Minimum k-Connectivity Maintenance for Robust Multi-Robot Systems

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***Abstract*— In many multi-robot applications it is critical to maintain connectivity within robotic team to allow for information exchange and coordination. In presence of possible failure of robots, the maintained multi-robot network should be able to stay connected even with a loss of certain robot members to ensure robustness and resilience of the multi-robot systems. In this paper, we consider the problem of robust connectivity maintenance that seeks to maintain $k-$*connectivity* such that the multi-robot network could stay connected with removal of fewer than k robots. We propose a provably minimum $k-$connectivity maintenance algorithm for multi-robot systems. This ensures the robustness of connectivity network at all time and also in a flexible and optimal way to provide highest freedom for robots task-related controllers. Particularly, we propose a $k-$Connected Minimum Constraints Subgraph algorithm that activates the minimum $k-$connectivity constraints to the original controllers, and then revise the original controllers in a *minimally invasive fashion*. We demonstrate the effectiveness of our approach via simulations of up to 40 robots in presence of multiple behaviors.**

I. Introduction and Related Work

Multi-robot systems have been widely studied for extending its capability of doing complex tasks through cooperative behaviors in a number of applications. The ability of collaboration in multi-robot systems often relies on the local information sharing and interaction among networked robot members through connected communication graph. It is necessary to consider connectivity maintenance that ensures robots stay connected by constraining inter-robot distance while executing original tasks. On the other hand, it is critical to consider the robustness of the multi-robot network as the expected number of robot failures could grow along with the increasing number of robots.

The problem of connectivity maintenance is particularly challenging for existing work since (a) the additional connectivity maintenance brings increased complexity for global connectivity control algorithms [1]–[3] due to the discontinuity from dynamic topology changes as pointed out in [4], and (b) there are no theoretical guarantee on the optimality of imposed connectivity constraints, e.g. [5]–[8] nor the perturbation to the original behavior-prescribed controllers due to the constraints, e.g. [1], [9]–[11]. Such issues could lead to overly conservative robots motion and thus behavior failure, for example, dead locks that might prevent the desired execution of behaviors, and inefficiency incurred by the perturbation of connectivity on control outputs between different behaviour groups. Hence, it is desired to derive an approach to maintain minimum satisfying connectivity so as to provide highest freedom for robots' original controllers while ensuring robustness of the multi-robot network.

The objective of this paper is thus to develop provably optimal algorithms for robust but flexible multi-robot connectivity control, by proposing minimum $k-$connectivity maintenance methods that achieve global redundant network connectivity while enabling the robot team to perform various behaviors *at best*. Note that we are not optimizing multi-robot task allocation that determines how to assign different behavior controllers to the robots. We assume the behavior allocation has been done and each robot already knows its real-time behavior-prescribed controller before revising it to accommodate the connectivity and collision avoidance constraints. We propose to achieve provably optimal robust connectivity maintenance by developing: 1) a novel quantifiable relationship between original behavior-prescribed controllers and the candidate connectivity constraints, 2) a novel *$k-$Connected Minimum Constraints Subgraph ($k-$CMCS)* method to activate dynamic quantified minimum connectivity constraints, which are *least violated* by the original unrevised behavior-prescribed controllers, and 3) a unified optimization framework to revise the robots controllers in presence of activated connectivity and collision avoidance constraints that are minimally invasive to the original behavior-prescribed controllers. This enables the multi-robot systems to execute different behaviors simultaneously on a single connected robot team with required robust connectivity.

II. Problem Formulation

Consider a heterogeneous robotic team $\mathcal{S}$ consisting of N mobile robots in a planar space, with the position and single integrator dynamics of each robot $i \in \{1, \ldots, N\}$ denoted by $x_i \in \mathbb{R}^2$ and $\dot{x}_i = u_i \in \mathbb{R}^2$ respectively. Each robot can connect and communicate directly with other robots within its spatial proximity. The communication graph of the robotic team is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each node $v \in \mathcal{V}$ represents a robot. If the spatial distance between robot $v_i \in \mathcal{V}$ and robot $v_j \in \mathcal{V}$ is less or equal to the communication radius R_c (i.e. $\|x_i - x_j\| \leq R_c$), then we assume the two can communicate and edge $(v_i, v_j) \in \mathcal{E}$ is undirected (i.e. $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$).

We assume each robot i has been assigned to a subgroup with some behavior-prescribed controller $u_i = \hat{u}_i$. To ensure successful multi-robot coordination and information exchange, it is required that the communication/connectivity graph $\mathcal{G}$ is connected. Moreover, in presence of possible robots failure, the graph should be *robust* in that the removal

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of certain number of robot nodes won't disconnect the connectivity graph for the remaining robot team, which leads to the following definition of *k-node connected graph* [12].

Definition 1. *(k-node connected graph) A connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be k-node connected (or k-connected) if it has more than k nodes and remains connected whenever fewer than k nodes are removed.*

Given a desired number of k due to robustness requirements on robots failure, assuming the current multi-robot graph $\mathcal{G}$ is already $k-$connected (for the rest of the paper, $k-$connected refer to $k-$node connected), we would like to enforce such constraint as robots execute their behavior-prescribed controllers such that the time-varying connectivity graph $\mathcal{G}(t)$ is $k-$connected at all time. The inter-robot collisions and velocity constraint should also be considered.

A. Safety and Connectivity Constraints using Barrier Certificates

During movements of multi-robot systems, the robots should avoid collisions with each other to remain safe. Consider the joint robot states $\mathbf{x} = \{x_1, \ldots, x_N\} \in \mathbb{R}^{2N}$ and define the minimum inter-robot safe distance as R_s, for any pair-wise inter-robot collision avoidance constraint between robots i and j. We have the following condition defining the safe set of $\mathbf{x}$.

$$\begin{aligned} h^s_{i,j}(\mathbf{x}) &= \|x_i - x_j\|^2 - R_s^2, \qquad \forall i > j \\ \mathcal{H}^s_{i,j} &= \{\mathbf{x} \in \mathbb{R}^{2N} : h^s_{i,j}(\mathbf{x}) \geq 0\} \end{aligned} \tag{1}$$

The set of $\mathcal{H}^s_{i,j}$ indicates the safety set from which robot i and j will never collide. For the entire robotic team, the safety set can be composed as follows.

$$\mathcal{H}^s = \bigcap_{\{v_i, v_j \in \mathcal{V} : i > j\}} \mathcal{H}^s_{i,j} \tag{2}$$

[13] proposed the safety barrier certificates $\mathcal{B}^s(\mathbf{x})$ that map the constrained safety set (2) of $\mathbf{x}$ to the admissible joint control space $\mathbf{u} \in \mathbb{R}^{2N}$. The result is summarized as follows.

$$\mathcal{B}^s(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^{2N} : \dot{h}^s_{i,j}(\mathbf{x}) + \gamma h^s_{i,j}(\mathbf{x}) \geq 0, \forall i > j\} \tag{3}$$

where γ is a user-defined parameter to confine the available sets. It is proven in [13] that the forward invariance of the safety set $\mathcal{H}^s$ is ensured as long as the joint control input $\mathbf{u}$ stays in set $\mathcal{B}^s(\mathbf{x})$. In other words, the robots will always stay safe if they are initially inter-robot collision free and the control input lies in the set $\mathcal{B}^s(\mathbf{x})$. Note that at any time point t with known current robot states $\mathbf{x}(t)$, the constrained control space in (3) corresponds to a class of linear constraints over pair-wise control inputs u_i and u_j for $\forall i > j$. Note that static obstacles may also be modelled in the same manner if treated as robots with zero velocity.

Next, we consider the pair-wise connectivity constraints among the robotic team. If the connectivity constraint is enforced between pair-wise robots i and j to ensure inter-robot distance not larger than communication range R_c, we have the following condition.

$$\begin{aligned} h^c_{i,j}(\mathbf{x}) &= R_c^2 - \|x_i - x_j\|^2 \\ \mathcal{H}^c_{i,j} &= \{\mathbf{x} \in \mathbb{R}^{2N} : h^c_{i,j}(\mathbf{x}) \geq 0\} \end{aligned} \tag{4}$$

The set of $\mathcal{H}^c_{i,j}$ indicates the feasible set on $\mathbf{x}$ from which robot i and j will never lose connectivity. Consider any connectivity graph $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c) \subset \mathcal{G}$ to enforce, the corresponding constrained set can be composed as follows.

$$\mathcal{H}^c(\mathcal{G}^c) = \bigcap_{\{v_i, v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}^c\}} \mathcal{H}^c_{i,j} \tag{5}$$

Similar to the safety barrier certificates in (3), the connectivity barrier certificates are defined as follows and indicate another class of linear constraints over pair-wise control inputs u_i and u_j for $(v_i, v_j) \in \mathcal{E}^c$ at any time point t.

$$\mathcal{B}^c(\mathbf{x}, \mathcal{G}^c) = \{\mathbf{u} \in \mathbb{R}^{2N} : \dot{h}^c_{i,j}(\mathbf{x}) + \gamma h^c_{i,j}(\mathbf{x}) \geq 0, \forall (v_i, v_j) \in \mathcal{E}^c\} \tag{6}$$

B. Objective Function

Consider that a task-related primary behavior control input $\hat{u}_i$ has been computed for each robot i before considering the mentioned constraints. The robotic team needs to determine whether and how to best modify its primary control input in a minimally invasive manner so as to achieve task-related behaviors while ensuring safety and $k-$connectivity. With the defined forms of constraints in (3) and (6), we formally define the *minimum $k-$connectivity maintenance* problem as follows given k.

$$\mathbf{u}^* = \underset{\mathcal{G}^c, \mathbf{u}}{\arg\min} \sum_{i=1}^{N} \|u_i - \hat{u}_i\|^2 \tag{7}$$

$$\text{s.t.} \quad \mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c) \subseteq \mathcal{G} \quad \text{is } k-\text{connected} \tag{8}$$

$$\mathbf{u} \in \mathcal{B}^s(\mathbf{x}) \bigcap \mathcal{B}^c(\mathbf{x}, \mathcal{G}^c), \quad \|u_i\| \leq \alpha_i, \forall i = 1, \ldots, N \tag{9}$$

The above Quadratic Programming (QP) optimization problem is to find the optimal active connectivity spanning subgraph $\mathcal{G}^c$ from current connected multi-robot connectivity graph $\mathcal{G}$ and the alternative control inputs $\mathbf{u}^* \in \mathbb{R}^{2N}$ bounded by maximum velocity α_i for each robot, so that $k-$connectivity, safety and velocity constraints described in (8) and (9) are satisfied while ensuring minimally invasive to the primary controller as shown in (7).

III. Maintaining Minimum $k-$Connectivity

First we consider the sub-problem of selecting optimal $k-$connectivity spanning subgraph $\mathcal{G}^{c*} = \mathcal{G}^*_k(\mathcal{V}, \mathcal{E}^*_k) \subseteq \mathcal{G}$ in (7) that introduces minimum $k-$connectivity constraints. As each edge $(v_i, v_j) \in \mathcal{E}^c$ in a candidate graph $\mathcal{G}^c$ enforces one pair-wise linear constraint over primary control inputs $\hat{u}_i$ and $\hat{u}_j$ for robot i and j as shown in (4), the graph $\mathcal{G}^c$ whose edges define the minimum connectivity constraints must exist among the set of all minimum $k-$connected spanning subgraph from current connectivity graph $\mathcal{G}$ that cover all the vertices $\mathcal{V}$ with minimum number of $k-$node connected edges.

Recall that resultant connectivity constraints due to enforced edges are in the form of (6) over the robots' controllers. Thus, to quantify the strength of connectivity constraint by an edge $(v_i, v_j) \in \mathcal{E}$, we introduce the weight assignment defined as follows.

$$w_{i,j} = \dot{h}^c_{i,j}(\mathbf{x}, \hat{u}_i, \hat{u}_j) + \gamma h^c_{i,j}(\mathbf{x}), \forall (v_i, v_j) \in \mathcal{E} \tag{10}$$

Compared to the connectivity constraint in (6), $w_{i,j}$ indicates the violation of the pair-wise connectivity constraint between the two robots, with the higher value of $w_{i,j}$ the less likely the connectivity constraint being violated. To that end, the present connectivity graph $\mathcal{G}$ can be converted to a weighted connectivity graph $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with $w_{i,j} \in \mathcal{W}$. Given such undirected connected graph $\hat{\mathcal{G}}$ and k, we propose the following summarized algorithm that outputs a weighted min-size $k-$connected spanning subgraph $\hat{\mathcal{G}}_k^*$, which is formally defined as *$k-$Connected Minimum Constraints Subgraph ($k-$CMCS).*

Algorithm 1 $k-$Connected Minimum Constraints Subgraph ($k-$CMCS)

Input: $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, k
Output: $\hat{\mathcal{G}}_k^*$
1: find a min-size $k-1$ edge cover $M \leftarrow \arg\min\{\beta|M| - \Sigma_{(v_i,v_j)\in M}\{w_{i,j}\} : \deg_M(v) \geq k-1, \forall v \in \mathcal{V}, M \subseteq \mathcal{E}\}$ with $\beta >> 2 \cdot \Sigma_{\forall w_{i,j}\in\mathcal{W}}|w_{i,j}|$ as any staisfying user-defined parameter.
2: find an inclusionwise minimal edge set $F \subseteq \mathcal{E} \setminus M$ such that $(\mathcal{V}, M \cup F)$ is $k-$connected
3: **return** $\hat{\mathcal{G}}_k^* \leftarrow (\mathcal{V}, M \cup F)$

With the Algorithm 1, we have the following theorem regarding its known approximation of the derived $k-$connected spanning subgraph $\mathcal{G}_k^*$.

Theorem 2. *Given weighted undirected graph $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ of node connectivity $\geq k$. Then the Algorithm 1 finds the $k-$CMCS $\hat{\mathcal{G}}_k^* = (\mathcal{V}, \mathcal{E}'_k, \mathcal{W}_k)$ such that $|\mathcal{E}'_k| \leq (1+\frac{1}{k})|\mathcal{E}_{opt}|$, where $\mathcal{E}_{opt}$ denotes the cardinality of the optimal solution.*

Due to limited space we skip the proof here but would refer to the work [14], [15] for concluding the proof. Hence Algorithm 1 provides a bounded solution to find a $k-$CMCS $\hat{\mathcal{G}}_k^* \subseteq \mathcal{G}$ with minimum number of edges that are least restrictive and can be used to define active pairwise connectivity constraints for ensuring $k-$connectivity.

With the final $k-$CMCS $\hat{\mathcal{G}}_k^*$ obtained from our Algorithm 1 as the optimal $k-$connectivity subgraph $\mathcal{G}^{c*} = \hat{\mathcal{G}}_k^*$ in (9), we can specify the safety and connectivity barrier certificates (3) and (6) to invoke linear constraints and efficiently solve the original quadratic programming (QP) problem in (7) to get optimal revised robot controllers satisfying safety and $k-$connectivity constraints with minimum invasion to the original controllers.

IV. Results

To evaluate our proposed k-CMCS method for robust connectivity maintenance, we designed experiments in simulation: $n = 40$ robots divided into $m = 3$ subgroups simultaneously performing 3 behaviors such as rendezvous to goal and circle formation behaviors. As shown in Figure 1 for $2-$connectivity requirement ($k = 2$) with the enforced connectivity constraints (preserved red edges), taking out no more than $k-1 = 1$ robots will not disconnect the rest multi-robot connectivity graph. In Figure 1a-c, our k-CMCS approach is able to generate minimum connectivity graph (red edges) from the present connectivity graph (grey edges) so that the invoked connectivity constraints are minimally invasive to the primary behavior controllers.

In comparison, we present converged results of other three methods shown in Figure 1d-f: which are i) always preserving initial connected edges (grey edges in Figure 1a) with converged result depicted in Figure 1d, ii) preserving edges in present k-Node Connected Spanning Subgraph (k-NCSS) [14], [15] that seeks to select minimum number of edges without consideration of robot motions (result depicted in Figure 1e), and iii) always preserving edges in initial k-CMCS (red edges in Figure 1a) without updating (result shown in Figure 1f). For results in Figure 1d and f, due to the rigid invoked connectivity graph as the robots move, they can hardly achieve circle formation and could fall into deadlock before reaching the target regions. Without considering the robots' original controllers, k-NCSS method in Figure 1e imposes overly constrained edges even if the number of them are minimum. In contrast, our k-CMCS method selects minimum number of edges and at the same time ensures they are most in favor of the robots original controllers, thus leading to more flexible motions. Numerical results are also provided in Figure 2 showing our method ensures safety and robust $k-$connectivity, while having minimal control perturbation due to connectivity as compared to other mentioned methods above.

V. Conclusion

In this paper, we considered the problem of minimum $k-$connectivity maintenance for flexible multi-robot behaviors. In particular, we proposed a $k-$Connected Minimum Constraints Subgraph ($k-$CMCS) algorithm to compute provably minimum $k-$connectivity constraints as to the robots behavior-prescribed controllers. In this way, the robots controllers will only be revised as necessary in a minimally invasive manner with dynamic and possibly discontinuous communication topology. This algorithm enables simultaneous behaviors at best while maintaining constraints due to collision avoidance and required redundant connectivity that is robust to robots failure.

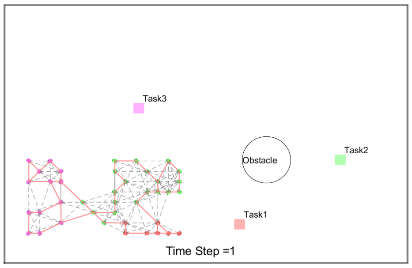
References

[1] L. Sabattini, N. Chopra, and C. Secchi, "Decentralized connectivity maintenance for cooperative control of mobile robotic systems," *The International Journal of Robotics Research*, vol. 32, no. 12, pp. 1411–1423, 2013.

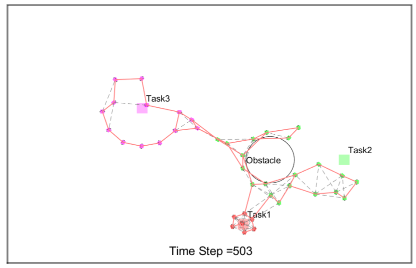
[2] L. Sabattini, C. Secchi, N. Chopra, and A. Gasparri, "Distributed control of multirobot systems with global connectivity maintenance," *IEEE Transactions on Robotics*, vol. 29, no. 5, pp. 1326–1332, 2013.

[3] P. Yang, R. A. Freeman, G. J. Gordon, K. M. Lynch, S. S. Srinivasa, and R. Sukthankar, "Decentralized estimation and control of graph connectivity for mobile sensor networks," *Automatica*, vol. 46, no. 2, pp. 390–396, 2010.

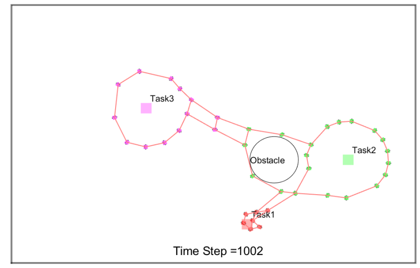
[4] R. K. Williams, A. Gasparri, G. S. Sukhatme, and G. Ulivi, "Global connectivity control for spatially interacting multi-robot systems with unicycle kinematics," in *Robotics and Automation (ICRA), 2015 IEEE International Conference on*. IEEE, 2015, pp. 1255–1261.



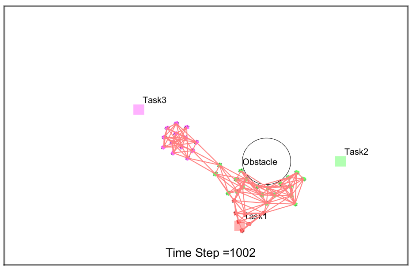
(a) Time Step = 1



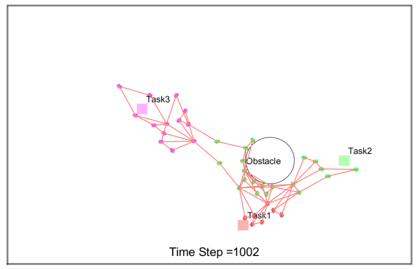
(b) Time Step = 503 (k-CMCS)



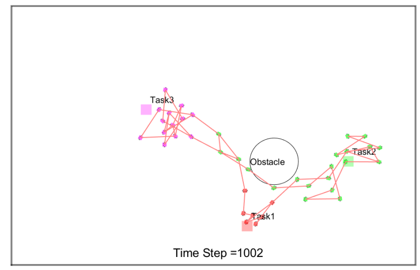
(c) Time Step = 1002 (k-CMCS, Converged)



(d) Time Step = 1002 (Fixed Initial Connectivity, Converged)

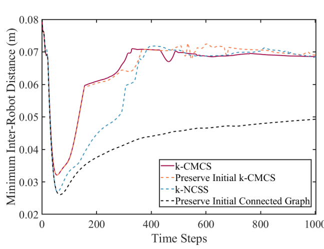


(e) Time Step = 1002 (k-NCSS, Converged)

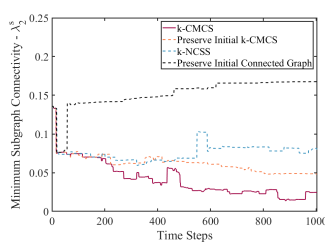


(f) Time Step = 1002 (Fixed Initial k-CMCS, Converged)

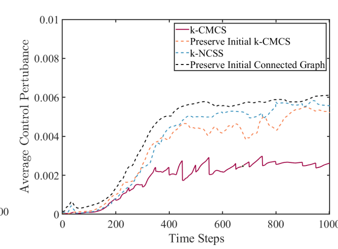
Fig. 1: Simulation example of 40 robots tasked to three different places simultaneously with 2-connectivity maintenance ($k = 2$): red robots rendezvous to red task 1 region, while green robots and magenta robots move to region of green task 2 and magenta task 3 and keep orbiting around the regions. Grey dashed lines in (a),(b) denote current connectivity edges and red lines in (a)-(f) denote current active k-connectivity graph invoking pair-wise connectivity constraints. Compared to inter-robot connectivity constraints from (d) initial connectivity graph, (e) minimum k-Node Connected Spanning Subgraph (k-NCSS) [14], [15], and (f) fixed initial k-CMCS (only computes k-CMCS once), converged result (c) of our proposed k-CMCS approach shows best task performance due to invoked minimum connectivity constraints on the robots.



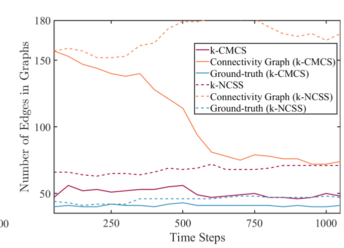
(a) Minimum inter-robot distance



(b) Minimum Algebraic Connectivity of Subgraph



(c) Average control perturbation



(d) Number of Edges in Graphs

Fig. 2: Performance comparison of simulation example in Figure 1 w.r.t. different metrics: (a) Minimum inter-robot distance (safety distance is 0.025m), (b) Minimum subgraph algebraic connectivity evaluated by second smallest eigenvalue of laplacian matrix with $k-1 = 1$ robot being taken out. Positive meaning connectivity ensured. (c) Control perturbation computed by $\frac{1}{N}\sum_{i=1}^{N}\left\|u_i^* - \hat{u}_i\right\|^2$, (d) Number of edges in the corresponding graphs. Note our k-CMCS approach activates less number of k-connectivity edges than k-NCSS [14], [15], and always stay within the ratio of $1 + \frac{1}{k}$ to the groud-truth minimum k-connected subgraph (Benchmarked by brute-force algorithm that exhaustively checks combinations of all edges).

[5] A. Li, L. Wang, P. Pierpaoli, and M. Egerstedt, "Formally correct composition of coordinated behaviors using control barrier certificates," in *Intelligent Robots and Systems (IROS), IEEE/RSJ International Conference on*. IEEE, 2018.

[6] L. Wang, A. D. Ames, and M. Egerstedt, "Multi-objective compositions for collision-free connectivity maintenance in teams of mobile robots," in *Decision and Control (CDC), 2016 IEEE 55th Conference on*. IEEE, 2016, pp. 2659–2664.

[7] ——, "Safe certificate-based maneuvers for teams of quadrotors using differential flatness," *arXiv preprint arXiv:1702.01075*, 2017.

[8] M. M. Zavlanos, M. B. Egerstedt, and G. J. Pappas, "Graph-theoretic connectivity control of mobile robot networks," *Proceedings of the IEEE*, vol. 99, no. 9, pp. 1525–1540, 2011.

[9] P. R. Giordano, A. Franchi, C. Secchi, and H. H. Bülthoff, "Bilateral teleoperation of groups of uavs with decentralized connectivity maintenance." in *Robotics: Science and Systems*. Citeseer, 2011.

[10] J. Banfi, N. Basilico, and S. Carpin, "Optimal redeployment of multirobot teams for communication maintenance," in *2018 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2018, pp. 3757–3764.

[11] N. Majcherczyk, A. Jayabalan, G. Beltrame, and C. Pinciroli, "Decentralized connectivity-preserving deployment of large-scale robot swarms," *arXiv preprint arXiv:1806.00150*, 2018.

[12] R. Diestel, "Graph theory, volume 173 of," *Graduate texts in mathematics*, p. 7, 2012.

[13] U. Borrmann, L. Wang, A. D. Ames, and M. B. Egerstedt, "Control barrier certificates for safe swarm behavior." Georgia Institute of Technology, 2015.

[14] Z. Nutov, "The k-connected subgraph problem," 2018.

[15] J. Cheriyan and R. Thurimella, "Approximating minimum-size k-connected spanning subgraphs via matching," *SIAM Journal on Computing*, vol. 30, no. 2, pp. 528–560, 2000.