Names and Symmetry in Computer Science

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LICS 2015 Tutorial
An introduction to nominal techniques motivated by Programming language semantics

Automata theory

Constructive type theory
Compositionality in semantics

The meaning of a compound phrase should be a well-defined function of the meanings of its constituent subphrases.

Although one can define behaviour of whole C/Java/OCaml/Haskell... programs (e.g. via an abstract machine) it’s often more illuminating, but harder, to give a semantics for individual C/Java/OCaml/Haskell... statement-forming constructs as operations for composing suitable mathematical structures.
The meaning of a compound phrase should be a well-defined function of the meanings of its constituent subphrases.

**Denotational semantics** are compositional:

- Each program phrase $P$ is given a denotation $\llbracket P \rrbracket$—a mathematical object representing the contribution of $P$ to the meaning of any complete program in which it occurs.
- The denotation of a phrase is a function of the denotations of its subphrases.
Question: are these OCaml expressions behaviourally equivalent?

\[
\begin{align*}
\text{let } a & = \text{ref } 42 \text{ in} \\
\text{fun } x \to a := (!a + x); \\
& \quad !a \\
\text{let } b & = \text{ref } (-42) \text{ in} \\
\text{fun } y \to b := (!b - y); \\
& \quad -(!b)
\end{align*}
\]
**Question:** are these OCaml expressions behaviourally equivalent?

| let \( a = \text{ref} \, 42 \) in 
| --- |
| \( \text{fun} \, x \rightarrow a := (!a + x) \); 
| \( !a \) |
| let \( b = \text{ref}(-42) \) in 
| --- |
| \( \text{fun} \, y \rightarrow b := (!b - y) \); 
| \( -(!b) \) |

**Answer:** yes, but devising denotational semantics that exactly match the execution behaviour of

higher-order functions + locally scoped names (of storage locations, of exceptions, of . . . )

has proved hard.

Current state-of-the-art: nominal game semantics (see forthcoming tutorial article by Murawski & Tzevelekos in *Foundations & Trends in Programming Languages*).
What does it mean for mathematical structures [needed for denotational semantics] to

\[
\begin{align*}
& \text{depend upon some names?} \\
& \text{be independent of some names?}
\end{align*}
\]

- Conventional answer: parameterization (explicit dependence via functions).
  Can lead to ‘weakening hell’.
Dependence & Symmetry

What does it mean for mathematical structures [needed for denotational semantics] to
depend upon some names? be independent of some names?

- Conventional answer: parameterization (explicit dependence via functions).
  Can lead to ‘weakening hell’.

- *Nominal techniques* answer: independence via invariance properties of symmetries.
Name permutations

- $A = \text{fixed countably infinite set of atomic names } (a, b, \ldots )$
- $S_A = \text{group of all (finite) permutations of } A$

Typical elements:

$$
\begin{array}{ccc}
    a & \circlearrowleft & b \\
    c & \circlearrowright & e \\
    d & \circlearrowright & \ldots \\
    f & \circlearrowleft & g \\
    h & \circlearrowleft & \\
\end{array}
$$

(but NOT $f, g$)
Name permutations

- $\mathcal{A} = \text{fixed countably infinite set of atomic names } (a,b,\ldots)$
- $S_\mathcal{A} = \text{group of all (finite) permutations of } \mathcal{A}$
  - each $\pi$ is a bijection $\mathcal{A} \cong \mathcal{A}$ (injective and surjective function)
  - $\pi$ finite means: $\{ a \in \mathcal{A} \mid \pi(a) \neq a \}$ is finite.
  - group: multiplication is composition of functions $\pi' \circ \pi$; identity is identity function $\text{id}$; inverses are inverse functions $\pi^{-1}$. 


A $S_A$-action on a set $X$ is a function

$$\pi \in S_A, \ x \in X \mapsto \pi \cdot x \in X$$

satisfying

- $\pi' \cdot (\pi \cdot x) = (\pi' \circ \pi) \cdot x$
- $\text{id} \cdot x = x$
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- $id \cdot x = x$

Simple example: $S_A$ acts on sets of names $A \subseteq A$ via

$$\pi \cdot A = \{\pi(a) \mid a \in A\}$$

E.g.

$$\left( a \xrightarrow{b} a \right) \cdot \{c \mid c \neq a\} = \{c \mid c \neq b\}$$
Support – the key definition

Suppose $S_A$ acts on a set $X$ and $x \in X$.

A set of names $A \subseteq A$ supports $x$ if for all $\pi \in S_A$

$$(\forall a \in A. \pi(a) = a) \Rightarrow \pi \cdot x = x$$

$X$ is a nominal set if every $x \in X$ has a finite support.
[AMP-Gabbay, LICS 1999]
Suppose $S_A$ acts on a set $X$ and $x \in X$.

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E.g. for $X = \{A \mid A \subseteq A\}$,

- $\{b \mid b \neq a\}$ is infinite, but is supported by $\{a\}$ because if $\pi(a) = a$, then $\pi$ is a permutation of $A - \{a\}$
Support – the key definition

Suppose $S_A$ acts on a set $X$ and $x \in X$.

A set of names $A \subseteq A$ supports $x$ if for all $\pi \in S_A$

$$\left( \forall a \in A. \pi(a) = a \right) \Rightarrow \pi \cdot x = x$$

$X$ is a nominal set if every $x \in X$ has a finite support.

[AMP-Gabbay, LICS 1999]

E.g. for $X = \{A \mid A \subseteq A\}$,

- $\{b \mid b \neq a\}$ is infinite, but is supported by $\{a\}$
- no finite set of names supports $\{a_0, a_2, a_4, \ldots\}$ (supposing $a_0, a_1, a_2, a_3, \ldots$ enumerates $A$), so $\{A \mid A \subseteq A\}$ is not a nominal set
- $\{A \subseteq A \mid A$ finite, or $A - A$ finite$\}$ is a nominal set
Support – the key definition

Suppose $S_A$ acts on a set $X$ and $x \in X$.

A set of names $A \subseteq A$ supports $x$ if for all $\pi \in S_A$

$$(\forall a \in A. \pi(a) = a) \Rightarrow \pi \cdot x = x$$

$X$ is a nominal set if every $x \in X$ has a finite support. [AMP-Gabbay, LICS 1999]

E.g. for $\lambda$-terms with (free & bound) variables from $A$

$$t ::= a \mid tt \mid \lambda a.t$$

modulo $\alpha$-equivalence

with $S_A$-action:

$$\begin{cases}
\pi \cdot a &= \pi(a) \\
\pi \cdot (tt') &= (\pi \cdot t)(\pi \cdot t') \\
\pi \cdot (\lambda a.t) &= \lambda \pi(a).(\pi \cdot t)
\end{cases}$$

FACT: each $t$ is supported by its finite set of free variables.
Category of nominal sets, $\textbf{Nom}$

**objects** are nominal sets

$= \text{sets equipped with an action of all (finite) permutations of } A$, all of whose elements have finite support

**morphisms** are *equivariant* functions

$= \text{functions preserving the permutation action.}$

**identities and composition**

$= \text{as usual for functions}$
Why use category theory?

- **equivalence of categories** from different mathematical realms (or even just functors between them) can tell us a lot. For example, the following are all equivalent:
  - Nom
  - the Schanuel topos (from Grothendieck’s generalized Galois theory)
  - the category of named sets (from the work of Montanari et al on model-checking $\pi$-calculus)

- **universal properties** (adjoint functors) can characterize a mathematical construction uniquely up to isomorphism and help predict its properties. For example...
Nominal exponentials

\[ X \to_{fs} Y \triangleq \{ f \in Y^X \mid f \text{ is finitely supported w.r.t. the action } \pi \cdot f = \lambda x \to \pi \cdot (f(\pi^{-1} \cdot x)) \} \]

This is characterised uniquely up to isomorphism by the fact that it give the right adoint to \((\_ \times X) : \mathbf{Nom} \to \mathbf{Nom}:

\[
\frac{Z \times X \to Y}{Z \to (X \to_{fs} Y)}
\]

(Products in \(\mathbf{Nom}\) are created by the forgetful functor to the category of sets.)
Nominal exponentials

\[ X \xrightarrow{\text{fs}} Y \triangleq \left\{ f \in Y^X \mid f \text{ is finitely supported w.r.t. the action } \pi \cdot f = \lambda x \rightarrow \pi \cdot (f(\pi^{-1} \cdot x)) \right\} \]

This is characterised uniquely up to isomorphism by the fact that it give the right adoint to \((\_ \times X) : \text{Nom} \rightarrow \text{Nom}:

\[
\begin{array}{c}
Z \times X \rightarrow Y \\
\hline
\hline
Z \rightarrow (X \xrightarrow{\text{fs}} Y)
\end{array}
\]

(Products in \text{Nom} are created by the forgetful functor to the category of sets.)

N.B. permutations have inverses
Name abstraction

Each $X \in \text{Nom}$ yields a nominal set $[A]X$ of name-abstractions $\langle a \rangle x$ are $\sim$-equivalence classes of pairs $(a, x) \in A \times X$, where

$$(a, x) \sim (a', x') \iff \exists b \# (a, x, a', x')$$

$$(b \ a) \cdot x = (b \ a') \cdot x'$$
Name abstraction

Each \( X \in \text{Nom} \) yields a nominal set \([A]X\) of name-abstractions \( \langle a \rangle x \) are \( \sim \)-equivalence classes of pairs \( (a, x) \in A \times X \), where

\[
(a, x) \sim (a', x') \iff \exists b \# (a, x, a', x') \quad (b \ a) \cdot x = (b \ a') \cdot x'
\]

**Freshness relation:** \( a \# x \) means \( a \not\in A \) for some finite support \( A \) for \( x \)

The freshness relation gives a well-behaved, syntax-independent notion of freeness, or non-occurrence.
Name-abstraction

Name abstraction $[\mathcal{A}](\_): \text{Nom} \to \text{Nom}$ is right adjoint to adjoint ‘separated tensor’ $\_ \ast \mathcal{A}$

$$X \ast \mathcal{A} \triangleq \{(x, a) \in X \times \mathcal{A} \mid a \not\approx x\}$$

$$X \ast \mathcal{A} \to Y$$

$$X \to [\mathcal{A}]Y$$

So $[\mathcal{A}]X$ is a kind of (affine) linear function space from $\mathcal{A}$ to $X$, but with great properties – e.g. it too has a right adjoint

$$([\mathcal{A}]X) \to Y$$

$$X \to \{f \in Y^\mathcal{A} \mid (\forall a \in \mathcal{A}) \ a \not\approx f(a)\}$$
Name-abstraction

Name abstraction $\lbrack A \rbrack(\_): \text{Nom} \to \text{Nom}$ is right adjoint to adjoint ‘separated tensor’ $\_ \ast A$

$$X \ast A \triangleq \{(x, a) \in X \times A \mid a \neq x\}$$

$$X \ast A \to Y$$

$$\overline{X \to [A]Y}$$

So $[A]X$ is a kind of (affine) linear function space from $A$ to $X$, but with great properties – e.g. it too has a right adjoint

$\leadsto$ an initial algebra semantics for syntax with binders using $[A](\_)$ to model binding operations with user-friendly inductive/recursive properties. . .
Initial algebras

- \( [\mathcal{A}] (\_ ) \) can be combined with \( \_ \times \_ \) and \( \_ + \_ \) to give functors \( T : \text{Nom} \to \text{Nom} \) that have initial algebras \( I : T D \to D \)

\[
\begin{array}{c}
T D \\
\downarrow I \\
D
\end{array}
\quad
\begin{array}{c}
T X \\
\downarrow F \\
X
\end{array}
\]

for all
Initial algebras

- $\text{[A]}(\_)$ can be combined with $\_ \times \_$ and $\_ \oplus \_$ to give functors $T : \text{Nom} \to \text{Nom}$ that have initial algebras $I : TD \to D$
Initial algebras

- $[\mathcal{A}](_\_)$ can be combined with $\_ \times \_ \text{ and } \_ + \_ \text{ to give functors } T : \text{Nom} \to \text{Nom} \text{ that have initial algebras } I : T D \to D$

- For a wide class of such functors (‘nominal algebraic functors’) the initial algebra $D$ coincides with sets of abstract syntax trees modulo $\alpha$-equivalence.

E.g. the initial algebra for

$$T(_\_) \triangleq \mathcal{A} + (_\_ \times _\_) + [\mathcal{A}](_\_)$$

is the usual set of untyped $\lambda$-terms and the initial-algebra universal property yields. . . .
Theorem.
Given any \( X \in \text{Nom} \) and
\[
\begin{align*}
f_1 & \in \Lambda \rightarrow_{fs} X \\
f_2 & \in X \times X \rightarrow_{fs} X \\
f_3 & \in [\Lambda]X \rightarrow_{fs} X
\end{align*}
\]
\[
\exists! \hat{f} \in \Lambda \rightarrow_{fs} X \\
\begin{aligned}
\hat{f} a &= f_1 a \\
\hat{f} (t_1 \ t_2) &= f_2 (\hat{f} t_1, \hat{f} t_2) \\
\hat{f} (\lambda a.t) &= f_3 (\langle a \rangle (\hat{f} t)) & \text{if } a \not\in (f_1, f_2, f_3)
\end{aligned}
\]

untyped \( \lambda \)-terms:
\[
t ::= a \mid tt \mid \lambda a.t
\]
α-Structural recursion

**Theorem.**

Given any $X \in \text{Nom}$ and

\[
\begin{align*}
  f_1 &\in \mathbb{A} \rightarrow_{fs} X \\
  f_2 &\in X \times X \rightarrow_{fs} X \\
  f_3 &\in \mathbb{A} \times X \rightarrow_{fs} X
\end{align*}
\]

s.t.

\[(\forall a) \ a \not\in (f_1, f_2, f_3) \Rightarrow (\forall x) \ a \not\in f_3(a, x) \quad (\text{FCB})\]

\[\exists! \ \hat{f} \in \Lambda \rightarrow_{fs} X\]

\[
\begin{align*}
  \hat{f} a &= f_1 a \\
  \hat{f}(t_1 t_2) &= f_2(\hat{f} t_1, \hat{f} t_2) \\
  \hat{f}(\lambda a.t) &= f_3(a, \hat{f} t) \quad \text{if } a \not\in (f_1, f_2, f_3)
\end{align*}
\]
Theorem. Given any $X \in \text{Nom}$ and \[
\begin{align*}
  f_1 & \in \mathcal{A} \rightarrow_{fs} X \\
  f_2 & \in X \times X \rightarrow_{fs} X \\
  f_3 & \in \mathcal{A} \times X \rightarrow_{fs} X
\end{align*}
\] s.t.

\[
(\forall a) \ a \# (f_1, f_2, f_3) \Rightarrow (\forall x) \ a \# f_3(a, x) \quad \text{(FCB)}
\]

\[\exists! \ \hat{f} \in \Lambda \rightarrow_{fs} X\]

\[
\begin{cases}
  \hat{f} \ a = f_1 \ a \\
  \hat{f} (t_1 t_2) = f_2 (\hat{f} t_1, \hat{f} t_2) \\
  \hat{f}(\lambda a. t) = f_3 (a, \hat{f} t) \quad \text{if } a \# (f_1, f_2, f_3)
\end{cases}
\]

E.g. capture-avoiding substitution $(\_)[t'/a']$ is the $\hat{f}$ for

\[
\begin{align*}
  f_1 \ a & \triangleq \text{if } a = a' \text{ then } t' \text{ else } a \\
  f_2(t_1, t_2) & \triangleq t_1 \ t_2 \\
  f_3(a, t) & \triangleq \lambda a. t
\end{align*}
\]

for which (FCB) holds, since $a \# \lambda a. t$
The notion of ‘\( \alpha \)-structural recursion’ generalizes smoothly from \( \lambda \)-terms to any nominal algebraic signature, giving a version of Gödel’s System T for nominal data types [J ACM 53(2006)459–506].

Urban & Berghofer’s Nominal package for Isabelle/HOL (interactive theorem prover for classical higher-order logic) implements this, and more.

Seems to capture informal practice quite well.
Applications: PL semantics

- operational semantics of ‘nominal’ calculi
- game semantics; domain theory
- equational logic & rewriting modulo $\alpha$-equivalence
- logic programming (Cheney-Urban: $\alpha$Prolog)
- functional metaprogramming (AMP-Shinwell: Fresh Ocaml)

Take-home message: permutation comes before substitution when dealing with the meta-theory of names and binding operations in syntactical structures.
An introduction to nominal techniques motivated by

Programming language semantics

Automata theory

Constructive type theory
Support – the key definition

Suppose $S_A$ acts on a set $X$ and $x \in X$.

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$$(\forall a \in A. \pi(a) = a) \Rightarrow \pi \cdot x = x$$

$X$ is a nominal set if every $x \in X$ has a finite support.

[AMP-Gabbay, LICS 1999]

Consider replacing $S_A$ by a subgroup $G$
i.e. use a more restrictive notion of symmetry
Different symmetries

Three interesting examples:

1. **Equality**: \( G \) = all (finite) permutations.
2. **Linear order**: \( A = \mathbb{Q} \) and \( G = \) order-preserving perms.
3. **Graphs**: \( A \) = vertices of the Rado graph and \( G \) = graph automorphisms.
Different symmetries

Three interesting examples:

1. **Equality**: $G = \text{all (finite) permutations.}$
2. **Linear order**: $A = \mathbb{Q}$ and $G = \text{order-preserving perms.}$
3. **Graphs**: $A = \text{vertices of the Rado graph and } G = \text{graph automorphisms.}$

In general [Bojańczyk, Klin, Lasota]:

- $A = \text{Carrier of the universal homogeneous structure (Fraïssé limit) for a finite relational signature}$
- $G = \text{automorphisms w.r.t. the signature}$

yields a universe $\mathcal{U} = \mathbb{P}_{fs}(A + \mathcal{U})$ with interesting applications for nominal computation.
Applications: automata theory

Automata over infinite alphabets:

- HD-automata: $\pi$-calculus verification [Montanari et al]

- fresh-register automata [Tzevelekos] (extending finite-memory automata of Kaminski & Francez): verifying programs with local names

- automata & Turing machines in sets with atoms [Bojańczyk et al]; CSP with atoms [Klin et al, LICS 2015]; ...
Applications: automata theory

Automata over infinite alphabets:

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  [Bojańczyk et al]; CSP with atoms [Klin et al, LICS 2015]; . . .

General approach: try to do ‘ordinary’ computation theory inside the universe/category of nominal sets (for a Fraïssé symmetry), but replace finite-state notions by orbit-finite ones. . .
Orbit finiteness

= having finitely many equivalence classes for

\[ x \sim y \triangleq \exists \pi. \, \pi \cdot x = y \]

E.g. for the equality symmetry, $A^n$ is orbit-finite, $A^*$ is not.
Orbit finiteness

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E.g. for the equality symmetry, \( A^n \) is orbit-finite, \( A^* \) is not.

Orbit-finite elements of \( U \) have good closure properties, except for powerset. Nominal automata (NA) satisfy:

- deterministic (D) \( \neq \) non-deterministic (ND)
- emptiness for ND-NA is decidable
- equivalence for D-NA is decidable
- D-NAs can be minimized

See [Bojańczyk-Klin-Lasota, LMCS 10(3) 2014].
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Nominal System T

Questions:

- Martin-Löf Type Theory generalizes Gödel’s System T to inductively defined families of dependent types. What is the right version of the notions of \textit{nominal set}, \textit{freshness} and \textit{name abstraction} within constructive type theory?

Stark-Schöpp CSL 2004 [extensional]
Cheney LMCS 2012 [missing locally scoped names]
AMP-Matthiesen-Derikx LSFA 2014 [locally scoped, judgementally fresh names]
Applications: Homotopy Type Theory

Cubical sets [Bezem-Coquand-Huber] model of Voevodsky’s axiom of univalence can be described using nominal sets equipped with an operation of substitution $x \mapsto x(i/a)$ where $i \in \{0, 1\}$.

- names are names of directions (cartesian axes) (so e.g., if an object has support \{a, b, c\} it is 3-dimensional)
- freshness $(a \# x) = \text{degeneracy} (x(i/a) = x)$
- identity types are modelled by name-abstraction: $\langle a \rangle x$ is a proof that $x(0/a)$ is equal to $x(1/a)$.

HoTT is about (proof-relevant) mathematical foundations (a topic no longer very popular with mathematicians). That’s where the mathematics of nominal sets came from...
Impact can take a long time

The mathematics behind nominal sets goes back a long way...


Impact can take a long time

The mathematics behind nominal sets goes back a long way...

...and it’s still too early to tell what will be the impact of the applications of it to CS developed over the last 15 years.

Take-home messages:

- Computation modulo symmetry deserves further exploration.
- Permutation comes before substitution and (hence) name-abstraction before lambda-abstraction... but it seems that constructive type theory and nominal techniques can coexist (wts).
Homework

Nominal Sets
Names and Symmetry in Computer Science