Equivariant Syntax and Semantics

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Can sum up the subject of this talk in three words:

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syntax!

The mathematics of syntax

- Seems of no interest to mathematicians and of little interest to logicians. (?)
- Vital for computer science — because of *symbolic computation* and *automated reasoning*.
- Has yet to reach an intellectual fixpoint for syntax involving name-binding and freshness of names.

Plan

- Review initial algebra view of abstract syntax.
- Abstract syntax is not abstract enough for name-binding and freshness of names.
- Category theory to the rescue!
- Equivariant initial algebra semantics for ‘nominal’ signatures.
- Applications to metaprogramming.

How to represent syntax?

$$\int_0^x \left(\int_1^y xy \, dx \right) + y \, dy$$

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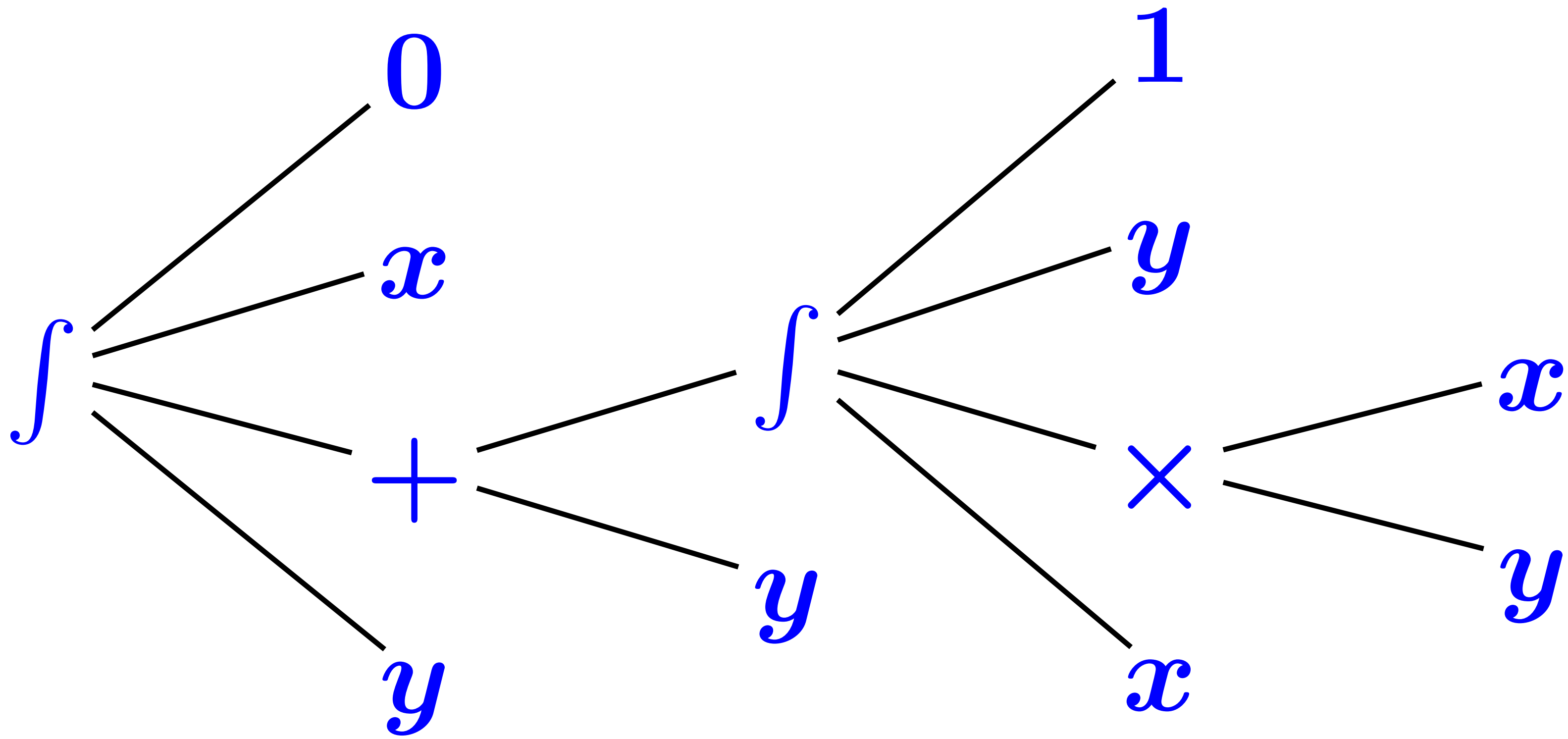
\int	_	0	^	x	(	\int	_	1	^	y		
{	x	y	}	d	x	)	+	y	d	y		

"Concrete syntax" — sequences of tokens generated by context free grammars, etc, etc.

Not *structurally* abstract.

How to represent syntax?



"Abstract syntax" — parse trees.

Initial algebra semantics

A signature Σ determines a functorial, sum-of-products construction on sets X:

$$X \mapsto T_{\Sigma}(X) \triangleq \sum_{F \in \Sigma} X^{\mathrm{ar}(F)}$$

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single-sorted, for simplicity; so arity of each operator $F \in \Sigma$ is just the number $\mathrm{ar}(F) \in \mathbb{N}$ of its arguments

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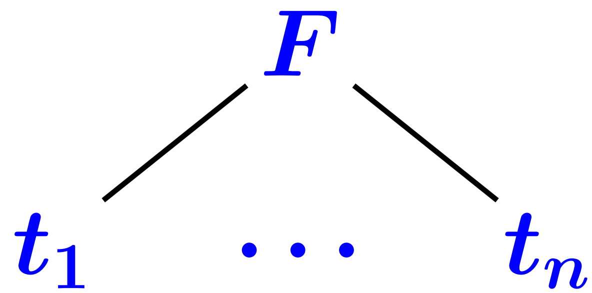
typical element $(F, (x_1, \ldots, x_n))$, where operator $F \in \Sigma$ has arity $\mathrm{ar}(F) = n$ and $x_1, \ldots, x_n \in X$

Initial algebra semantics

- set $I_\Sigma \triangleq \{\text{parse trees over } \Sigma\}$
- bijection $T_\Sigma(I_\Sigma) \longrightarrow I_\Sigma$
  between $(F, (t_1, \ldots, t_n))$ in $T_\Sigma(I_\Sigma)$
  and trees in I_Σ



are determined uniquely up to bijection by their initial algebra property...

Initial algebra property

For any $T_\Sigma(X) \xrightarrow[f]{} X$

Initial algebra property

$$\begin{array}{ccc} & & I_\Sigma \\ & & \big\downarrow \overline{f} \\ T_\Sigma(X) & \xrightarrow[f]{} & X \end{array}$$

there is a unique $\overline{f}$ such that

Initial algebra property

$$
\begin{array}{ccc}
T_\Sigma(I_\Sigma) & \xrightarrow{\ \cong\ } & I_\Sigma \\
{\scriptstyle T(\overline{f})}\Big\downarrow & \text{commutes} & \Big\downarrow{\scriptstyle \overline{f}} \\
T_\Sigma(X) & \xrightarrow[\ f\ ]{} & X
\end{array}
$$

$\overline{f}(t)$ applies f iteratively, according to the structure of the tree t.

Initial algebra property

- Encompasses useful principles of

  structural recursion
  and
  structural induction

  for parse trees over Σ.

  (Generalises *primitive recursion* and *mathematical induction* for the natural numbers.)

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- 'Arrow-theoretic' rather than 'element-theoretic' characterisation of parse trees—important later.

Abstract syntax is not sufficiently abstract

Parse trees take no account of variable binding.

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semantically equal expressions, represented by different, but α-convertible parse trees.

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Isn't this a matter of *semantics* rather than syntax? No, because...

Substitution

E.g. in order to respect meaning, the result of the syntactic operation

> *substitute* $2x$ *for the free occurrence of* y *in*
> $\int_0^1 (x + y) dx$

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> $\int_0^1 (x + y)dx$ = y + 0.5

Substitution

E.g. in order to respect meaning, the result of the syntactic operation

> *substitute $2x$ for the free occurrence of y in*
> $\int_0^1 (x + y) dx$

is not $\int_0^1 (x + 2x) dx$, but rather, is $\int_0^1 (u + 2x) du$, where u is fresh.

The problem

Hand-coding notions of

- free and bound variables,
- renaming of bound variables,
- freshness of variables
- substitution for free variables, etc

is painful and error-prone for complex languages, or large programs.

Need better mathematical foundations leading to better automatic support for these tasks.

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Generalisation of initial algebra semantics yielding useful principles of structural recursion/induction for parse trees modulo α-conversion over a nominal signature.

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- lead to improved languages for metaprogramming

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Generalisation of initial algebra semantics yielding useful principles of structural recursion/induction for parse trees modulo α-conversion over a nominal signature.

extension of usual notion of many-sorted algebraic signature to treat parse trees with lexically scoped binders modulo α-equivalence

Nominal signatures

- Sorts partitioned in two: sorts of bindable names (ν) and sorts of data (δ).
- Operators (F) have arities $\tau \longrightarrow \delta$, where

$$\tau ::= \nu \mid \delta \mid 1 \mid \tau,\tau \mid \nu.\tau$$

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type of pairs

type of name-abstractions

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here, for simplicity, we will assume there's just one

Nominal signatures

Closely related notions:

- *binding signatures* of Fiore, Plotkin & Turi (LICS 1999)
- *nominal algebras* of Honsell, Miculan & Scagnetto (ICALP 2001)

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N.B. all these notions of signature restrict attention to iterated, but *unary* name-binding—there are other kinds of lexically scoped binder.

Example: π-calculus

sort of bindable names: ν (channels)

sort of data: π (processes)

operators:

$$\begin{aligned}
0 &: 1 \longrightarrow \pi \\
Par &: \pi, \pi \longrightarrow \pi \\
Sum &: \pi, \pi \longrightarrow \pi \\
In &: \nu, (\nu.\pi) \longrightarrow \pi \\
Out &: \nu, \nu, \pi \longrightarrow \pi \\
Tau &: \pi \longrightarrow \pi \\
Nu &: \nu.\pi \longrightarrow \pi \\
Guard &: \nu, \nu, \pi \longrightarrow \pi
\end{aligned}$$

Example: an untyped FPL

sort of bindable names: var (variables)

sort of data: exp (expressions)

operators:

$$\begin{aligned} Var &: var \longrightarrow exp \\ App &: exp, exp \longrightarrow exp \\ Fun &: var.exp \longrightarrow exp \\ Let &: exp, (var.exp) \longrightarrow exp \\ Letrec &: var.(exp, exp) \longrightarrow exp \end{aligned}$$

Example: an untyped FPL

sort of bindable names: var (variables)

sort of data: exp (expressions)

$$Var : var \longrightarrow exp$$
$$App : exp, exp \longrightarrow exp$$
$$Fun : var.exp \longrightarrow exp$$
$$Let : exp, (var.exp) \longrightarrow exp$$
$$Letrec : var.(exp, exp) \longrightarrow exp$$

$Let(t, (x.t'))$
stands for
$let\ x = t\ in\ t'$

$Letrec(x.(t, t'))$
stands for
$letrec\ x = t\ in\ t'$

Parse trees and their types over a nominal signature:

- infinitely many atoms $a : \nu$ for each sort ν of bindable names
- $() : 1$ and $\dfrac{t : \tau \qquad t' : \tau'}{(t, t') : \tau, \tau'}$
- $\dfrac{t : \tau}{a.t : \nu.\tau}$ for each atom $a : \nu$
- $\dfrac{t : \tau}{F\,t : \delta}$ if F has arity $\tau \longrightarrow \delta$

α-Equivalence, $=_\alpha$

least congruence identifying $a.t$ with $b.[a{\mapsto}b]t$ if b does not occur (at all) in t

where

$[a{\mapsto}b]t$ = rename all free occurrences of a to be b in t.

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The notions underlying initial algebra semantics have purely arrow-theoretic definitions, so change category from *Set* to one with a suitable functorial construction for modelling name-abstraction $\nu.\tau$

> not a new idea—cf. initial algebra semantics in categories of domains, in order to treat fixpoint recursion

Two candidates to replace the category of sets and functions (both in Proc. LICS'99):

- Fiore-Plotkin-Turi: category of presheaves on finite sets & functions

  —nice categorical analysis of de Bruijn indices/levels; not so nice (?) for applications

Two candidates to replace the category of sets and functions (both in Proc. LICS'99):

- Fiore-Plotkin-Turi: category of presheaves on finite sets & injections
- Gabbay-AMP: category of FM-sets ($\simeq$ 'Schanuel topos')
  —a semantics for *name-abstraction* and *freshness of names* via use of permutation actions

Why use name-permutation/swapping?

Problem of 'capture': as a total operation on parse trees, $[a \mapsto b](-)$ doesn't respect $=_\alpha$, so can't be part of a theory of terms modulo $=_\alpha$.

E.g. $b.a =_\alpha c.a$, but applying $[a \mapsto b]$

$[a \mapsto b](b.a) = b.b \neq_\alpha c.b = [a \mapsto b](c.a)$.

Why use name-permutation/swapping?

Problem of 'capture': as a total operation on parse trees, $[a \mapsto b](-)$ doesn't respect $=_\alpha$, so can't be part of a theory of terms modulo $=_\alpha$.

Traditional solution: replace $[a \mapsto b]t$ by a *more complicated*, capture-avoiding form of renaming (and substitution).

Why use name-permutation/swapping?

Problem of 'capture': as a total operation on parse trees, $[a \mapsto b](-)$ doesn't respect $=_\alpha$, so can't be part of a theory of terms modulo $=_\alpha$.

A nice alternative: use a *less complicated* form of renaming

$(a\,b)\cdot t$ = swap *all* occurrences of a and b in t

Inductive definition of $=_\alpha$

$$a =_\alpha a$$

$$() =_\alpha ()$$

$$\frac{t_1 =_\alpha t'_1 \quad t_2 =_\alpha t'_2}{(t_1, t_2) =_\alpha (t'_1, t'_2)}$$

$$\frac{t =_\alpha t'}{a.t =_\alpha a.t'}$$

$$\frac{a' \neq a \quad a' \# t \quad (a\,a')\cdot t =_\alpha t'}{a.t =_\alpha a'.t'}$$

$$\frac{t =_\alpha t'}{F\,t =_\alpha F\,t'}$$

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Freshness: "a' does not occur in t"

Category of FM-sets

Fix an infinite set $\mathbb{A}$ of 'atoms' a, b, c ...

- Objects: sets X equipped with an $\mathbb{A}$-permutation action, all of whose elements have the finite support property

  each $x \in X$ satisfies $(a\,b)\cdot x = x$ *for all but finitely many* $a, b \in \mathbb{A}$.

Category of FM-sets

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- Objects: sets X equipped with an $\mathbb{A}$-permutation action, all of whose elements have the finite support property
- Morphisms: equivariant functions

  $$f((a\,b)\cdot x) = (a\,b)\cdot(f\,x)$$

Category of FM-sets

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- Objects: sets X equipped with an $\mathbb{A}$-permutation action, all of whose elements have the finite support property
- Morphisms: equivariant functions
- Freshness $a \# x$ ("a is fresh for x") is a derived notion:

  $a \# x$ iff $(a\, b)\cdot x = x$ for *all but finitely many* $b \in \mathbb{A}$.

Atom-abstractions, $\mathbb{A}.X$

quotient of $\mathbb{A} \times X$ by equivalence relation identifying (a, x) and (a', x')

iff either $a = a'$ and $x = x'$,
or $a' \# x$ and $(a\,a')\cdot x = x'$.

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Functor $\mathbb{A}.(-) :$ FM-Set $\longrightarrow$ FM-Set has excellent properties—in particular it can be used with sums and products in inductive definitions of FM-sets.

Theorem

For nominal signature Σ,

$\{$parse trees over $\Sigma\}/{=_\alpha}$
with its natural FM-sets structure

is initial algebra for associated functor
T_Σ : FM-Set $\longrightarrow$ FM-Set.

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for simplicity, assume Σ has a single data sort δ and a single sort of bindable names ν

Theorem

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T_Σ : FM-Set $\longrightarrow$ FM-Set.

Lemma: support of α-equivalence class of a parse tree coincides with the set of free names of (any representative) parse tree.

Theorem

For nominal signature Σ,

$\{$parse trees over $\Sigma\}/{=_\alpha}$
with its natural FM-sets structure

is initial algebra for associated functor T_Σ : FM-Set $\longrightarrow$ FM-Set.

generalises usual 'sum-of-products' functor by interpreting name-abstraction arities $\nu.(-)$ as atom-abstraction functors $\mathbb{A}.(-)$

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Generalisation of initial algebra semantics yielding useful principles of structural recursion/induction for parse trees modulo α-conversion over a nominal signature.

- close to informal practice ("Barendregt Variable Convention")
- lead to improved languages for metaprogramming

Close to informal practice

- FM-Set models classical logic $+ \mathrm{ZFA} + \neg \mathrm{AC}$.
- Equivariance becomes part of the implicit mathematical infra-structure—no need to prove it case-by-case.
- Initial algebra property $\Rightarrow$ structural induction involving freshness quantifier—formalises a common informal logical pattern.

Close to informal practice

- FM-Set models classical logic $+ \mathrm{ZFA} + \neg\mathrm{AC}$.
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"for some/any fresh name. . . "

- $\Rightarrow$ structural induction involving freshness quantifier—formalises a common informal logical pattern.

Close to informal practice

- FM-Set models classical logic $+ \mathbf{ZFA} + \neg\mathbf{AC}$.
- Equivariance becomes part of the implicit mathematical infra-structure—no need to prove it

See it at work in the Cardelli-Caires spatial process logic (TACS 2001 & CONCUR 2002)

- freshness quantifier—formalises a common informal logical pattern.

Applications to metaprogramming

Shinwell, Gabbay, AMP: FreshML = ML +

- bindable names and name-abstraction types
- name-abstraction patterns
- static freshness checking, guarantees run-time behaviour respects $=_\alpha$

Applications to metaprogramming

Shinwell, Gabbay, AMP: FreshML

See
⟨www.cl.cam.ac.uk/users/amp12/freshml/⟩

Applications to metaprogramming

Urban, Gabbay, AMP:

extension of first-order unification to parse trees mod $=_\alpha$ over a nominal signature

with applications to term-rewriting & logic programming (work in progress).

‘Syntax modulo’

Here: initial algebra semantics for syntax modulo $=_\alpha$.

Use of name-permutation (rather than renaming) leads to a rich theory with good structural recursion/induction principles for syntax modulo $=_\alpha$.

‘Syntax modulo’

Other important ways of making syntax more abstract:

- quotient by ‘structural congruence’ in process calculus (cf. the ‘Chemical abstract machine’)
- graph structures (e.g. semistructured data with references)

Are there useful notions of *structural* recursion/induction for these?

Final

¡Gracias por su atención!