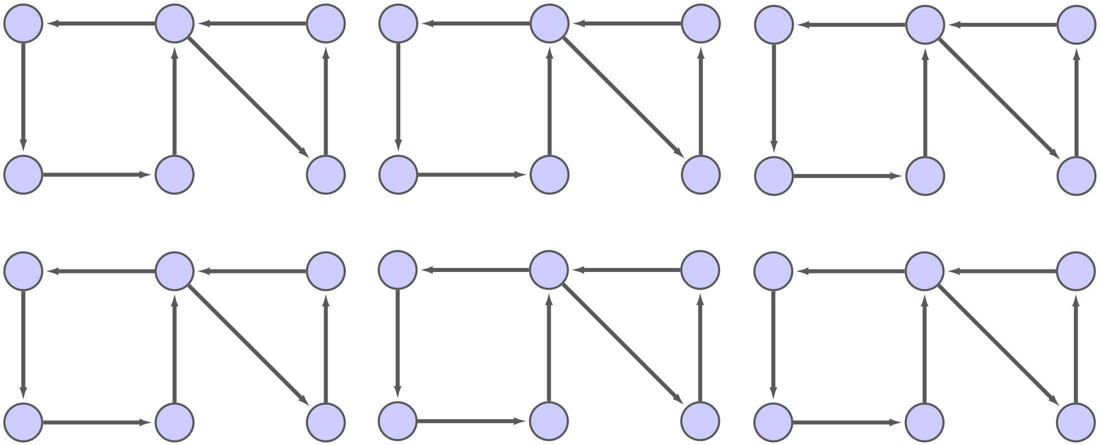

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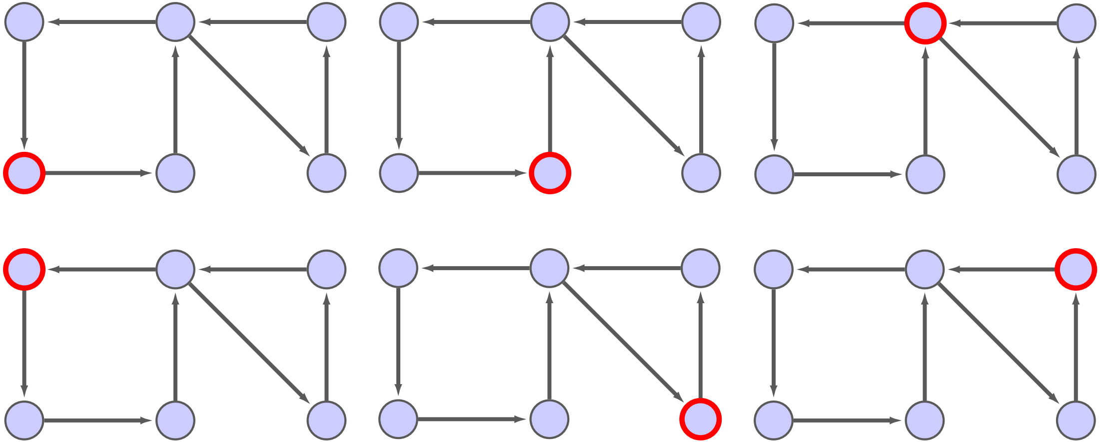

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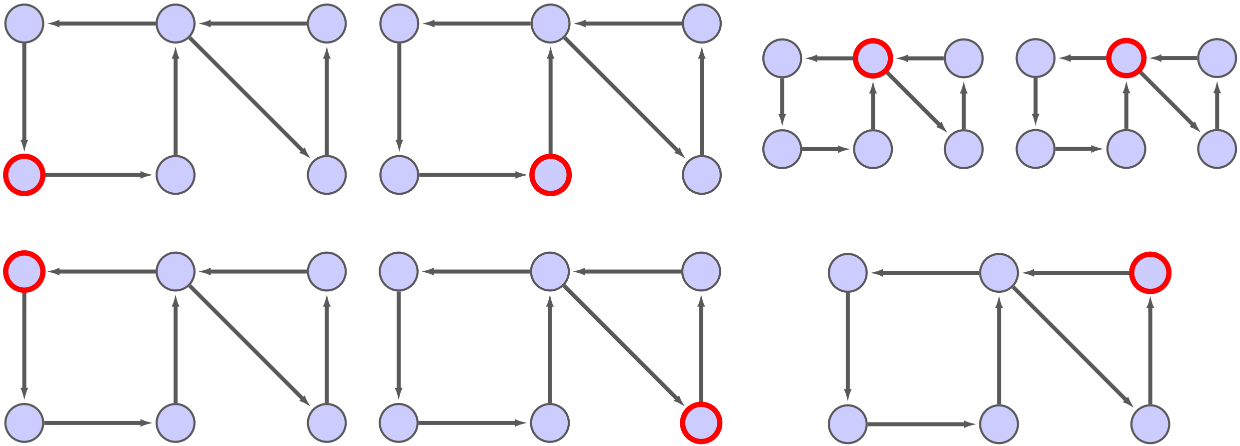

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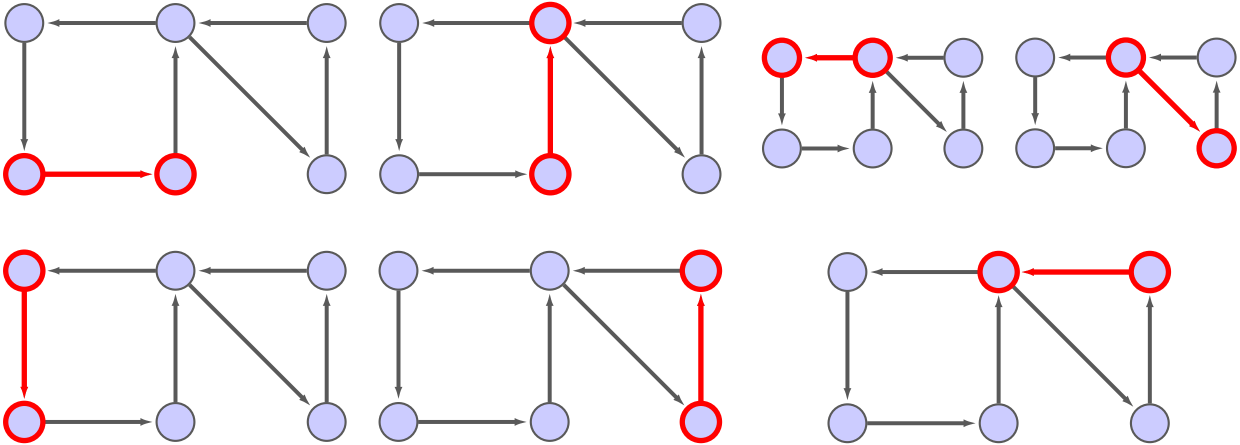

1. Original definition: abstract state **machine model** [Blass, Gurevich, Shelah, 1999].
2. **BGS-logic** [Rossman, 2010].
3. **Polynomial Interpretation Logic** [Grädel, Pakusa, Schalthöfer, Kaiser, 2015].

Definition (FO-interpretation)

A σ-structure $\mathfrak{B}$ is **FO-interpretable** in a τ-structure $\mathfrak{A}$ if there exist formulas $\varphi_\delta(\bar{x}), \varphi(\bar{x}, \bar{y})_\approx, (\varphi_R)_{R \in \sigma}$ and a $k \in \mathbb{N}$ such that

- $B = \{[\bar{a}]_\approx \mid \bar{a} \in A^k, \mathfrak{A} \models \varphi_\delta(\bar{a})\}$
- For each r-ary $R \in \sigma$, $R^\mathfrak{B} = \{(\bar{a}_1, \ldots, \bar{a}_r) \mid \mathfrak{A} \models \varphi_R(\bar{a}_1, \ldots, \bar{a}_r)\}$.

Polynomial Interpretation Logic

Definition (PIL, simplified)

Sentences of PIL are of the form $(\mathcal{I}_{\text{step}}, \psi_{\text{end}}, \psi_{\text{out}}, p(n))$, where

- $\mathcal{I}_{\text{step}}$ is an FO-interpretation,
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$(\mathcal{I}_{\text{step}}, \psi_{\text{end}}, \psi_{\text{out}}, p(n))$ defines a **run** in any structure $\mathfrak{A}$:

$$\mathfrak{A}, \mathcal{I}_{\text{step}}(\mathfrak{A}), \mathcal{I}_{\text{step}}(\mathcal{I}_{\text{step}}(\mathfrak{A})), \dots, \mathfrak{B}.$$

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where $\mathfrak{B}$ is the first structure in the run with $\mathfrak{B} \models \psi_{\text{end}}$.
If the number of steps or size of the structures *exceeds $p(|\mathfrak{A}|)$*, it is *aborted*.

$$\mathfrak{A} \models (\mathcal{I}_{\text{step}}, \psi_{\text{end}}, \psi_{\text{out}}, p(n)) \iff \text{the run terminates with } \mathfrak{B}, \text{ s.t. } \mathfrak{B} \models \psi_{\text{out}}.$$

Power of PIL: Using higher-dimensional interpretations, a PIL sentence can *grow the input structure* arbitrarily.

Computing linear orders

$$\mathcal{I}_{\mathsf{step}} := (\varphi_\delta(x_1, x_2) := \mathsf{true},$$
$$\varphi_<(x_1, x_2, y_1, y_2) := (x_1 = x_2 \wedge y_1 = y_2 \wedge x_1 < y_1) \vee (x_1 = x_2 \wedge y_1 \neq y_2 \wedge y_1 = x_1)).$$

Ignore $\psi_{\mathsf{end}}, \psi_{\mathsf{out}}, p(n)$ for now.

Example: Defining a linear order in PIL

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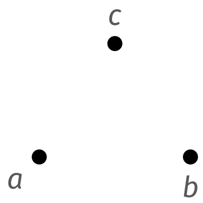

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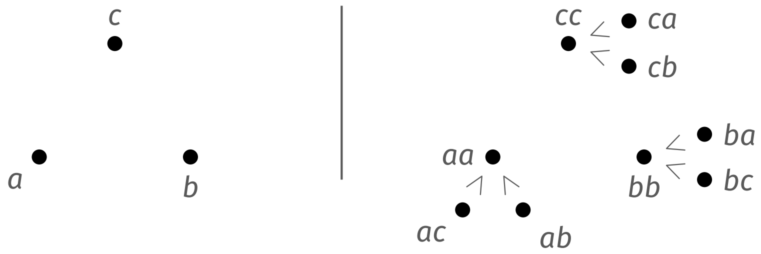

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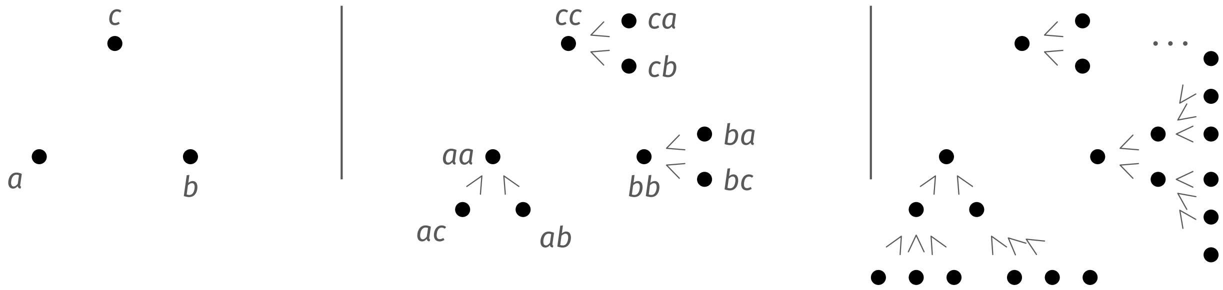

Weakness of CPT

The stages $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2, \dots$ of a PIL computation are closed under the group $\mathbf{Aut}(\mathfrak{A})$.

If $\mathfrak{A}$ is very symmetric, then intuitively, the structures $\mathfrak{A}_i$ quickly become super-polynomially large, and the computation cannot be carried out in PIL.

Presentations of CPT

1. Original definition: abstract state machine model.
2. **BGS-logic**: Useful for lower bounds.
3. Polynomial Interpretation Logic: Most understandable.

BGS-logic is FO augmented with terms for the creation of *hereditarily finite sets*:

Syntax and semantics of set-terms:

Let $\mathfrak{A} = (A, \tau)$ be a structure.

- Universe of structure: $[\![\texttt{Atoms}]\!]^{\mathfrak{A}} = A$.

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- **Comprehension term:** $[\![\{s : x \in r : \varphi\}]\!]^{\mathfrak{A}} = \{[\![s(x)]\!]^{\mathfrak{A}} \mid x \in [\![r]\!]^{\mathfrak{A}}, \mathfrak{A} \models \varphi(x)\}$.

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- **Iteration term:** $[\![s(x)^*]\!]^{\mathfrak{A}}$ is the least fixed-point of the sequence $[\![s(\emptyset)]\!]^{\mathfrak{A}}, [\![s(s(\emptyset))]\!]^{\mathfrak{A}}, [\![s(s(s(\emptyset)))]\!]^{\mathfrak{A}}, \dots$.

Choiceless Polynomial Time as BGS-logic

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- Iteration terms are accompanied by a polynomial resource bound for time and space: (s^*, p).

CPT is strictly stronger than fixed-points

Theorem

$$\text{FPC} \lneq \text{CPT}.$$

Proof.

- FPC cannot distinguish CFI graphs.

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Proof.

- FPC cannot distinguish CFI graphs.
- This is also true if they are augmented with exponentially many isolated vertices (*padding*).
- CPT can distinguish the padded CFI structures by *computing all linear orders* on the non-padding part and running the PTIME-algorithm that distinguishes them.

Inexpressibility results for CPT?

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- But there is no known decision problem in PTIME that is not in CPT.
- Obvious idea: Try CFI graphs...

Definability on different classes of base graphs:

Linearly ordered base graphs [1]	✓
Preordered base graphs with log-size colour classes [2]	✓
Base graphs with linear degree [2]	✓
General unordered base graphs	?

[1] [Dawar, Richerby, Rossman, 2008]
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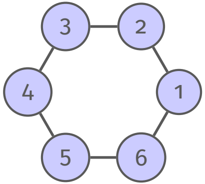
Is the CFI-query definable in Choiceless Polynomial Time?



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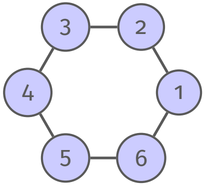

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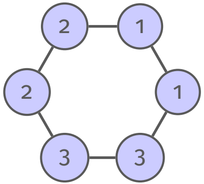

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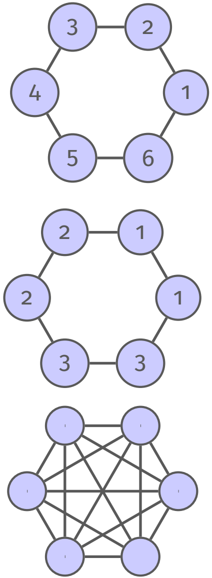

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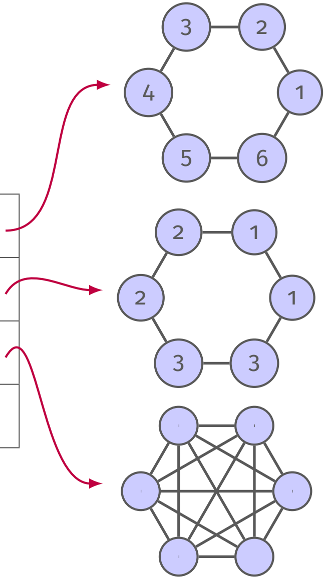

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Non-definability of preorders

Theorem (P., CSL 2021)

There exists a family of unordered base graphs (with sub-linear degree) where no preorder with log-size colour classes is CPT-definable.

$\Rightarrow$ There are CFI-instances that cannot be handled with any of the known CPT-algorithms.

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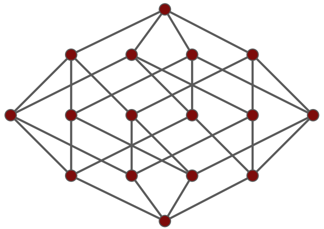
Theorem (technical version)

Any preorder with log-size colour classes on the n-dimensional hypercube has an orbit of super-polynomial size.



4-dimensional hypercube

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But: A property of the created h.f. sets controls the expressivity like the pebble number.

Definition (Support)

Let x a h.f. set over $\mathfrak{A}$. A **support** for x is a tuple α over $\mathfrak{A}$ such that “fixing α also fixes x”.

Formally: For every $\pi \in \mathbf{Aut}(\mathfrak{A})$ such that $\pi(\alpha) = \alpha$, it holds $\pi(x) = x$.

The support of a hereditarily finite set

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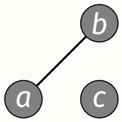
Example:



The structure $\mathfrak{A}$

- $\mathbf{Aut}(\mathfrak{A}) = \{\mathrm{id}, (a\ b)\}$.
- Let $x = \{a, \{c\}\}$.
- A trivial support for x: ac.
- A **smallest support**: a (or b).

An approach to prove indistinguishability in CPT

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- Since CPT is *polynomially bounded*, $|\mathbf{Orb}_{\mathfrak{A}}(x)|$ must be polynomial in $|\mathfrak{A}|$.

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- $\implies$ Any CPT-program distinguishing $\mathfrak{A}$ and $\mathfrak{B}$ must construct (on input $\mathfrak{A}$) a h.f. set x whose smallest support has size $\geq k$ [Dawar, Richerby, Rossman, 2008].
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- Since CPT is *polynomially bounded*, $|\mathbf{Orb}_{\mathfrak{A}}(x)|$ must be polynomial in $|\mathfrak{A}|$.
- $\implies$ One way to show CPT-indistinguishability is to prove that **large support** implies a **large orbit**.

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Generally: On any structure $\mathfrak{A}$ with $\mathbf{Aut}(\mathfrak{A}) = \mathbf{Sym}(A)$,
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$\implies$ Informally, on structures with $\mathbf{Aut}(\mathfrak{A}) = \mathbf{Sym}(A)$, CPT is not stronger than FPC.

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- **But:** It seems plausible that CPT cannot distinguish unordered CFI graphs, e.g. over n-dimensional hypercubes.
- In particular, the **rank operator** is probably not expressible in CPT (?)
- **Open problem:** Separate CPT from PTIME.

Proof complexity and finite model theory

Problem

Input: *A set* $\mathcal{F} = \{\varphi_1(\vec{x}), ...\varphi_m(\vec{x})\}$ *of propositional formulas.*

Question: *Is there a* $\{0, 1\}$*-assignment to the variables that satisfies all formulas?*

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- If every certificate has polynomial size, then $\mathrm{co\text{-}NP} = \mathrm{NP}$.
- **Proof complexity** seeks to establish proof size lower bounds against stronger and stronger proof systems, towards $\mathrm{co\text{-}NP} \neq \mathrm{NP}$.

Motivation: Transfer lower-bound techniques from finite model theory to proof complexity.

Problem

Input: *A set* $\mathcal{F} = \{f_1(\vec{x}), ...f_m(\vec{x})\}$ *of polynomials in* $\mathbb{F}[X]$.

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Typical certificate via **Hilbert's Nullstellensatz**: $\mathcal{F}$ is unsat iff there exist $g_1, ...g_m \in \mathbb{F}[X]$ such that $\sum_{i\in[m]} g_i \cdot f_i = 1$.

Definition (Grochow, Pitassi; 2016)

An **IPS certificate** of unsatisfiability of $\mathcal{F} = \{f_1(\vec{x}), \dots, f_m(\vec{x})\} \subseteq \mathbb{F}[X]$ is a polynomial $C(\vec{x}, y_1, \dots, y_m)$ such that:

1. $C(\vec{x}, \vec{0}) = 0$.
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To make IPS "*isomorphism-invariant*", we only allow circuits that are symmetric under the symmetries of $\mathcal{F}$.

Theorem (Dawar, Grädel, Kullmann, P., 2025)

Let $G \not\cong H$, $k \in \mathbb{N}$.

- *G and H k-**WL-distinguishable** $\Leftrightarrow$ there is a poly-size proof of non-isomorphism in deg_k sym-IPS.*
- *G and H **CPT-distinguishable** $\Rightarrow$ there is a poly-size proof of non-isomorphism in sym-IPS (possibly of unbounded degree).*

Logics with witnessed choice operators

- **Observation:** Suppose $C \subseteq A^k$ is an orbit of $\mathfrak{A}$, i.e. there is a tuple $\bar{a} \in A^k$ such that

$$C = \{\pi(\bar{a}) \mid \pi \in \mathbf{Aut}(\mathfrak{A})\}.$$

Then the computation outcome will be the same for every possible choice from C.

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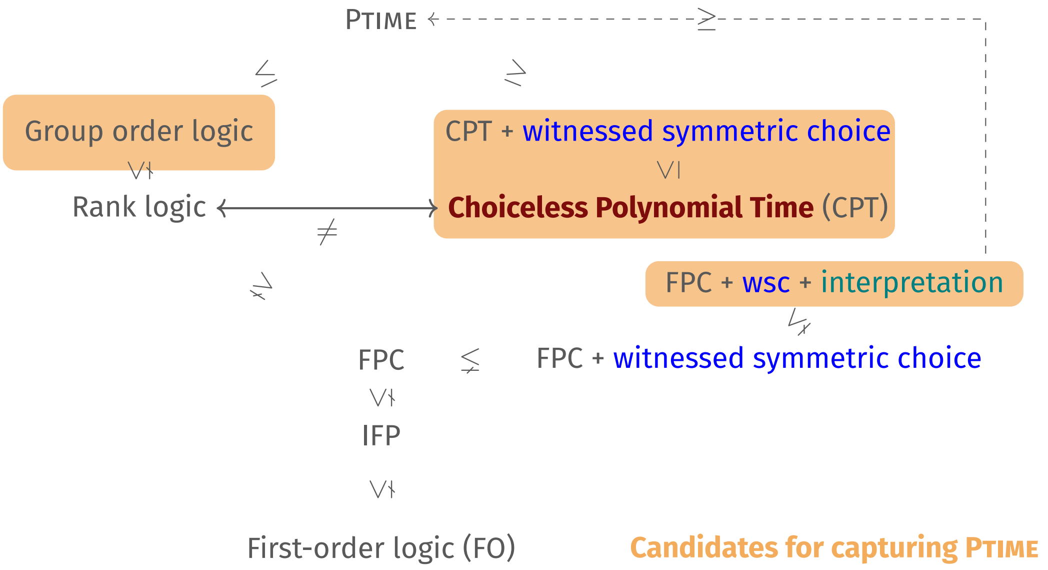
- Choice sets can be defined via formulas.
- **Problem:** If C is not an orbit, then the computation should be aborted because the choice would break isomorphism-invariance. How does the program know if C is an orbit?
- **Solution:** Not only C must be defined by a formula, but also the automorphisms witnessing the fact that C is an orbit.

CPT with witnessed symmetric choice

Theorem (Lichter, Schweitzer, 2024)

CPT with witnessed symmetric choice captures PTIME on every class of structures on which it defines isomorphism.

Landscape of polynomial time logics



[1] A. Blass, Y. Gurevich, and S. Shelah. **"On polynomial time computation over unordered structures".** In: The Journal of Symbolic Logic 67.3 (2002), pp. 1093–1125.

[2] Andreas Blass, Yuri Gurevich, and Saharon Shelah. **"Choiceless polynomial time".** In: Annals of Pure and Applied Logic 100.1-3 (1999), pp. 141–187.

[3] Anuj Dawar and David Richerby. **"A fixed-point logic with symmetric choice".** In: International Workshop on Computer Science Logic. Springer. 2003, pp. 169–182.

[4] Anuj Dawar, David Richerby, and Benjamin Rossman. **"Choiceless Polynomial Time, Counting and the Cai–Fürer–Immerman graphs".** In: Annals of Pure and Applied Logic 152.1-3 (2008), pp. 31–50.

[5] Anuj Dawar et al. **Symmetric Proofs in the Ideal Proof System.** 2025. arXiv: `2504.16820 [cs.LO]`. https://arxiv.org/abs/2504.16820.

[6] F. Gire and H.K. Hoang. **“An Extension of Fixpoint Logic with a Symmetry-Based Choice Construct”.** In: Information and Computation 144.1 (1998), pp. 40–65. ISSN: 0890-5401. https://doi.org/10.1006/inco.1998.2712.

[7] Erich Grädel and Martin Grohe. **“Is Polynomial Time Choiceless?”** In: Fields of Logic and Computation II. Springer, 2015, pp. 193–209.

[8] Erich Grädel et al. **“Characterising Choiceless Polynomial Time with First-Order Interpretations”.** In: 30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015. IEEE Computer Society, 2015, pp. 677–688. 10.1109/LICS.2015.68. https://doi.org/10.1109/LICS.2015.68.

[9] Joshua A. Grochow and Toniann Pitassi. **“Circuit Complexity, Proof Complexity, and Polynomial Identity Testing: The Ideal Proof System”.** In: J. ACM 65.6 (2018), 37:1–37:59. 10.1145/3230742. https://doi.org/10.1145/3230742.

[10] Moritz Lichter. **“Witnessed Symmetric Choice and Interpretations in Fixed-Point Logic with Counting”.** In: 50th International Colloquium on Automata, Languages, and Programming (ICALP 2023). Ed. by Kousha Etessami, Uriel Feige, and Gabriele Puppis. 261. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023, 133:1–133:20. ISBN: 978-3-95977-278-5. 10.4230/LIPIcs.ICALP.2023.133. https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.ICALP.2023.133.

[11] Moritz Lichter and Pascal Schweitzer. **“Choiceless Polynomial Time with Witnessed Symmetric Choice”.** In: J. ACM 71.2 (Apr. 2024). ISSN: 0004-5411. 10.1145/3648104. https://doi.org/10.1145/3648104.

[12] Benedikt Pago. **“Choiceless Computation and Symmetry: Limitations of Definability”.** In: 29th EACSL Annual Conference on Computer Science Logic (CSL 2021). 183. Leibniz International Proceedings in Informatics (LIPIcs). 2021, 33:1–33:21. 10.4230/LIPIcs.CSL.2021.33. https://drops.dagstuhl.de/opus/volltexte/2021/13467.

[13] Wied Pakusa, Svenja Schalthöfer, and Erkal Selman. **“Definability of Cai-Fürer-Immerman problems in Choiceless Polynomial Time”.** In: ACM Transactions on Computational Logic (TOCL) 19.2 (2018), pp. 1–27. 10.1145/3154456.

[14] Benjamin Rossman. **“Choiceless Computation and Symmetry”.** In: Fields of Logic and Computation. Springer, 2010, pp. 565–580.

[15] Faried Abu Zaid et al. **“Choiceless Polynomial Time on structures with small Abelian colour classes”.** In: International Symposium on Mathematical Foundations of Computer Science. Springer. 2014, pp. 50–62.