Linear-algebraic logics

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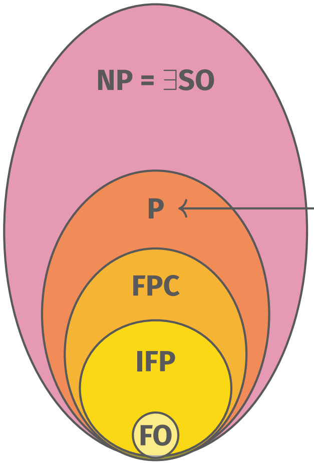
Strengthening FPC



Linear-algebraic problems over finite fields.

Importing linear algebra into logic

Goal: Enrich FO/IFP with an operator to *solve* unordered *systems of linear equations*.

An FO-formula $\varphi(\bar{x}, \bar{y})$ defines a system of equations (a **matrix**) $M(\mathfrak{A}, \varphi)$ in $\mathfrak{A}$ as follows.

- Row index set: $A^{\bar{x}}$.
- Column index set: $A^{\bar{y}}$.
- Entry $M(\mathfrak{A}, \varphi)[\bar{a}, \bar{b}] = \begin{cases} 1 & \text{if } \mathfrak{A} \models \varphi(\bar{a}, \bar{b}) \\ 0 & \text{if } \mathfrak{A} \not\models \varphi(\bar{a}, \bar{b}) \end{cases}$

Example:

Let $G = (V, E)$ be a graph. Then for $\varphi(x, y) = Exy$, $M(G, \varphi)$ is the $(V \times V)$-adjacency matrix of G.

Rank operators

Let p be a prime. If $\varphi(\bar{x}, \bar{y})$ is a formula (or a numeric term), then $\mathrm{rk}_p(\bar{x}, \bar{y})\varphi$ is a numeric term.

Semantics: For a two-sorted structure $\mathfrak{A}^*$,

$$[\![\mathrm{rk}_p(\bar{x}, \bar{y})\varphi]\!]^{\mathfrak{A}^*} = \text{the } \textit{rank} \text{ of } M(\varphi, \mathfrak{A}^*)\text{, interpreted as a matrix over } \mathbb{F}_p.$$

- $\mathrm{FO} + \mathbf{rk}_p$ is the extension of FO with rk_p.
- $\mathrm{FO} + \mathbf{rk}$ is the extension of FO with rank operators for all primes p.
- $\mathrm{IFP} + \mathbf{rk}_p$, $\mathrm{IFP} + \mathbf{rk}$ are the respective extensions of fixed-point logics.

- **Rank** simulates *counting*: For any prime p, $\mathrm{rk}_p(\bar{x}, \bar{y})\ (\bar{x} = \bar{y} \wedge \varphi(\bar{x}))$ is equivalent to the counting term $\#_{\bar{x}}[\varphi(\bar{x})]$.
- $\mathrm{FO} + \mathrm{rk}_p$ expresses whether a *system of linear equations* over $\mathbb{F}_p$ *has a solution*: $A \cdot \boldsymbol{x} = \boldsymbol{b}$ has a solution iff $\mathrm{rk}(A) = \mathrm{rk}(A|\boldsymbol{b})$.
- For any prime p, $\mathrm{FO} + \mathrm{rk}_p$ expresses *(s, t)-connectivity* in undirected graphs.
- $\mathrm{FO} + \mathrm{rk}_2$ *distinguishes* the *Cai-Fürer-Immerman graphs* that are indistinguishable in FPC.

(s,t)-connectivity

Input: An undirected graph $G = (V, E, s, t)$ with two distinguished vertices (constants) s and t.
Question: Is there a path between s and t?

$G = (V, E, s, t)$ has an s-t-path if and only if the following *system of equations* in $\mathbb{F}_p$ has *no solution*:

Variables: $\{x_v \mid v \in V\}$.

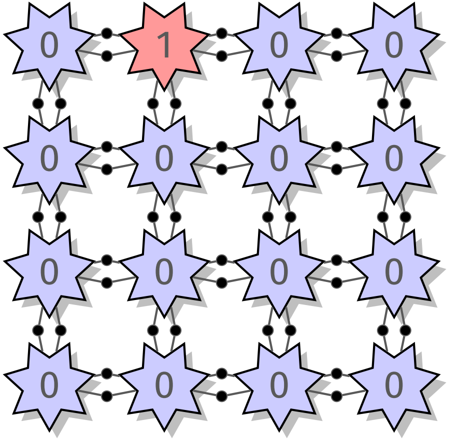
$$x_s = 1$$
$$x_t = 0$$
$$x_u - x_v = 0 \text{ for every edge } uv \in E$$

Coefficient matrix $M_G \in \mathbb{F}_p^{V^2 \times V}$ defined using formulas

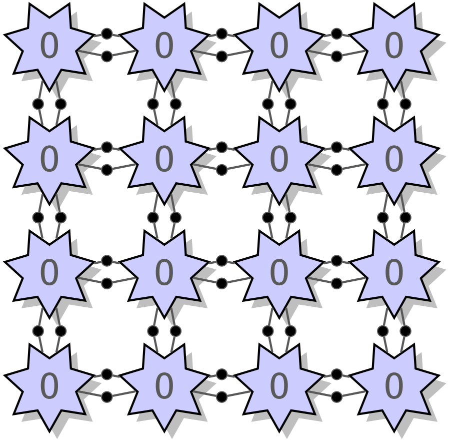
$$\varphi_{+1}(x_1, x_2, y) := ((x_1 = s \wedge x_2 = s \wedge y = s) \vee (x_1 = t \wedge x_2 = t \wedge y = t)) \vee (Ex_1x_2 \wedge x_1 = y)$$

$$\varphi_{-1}(x_1, x_2, y) := (Ex_1x_2 \wedge x_2 = y)$$

Distinguishing Cai-Fürer-Immerman graphs in $\mathsf{FO} + \mathsf{rk}_2$



$\mathsf{CFI}(G)$



$\mathsf{CFI}(H)$

Lemma (Cai, Fürer, Immerman, 1992)

Let G be a connected graph, and $\lambda_0, \lambda_1 \colon V \to \mathbb{Z}_2$ two node labellings.

$$\mathsf{CFI}(G, \lambda_0) \cong \mathsf{CFI}(G, \lambda_1) \iff \sum_{v \in V} \lambda_0(v) = \sum_{v \in V} \lambda_1(v) \mod 2.$$

In $\mathsf{CFI}(G, \lambda)$, one can FO-define a *system of linear equations* over $\mathbb{Z}_2$ which *has a solution* if and only if $\sum_{v \in V} \lambda(v) = 0 \mod 2$.

A history of rank logics

Infinitary linear-algebraic logic [Approximations of isomorphism and logics with linear-algebraic operators: Dawar, Grädel, Pakusa (2019)]

$\geq$

IFP + rk^* (“*rank logic*”) [Rank logic is dead, long live rank logic!: Grädel, Pakusa (2015)]

$\gneq$

IFP + rk [Logics with rank operators: Dawar, Grohe, Holm, Laubner (2009)]

$\gneq$

FPC

contained in PTIME

Rank logic is dead, long live rank logic!

Let rk^* denote the *uniform* rank operator that takes the prime p as input via a numerical term.

Theorem (Grädel, Pakusa, 2015)

$$\mathsf{IFP} + \mathsf{rk} \lneq \mathsf{IFP} + \mathsf{rk}^* \leq \textsc{Ptime}.$$

Proof.

- For contradiction, let $\psi \in \mathsf{IFP} + \mathsf{rk}$ be a sentence defining rk^*. There is a finite set Ω of primes p such that rk_p appears in ψ.
- Let $\mathcal{K}$ be a class of *CFI graphs over* $\mathbb{Z}_q$, for a prime $q \notin \Omega$.
- **Technical result:** On $\mathcal{K}$, ψ is equivalent to a sentence in FPC (coprimeness of the rank operators with q)
- FPC cannot distinguish CFI graphs $\implies$ ψ does not distinguish graphs in $\mathcal{K}$.
- But: rk^* does distinguish them.

A game characterisation of $\mathsf{IFP} + \mathsf{rk}^*$?

[Grädel, Pakusa]: *The uniform operator* rk^* *is the "right" rank operator.*

Problem: How to show limitations of $\mathsf{IFP} + \mathsf{rk}^*$?

There is a *game characterisation* for $\mathsf{IFP} + \mathsf{rk}^*$ [Dawar, Holm, 2012], but the "natural" game characterises a much richer logic.

Towards infinitary linear-algebraic logic

- **Rank** is just *one* example of an isomorphism-invariant property of matrices.
- What if we add an operator for *every* isomorphism-invariant matrix property?
- There can be operators that are not in PTIME or not even computable, but they are still limited by isomorphism-invariance.
- Equivalence in **infinitary FO** with **all linear-algebraic operators** turns out to have a useful game characterisation.

Isomorphism-invariant linear-algebraic operators

- An m-ary linear-algebraic operator is an $\mathbb{N}$-valued function $f(M_1, \dots, M_m)$.
- f is isomorphism-invariant if $f(M_1, \dots, M_m) = f(N_1, \dots, N_m)$ whenever "$(M_1, \dots, M_m) \cong (N_1, \dots, N_m)$".
- The matrices $(M_1, \dots, M_m)$ are viewed as linear transformations of a vector space $\mathbb{F}^A$, and $(N_1, \dots, N_m)$ are linear transformations of $\mathbb{F}^B$.
- "$(M_1, \dots, M_m) \cong (N_1, \dots, N_m)$" if there is a vector space isomorphism $S \colon \mathbb{F}^A \to \mathbb{F}^B$ that maps $(M_1, \dots, M_m)$ to $(N_1, \dots, N_m)$.

Definition (Simultaneous similarity)

We write $(M_1, \dots, M_m) \cong (N_1, \dots, N_m)$ if the tuples of matrices are *simultaneously similar*, which means: There exists an *invertible matrix* S such that $N_i \cdot S = S \cdot M_i$ for all $i \in [m]$.

Let f be an m-ary isomorphism-invariant linear-algebraic operator over a finite field $\mathbb{F}$, and $t \in \mathbb{N}$. Let $\varphi_1, \dots, \varphi_m$ be formulas. Then

$$\mathcal{Q}_f^t(\varphi_1(\bar{x}, \bar{y}), \dots, \varphi_m(\bar{x}, \bar{y}))$$

is a formula that is true in a structure $\mathfrak{A}^*$ if

$$f(M(\varphi_1, \mathfrak{A}^*), \dots, M(\varphi_m, \mathfrak{A}^*)) \geq t.$$

Definition (LA [Dawar, Grädel, Pakusa, 2019])

The logic LA is the closure of infinitary FO under quantifiers $\mathcal{Q}_f^t$ for all isomorphism-invariant linear-algebraic operators f, and all $t \in \mathbb{N}$.

For $k \in \mathbb{N}$, LA^k is the k-variable fragment.

Invertible-map equivalence

For $k \in \mathbb{N}$, Q a set of prime numbers, we write

$$\mathfrak{A} \equiv^{\mathsf{IM}}_{k,Q} \mathfrak{B}$$

if $\mathfrak{A}$ and $\mathfrak{B}$ agree on all sentences of $\mathsf{LA}^k(Q) \subseteq \mathsf{LA}^k$, the fragment containing only algebraic operators over fields $\mathbb{F}_p$ with $p \in Q$.

If $\mathfrak{A} \equiv^{\mathsf{IM}}_{k,\mathbb{P}} \mathfrak{B}$, then also no sentence in $\mathsf{IFP} + \mathsf{rk}^*$ distinguishes $\mathfrak{A}$ and $\mathfrak{B}$.

$\mathfrak{A} \equiv^{\mathrm{IM}}_{k,Q} \mathfrak{B}$ if and only if Duplicator has a winning strategy in the **invertible-map game**:

Definition (Dawar, Holm, 2012)

Let $\mathfrak{A}, \mathfrak{B}$ two structures, $k \in \mathbb{N}$ the number of pebbles.
The position after any round is $(\bar{a} \in A^\ell, \bar{b} \in B^\ell)$ with $\ell \leq k$. In each round,

- Spoiler announces a prime $p \in Q$ and picks up some number $2m \leq k$ of pebbles from each structure.
- Duplicator chooses all of the following:
  1. A partition $\boldsymbol{P}$ of $A^m \times A^m$ and a partition $\boldsymbol{Q}$ of $B^m \times B^m$ with the same number of parts.
  2. A bijection $\lambda \colon \boldsymbol{P} \to \boldsymbol{Q}$.
  3. An invertible matrix $S \in \mathbb{F}_p^{A^m \times B^m}$ such that for every $P \in \boldsymbol{P}$,
  $$\chi^P = S \cdot \chi^{\lambda(P)} \cdot S^{-1},$$
  where $\chi^P(\bar{u}, \bar{v}) = 1$ if $\bar{u}\bar{v} \in P$, and $\chi^P(\bar{u}, \bar{v}) = 0$, otherwise.
- Spoiler chooses $P \in \boldsymbol{P}$, and places the pebbles on a tuple $\bar{w} \in P$, and a tuple $\bar{w}' \in \lambda(P)$.

Efficient decidability of the IM-equivalences

For every $k \in \mathbb{N}$, Q a *finite* set of primes, the following problem is in PTIME.

IM-equivalence

Input: Two structures $\mathfrak{A}, \mathfrak{B}$.
Question: Is $\mathfrak{A} \equiv^{\mathsf{IM}}_{k,Q} \mathfrak{B}$?

The algorithm is a refinement of the *k-dimensional Weisfeiler Leman* graph isomorphism test.
It computes a **colouring of the k-tuples** according to their $\mathsf{LA}^k(Q)$**-type**.

A first limitation of invertible-map equivalences

Fact:
For every prime p, $\equiv^{\mathsf{IM}}_{k,p}$ is an *approximation to graph isomorphism* that is strictly *finer than* $\equiv_{\mathcal{C}^k}$.

Theorem (Dawar, Grädel, Pakusa, 2019)

If $Q \neq \mathbb{P}$, then there is no fixed $k \in \mathbb{N}$ such that $\equiv^{\mathsf{IM}}_{k,Q}$ is as fine as isomorphism on all structures.

Proof sketch. For a prime $p \notin Q$, non-isomorphic CFI graphs over $\mathbb{Z}_p$ are $\equiv^{\mathsf{IM}}_{k,Q}$-equivalent. This is shown with a sophisticated algebraic argument, but essentially the same "*coprimeness trick*" as in *Rank logic is dead, long live rank logic!*

Question: Is $\equiv^{\mathsf{IM}}_{k,\mathbb{P}}$ the same as isomorphism?

An inexpressibility result for IFP $+$ rk* and LA

Infinitary linear-algebraic logic [Approximations of isomorphism and logics with linear-algebraic operators: Dawar, Grädel, Pakusa (2019)]

$\geq$

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$\geq$ $\not\leq$

IFP $+$ rk [Logics with rank operators: Dawar, Grohe, Holm, Laubner (2009)]

$\geq$ $\not\leq$

FPC

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An inexpressibility result for $\mathsf{IFP} + \mathsf{rk}^*$ and LA

Theorem (Lichter, 2021)

$\mathsf{IFP} + \mathsf{rk}^*$ *does not capture* PTIME.

The **technical contribution** is this:

Theorem (Lichter, 2021)

For every fixed $k \in \mathbb{N}$, there are non-isomorphic CFI-structures over some ring $\mathbb{Z}_{2^{q(k)}}$ that are $\equiv^{\mathsf{IM}}_{k,2}$-equivalent.

The proof is a Duplicator winning strategy in the IM-game. Combining this with the already known "coprimeness argument" yields:

Theorem (Dawar, Grädel, Lichter, 2022)

There is no $k \in \mathbb{N}$ such that $\equiv^{\mathsf{IM}}_{k,\mathbb{P}}$ is isomorphism.

Winning the invertible-map game

Theorem (Lichter, 2021)

For every fixed $k \in \mathbb{N}$, there are non-isomorphic CFI-structures over some ring $\mathbb{Z}_{2^{q(k)}}$ that are $\equiv^{\mathsf{IM}}_{k,2}$-equivalent.

- The hard part is not the construction of the structures, but the construction of the invertible matrices in Duplicator's winning strategy.
- Recall that Spoiler moves $2m \leq k$ pebbles each round. The winning strategy is defined by induction on m, and the size of the ring $\mathbb{Z}_{2^{q(m)}}$ grows with m.
- In the case $m = 1$, CFI-structures over $\mathbb{Z}_4$ suffice.

Definition

Let $\mathfrak{A}, \mathfrak{B}$ two structures, $k \in \mathbb{N}$ the number of pebbles.
The position after any round is $(\bar{a} \in A^\ell, \bar{b} \in B^\ell)$ with $\ell \leq k$. In each round,

- Spoiler announces a prime $p \in Q$ and picks up some number $2m \leq k$ of pebbles from each structure.
- Duplicator chooses all of the following:
  1. A partition $\boldsymbol{P}$ of $A^m \times A^m$ and a partition $\boldsymbol{Q}$ of $B^m \times B^m$ with the same number of parts.
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$$\chi^P = S \cdot \chi^{\lambda(P)} \cdot S^{-1},$$

  where $\chi^P(\bar{u}, \bar{v}) = 1$ if $\bar{u}\bar{v} \in P$, and $\chi^P(\bar{u}, \bar{v}) = 0$, otherwise.
- Spoiler chooses $P \in \boldsymbol{P}$, and places the pebbles on a tuple $\bar{w} \in P$, and a tuple $\bar{w}' \in \lambda(P)$.
- Spoiler wins if the pebbles do not define a local isomorphism.

The picture now

$\textsc{Ptime}$

$\not\geq$

[Separating rank logic from polynomial time: Lichter (2021),
Limitations of the invertible-map equivalences: Dawar, Grädel, Lichter (2022)]

Infinitary linear-algebraic logic

$\geq$

IFP + rk* (“*rank logic*”) $\lneq$ Group order logic [Group Order Logic: Dahan (2025)]

$\gneq$

IFP + rk

$\gneq$

FPC

contained in PTIME

Group order logic

Group order logic is IFP extended by a **group order operator**.

	Rank logic	**Group order logic**
Definable object	Matrix $A \in \mathbb{F}_p^{I \times J}$ (i.e. generating set of a vector space)	Generating set of a permutation group $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$
Isomorphism-invariant property	$\mathbf{rk}(A)$	$\lvert \Gamma \rvert$

Theorem (Dahan, 2025)

- *Group order subsumes rank: $\mathrm{rk}(A)$ is definable from the size of the column space of A.*
- *Group order is more powerful than rank: It captures PTIME on CFI graphs, even over rings.*

Applications and open questions

Applications in CSP and Graph Isomorphism

Graph isomorphism: For every $k \in \mathbb{N}$, every finite set of primes Q, the algorithm deciding $\equiv^{\mathrm{IM}}_{k,Q}$ is a *polynomial time* graph isomorphism *heuristic* stronger than k-dimensional Weisfeiler-Leman.

Definition (Temporal CSP)

A *temporal* CSP has an infinite template structure that is FO-definable in $(\mathbb{Q}, <)$.

[Bodirsky, Rydval, Pakusa, 2020] fully classified the descriptive complexity of temporal CSPs:

$$\text{Datalog} \lneq \text{IFP} = \text{FPC} \lneq \text{IFP} + \text{rk}_2 \,.$$

In particular, there is a natural $\text{IFP} + \text{rk}_2$-algorithm that solves all these temporal CSPs up to Datalog-reductions.

Why not rank logic over the ring $\mathbb{Z}$?

- Rank logic apparently *cannot solve* equation system over *rings* (Lichter).
- Why not define a notion of "*rank*" over rings, or most generally, over $\mathbb{Z}$?
- Possible idea: Use the **Smith normal form** of integer matrices as "rank".
- Such a $\mathbb{Z}$-rank logic should be able to distinguish all CFI graphs over rings.
- Many interesting **CSP** and **graph isomorphism algorithms** solve systems of linear equations over $\mathbb{Z}$.

$\mathbb{Z}$-affine CSP algorithms

- Let $\mathfrak{A}$ be an instance of CSP($\mathfrak{B}$), and $k \in \mathbb{N}$. The *width-k relaxation* of "$\mathfrak{A} \to \mathfrak{B}$?" is a **system of linear equations** $\mathrm{L}^{k,\mathfrak{B}}_{\mathrm{CSP}}(\mathfrak{A})$ which asks to assign weights to solutions of subinstances of size k.
- $\mathrm{L}^{k,\mathfrak{B}}_{\mathrm{CSP}}(\mathfrak{A})$ has a $\{0, 1\}$-solution if and only if $\mathfrak{A} \to \mathfrak{B}$.
- Solving $\mathrm{L}^{k,\mathfrak{B}}_{\mathrm{CSP}}(\mathfrak{A})$ over $\mathbb{Z}$ is in polynomial time, and used in many **CSP heuristics** like BLP+AIP, BA^k, $\mathbb{Z}$-affine k-consistency, cohomological k-consistency.
- $\mathrm{L}^{k,\mathfrak{B}}_{\mathrm{CSP}}(\mathfrak{A})$ is *FO-definable* in $\mathfrak{A}$, so most of these algorithms could be expressed in a *rank logic over* $\mathbb{Z}$.

Lower bounds for $\mathbb{Z}$-affine algorithms using CFI structures

Theorem (Lichter, P., 2025)

There is a polynomial-time solvable CSP which is not solved by the $\mathbb{Z}$-affine algorithms, except by cohomological k-consistency.

The example is a **combination of CFI structures** over $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

Cohomological k-consistency

Cohomological k-consistency [O'Conghaile 2022]

Input: Instance $\mathfrak{A}$.
Let $\mathcal{H}_0(X) := \text{Hom}(\mathfrak{A}[X], \mathfrak{B})$ for every $X \in \binom{A}{\leq k}$.
repeat
  Let $\mathcal{H}'_i(X) \subseteq \mathcal{H}_i(X)$ be the partial homomorphisms
    that are not removed by the k-consistency procedure.
  Let $\mathcal{H}_{i+1}(X) \subseteq \mathcal{H}'_i(X)$ be the partial homomorphisms $f : X \to B$
    such that $\mathsf{L}^{k,\mathfrak{B}}_{\mathsf{CSP}}(\mathfrak{A})$, *augmented with the equation* $x_{X,f} = 1$, has a $\mathbb{Z}$-solution.
until $\mathcal{H}_{i+1} = \mathcal{H}_i$
If $\mathcal{H}_i(X) = \emptyset$ for some $X \in \binom{A}{\leq k}$, then return $\mathfrak{A} \not\to \mathfrak{B}$.

Open problems

- Can IFP $+$ rk_2 solve systems of linear equations over $\mathbb{Z}_4$? This is not ruled out by Lichter's result.
- Define a **rank logic over** $\mathbb{Z}$ and a useful game for it.
- *Inexpressibility* results for **group order logic**?
- Find a tractable CSP that is not solved by **cohomological** k**-consistency**.
- Can rank logic/group order logic simulate any group-theoretic graph isomorphism algorithm, for example for the class of bounded-degree graphs?

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