Nominal Semantics of Abstraction and Restriction

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Unofficial title:
The Joy of Name-Swapping

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Aim: describe the topos of nominal sets (a simple reformulation of Schanuel’s atomic topos) as a model for computations on syntactical structures involving

- **freshness** of names
  (as in “choose a fresh name”)

- **name-abstraction**
  (e.g. as in $\lambda$-calculus / $\alpha$-equivalence)

- **name-restriction**
  (e.g. as in the $\pi$-calculus / structural congruence)
Aim: describe the topos of nominal sets (a simple reformulation of Schanuel’s atomic topos) as a model for computations on syntactical structures involving

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- name-restriction (e.g. as in the π-calculus / structural congruence)

The key idea is equivariance: regard names as atoms and enforce anonymity in binding constructs through invariance under atom-permutations.
Atoms, permutations and actions

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- $G \triangleq$ group of all finite permutations of $A$. 

A function $f: X \rightarrow Y$ between $G$-sets is equivariant if $f(g \cdot x) = (f \cdot x)$, for all $g \in G$ and $x \in X$. 

CTCS2004, 3
Atoms, permutations and actions

- $A \triangleq$ fixed, countably infinite set, whose elements will be called atoms.
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- An action of $G$ on a set $X$ is a function
  
  $G \times X \rightarrow X$ written $(\pi, x) \mapsto \pi \cdot x$

  satisfying $\iota \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \pi') \cdot x$.
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- $G$-set $\triangleq$ set $X$ + action of $G$ on $X$. 
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- **G-set** $\triangleq$ set **X** + action of **G** on **X**.

- A function $f : X \rightarrow Y$ between **G**-sets is equivariant if $f(\pi \cdot x) = \pi \cdot (f x)$, for all $\pi \in G$ and $x \in X$. 
Languages are $G$-sets

For example, $\lambda$-terms modulo $\alpha$-equivalence

$$\{ t ::= a \mid \lambda a \, t \mid t \, t \} / =_{\alpha}$$

with $G$-action recursively defined by:

$$\pi \cdot a =_{\alpha} \pi(a)$$
$$\pi \cdot (\lambda a \, t) =_{\alpha} \lambda \pi(a) (\pi \cdot t)$$
$$\pi \cdot (t \, t') =_{\alpha} (\pi \cdot t) (\pi \cdot t')$$

Lemma: $a$ is not free in $t$ iff $t = t$ holds for all but finitely many atoms $a_0$.

So what? Lemmas suggest a syntax-independent notion of freshness ("not free in") relation...
Languages are $G$-sets

For example, $\lambda$-terms modulo $\alpha$-equivalence

$$\{ t ::= a \mid \lambda a \ t \mid t \ t \} /=_\alpha$$

with $G$-action recursively defined by:

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N.B. binding and non-binding constructs are treated just the same
Languages are $\mathcal{G}$-sets

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**Lemma:** $a$ is not free in $t$ iff $(a \ a') \cdot t =_\alpha t$ holds for all but finitely many atoms $a'$. 
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*permutation transposing $a$ and $a'$*
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**So what?**
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So what? Lemma suggests a syntax-independent notion of freshness (“not free in”) relation. . .
Finite support

Definition: a finite set of atoms $\overline{a} \subseteq A$ supports an element $x \in X$ of a $\mathbb{G}$-set $X$ if $(a a') \cdot x = x$ holds for all $a, a' \in A - \overline{a}$.

A nominal set is a $\mathbb{G}$-set all of whose elements have a finite support.
Finite support

Definition: a finite set of atoms $\overline{a} \subseteq A$ supports an element $x \in X$ of a $G$-set $X$ if $(a a') \cdot x = x$ holds for all $a, a' \in A - \overline{a}$.

A nominal set is a $G$-set all of whose elements have a finite support.

Lemma: If $x \in X$ has a finite support, then it has a smallest one, written $\text{supp}(x)$.

E.g. for a $\lambda$-term, $\text{supp}(t) = \{\text{free variables of } t\}$. 
Finite support

Definition: a finite set of atoms \( \bar{a} \subseteq A \) supports an element \( x \in X \) of a \( G \)-set \( X \) if \( (a a') \cdot x = x \) holds for all \( a, a' \in A \setminus \bar{a} \).

A nominal set is a \( G \)-set all of whose elements have a finite support.

\[ Nset = \text{category of nominal sets and equivariant functions.} \]
\( \mathcal{N} \text{set is a topos} \)
$\mathcal{N}set$ is a topos

- Terminal object: $1 \cong \{0\}$
  action $\pi \cdot 0 = 0$,
  support $supp(0) = \emptyset$. 
\textit{\( \mathcal{N}set \) is a topos}

- **Terminal object:** \( \mathbf{1} \triangleq \{0\} \)
  
  action \( \pi \cdot 0 = 0 \),
  
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- **Product:** \( X \times Y \triangleq \{(x, y) \mid x \in Y \ \& \ y \in Y\} \)
  
  action \( \pi \cdot (x, y) \triangleq (\pi \cdot x, \pi \cdot y) \),
  
  support \( \text{supp}(x, y) = \text{supp}(x) \cup \text{supp}(y) \).
\[ \mathcal{N}set \text{ is a topos} \]

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- **Powerobjects:** \( P_{fs}(X) = \) all subsets \( S \subseteq X \) that are finitely supported w.r.t. the action given by \( \pi \cdot S \triangleq \{\pi \cdot x \mid x \in S\} \). (\( \mathcal{N}set \) is boolean.)
Nset is a topos

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  - action \(\pi \cdot 0 = 0\),
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- Product: \(X \times Y \triangleq \{(x, y) \mid x \in Y \& y \in Y\}\)
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- Powerobjects: \(P_{fs}(X) = \) all subsets \(S \subseteq X\) that are finitely supported w.r.t. the action given by \(\pi \cdot S \triangleq \{\pi \cdot x \mid x \in S\}\). (Nset is boolean.)
- Exponentials: \(Y^{X} = \) all functions from \(X\) to \(Y\) that are finitely supported w.r.t. the action given by \(\pi \cdot f \triangleq \lambda x \in X. \pi \cdot (f(\pi^{-1} \cdot x))\).
First-order logic (and arithmetic) in $\mathcal{Nset}$ is just like for ordinary sets. For example:

- **Negation:** if $[\phi(x)] = S \in P_{fs}(X)$, then $[-\phi(x)] = X - S$. 
First-order logic (and arithmetic) in \( \mathcal{N}set \) is just like for ordinary sets. For example:

- **Negation:** if \( \llbracket \phi(x) \rrbracket = S \in P_{fs}(X) \), then \( \llbracket \neg \phi(x) \rrbracket = X - S \).

\((X - S) \) is in \( P_{fs}(X) \) because it is supported by any finite \( \bar{a} \) that supports \( S \) \.)
First-order logic (and arithmetic) in $\mathbb{N}\text{-}set$ is just like for ordinary sets. For example:

- **Negation:** if $[\phi(x)] = S \in P_{fs}(X)$, then $[-\phi(x)] = X - S$.

- **For all:** if $[\phi(x, y)] = S \in P_{fs}(X \times Y)$, then $[\forall x. \phi(x, y)] = \{y \in Y \mid \forall x \in X. (x, y) \in S\}$. 
First-order logic (and arithmetic) in $\mathcal{Nset}$ is just like for ordinary sets. For example:

- **Negation:** if $\llbracket \phi(x) \rrbracket = S \in P_{fs}(X)$, then $\llbracket \neg \phi(x) \rrbracket = X - S$.

- **For all:** if $\llbracket \phi(x, y) \rrbracket = S \in P_{fs}(X \times Y)$, then $\llbracket \forall x. \phi(x, y) \rrbracket = \{ y \in Y \mid \forall x \in X. (x, y) \in S \}$. 

  ($\{ y \in Y \mid \forall x \in X. (x, y) \in S \}$ is in $P_{fs}(Y)$, because it is supported by any finite $\overline{a}$ that supports $S$.)


Higher-order logic in $\mathcal{N}\text{set}$ is like higher-order logic for ordinary sets, except that we have to restrict to finitely supported sets and functions when forming powersets and exponentials.
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For example

Tarski Fixpoint Theorem: for any monotone and finitely supported function $\Phi$ from $P_{fs}(X)$ to itself, the usual least (pre)fixed point

$$\mu(\Phi) \triangleq \bigcap \{ S \in P_{fs}(X) \mid \Phi(S) \subseteq S \}$$

is again finitely supported, hence in $P_{fs}(X)$. 
Higher-order logic in $\mathcal{N}set$ is like higher-order logic for ordinary sets, except that we have to restrict to finitely supported sets and functions when forming powersets and exponentials.

This rules out the use of choice. For example

$$n \mapsto C(n) \triangleq \{ S \subseteq A \mid \text{card}(S) = n \}$$

is a finitely (indeed, emptily) supported function from $\mathbb{N}$ to non-empty elements of $P_{fs}(P_{fs}(A))$, but there is no finitely supported function $c$ from $\mathbb{N}$ to $P_{fs}(A)$ satisfying

$$\forall n \in \mathbb{N}. \ c(n) \in C(n)$$
Atom-abstraction
Nominal set of atom-abstractions, $[\mathcal{A}]X$

$$[\mathcal{A}]X \triangleq (\mathcal{A} \times X)/\sim$$
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with equivalence relation $(a, x) \sim (a', x')$ given by:

$(a a'') \cdot x = (a' a'') \cdot x'$ in $X$, for some (or indeed any) $a''$ s.t. $a'' \not\# (a, x, a', x')$
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given elements $x \in X$ and $y \in Y$ of nominal sets, we write $x \not\equiv y$ to mean $\text{supp}(x) \cap \text{supp}(y) = \emptyset$. 

Nominal set of atom-abstractions, $[\mathbb{A}]X$

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Write $[a]x$ for the $\sim$-equivalence class of $(a, x)$. 


Nominal set of atom-abstractions, $\left[ A \right] X$

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Write $\left[ a \right] x$ for the $\sim$-equivalence class of $(a, x)$.

Action: $\pi \cdot \left[ a \right] x = \left[ \pi(a) \right] \left( \pi \cdot x \right)$

(and it follows that $\text{supp}(\left[ a \right] x) = \text{supp}(x) - \{a\}$).
Atom-abstraction is extremely well-behaved:

<table>
<thead>
<tr>
<th>Equivariant functions $Y \to [A]X$</th>
</tr>
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<tbody>
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<td>naturally correspond to equivariant functions</td>
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<td>[(a, y) \in A \times Y \mid a \neq y] $\to X$</td>
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<td>$f : A \times X \to Y$ s.t. $a \neq f(a, x)$, all $a, x$</td>
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\[
[A](X \times Y) \cong [A]X \times [A]Y \\
[A](X + Y) \cong [A]X + [A]Y \\
[A](Y^X) \cong ([A]Y)^{[A]X}
\]
Atom-abstraction gives the non-recursive essence of $\alpha$-equivalence.

**Theorem:**
\[
\{ t ::= a \mid \lambda a \, t \mid t \, t \}/=_{\alpha}
\]
(quotient of an inductively defined set)
as an object of $\mathcal{Nset}$ is isomorphic to
\[\mu X \ (A + [A]X + X \times X)\]
(inductively defined nominal set).
Similarly for other languages involving binders.
Atom-abstraction gives the non-recursive essence of \(\alpha\)-equivalence.

**Theorem:**\[
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(inductively defined nominal set).

Similarly for other languages involving binders.

This observation was the starting point for **FreshML/Fresh O’Caml**—functional programming for nominal sets and equivariant functions (Shinwell, Gabbay & AMP, ICFP’03).
Atom-restriction
\[ \pi\text{-Calculus restriction, } \nu x . P \]

Its reduction

\[
P[a/x] \rightarrow P'[a/x] \quad a \notin fn(P, P')
\]

\[
\nu x . P \rightarrow \nu x . P'
\]

is specific to \(\pi\)-calculus, but structural congruences

\[\begin{align*}
(\alpha) & \quad \nu x . P \equiv \nu x'. P[x'/x] \quad \text{if } x' \notin fn(P) \\
(\gamma) & \quad \nu x . P \equiv P \quad \text{if } x \notin fn(P) \\
(\sigma) & \quad \nu x . \nu x'. P \equiv \nu x'. \nu x . P \\
(\varepsilon) & \quad (\nu x . P)|P' \equiv \nu x . (P|P') \quad \text{if } x \notin fn(P')
\end{align*}\]

are quite general properties of a “spatial” notion of scope restriction.

(Cf. Caires & Cardelli, A spatial logic for concurrency, CONCUR’02.)
\( \pi \)-Calculus restriction, \( \forall x. P \)

Its reduction

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P[a/x] \rightarrow P'[a/x] \quad a \notin \text{fn}(P, P')
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\( \forall x. P \rightarrow \forall x. P' \)

is specific to \( \pi \)-calculus, but structural congruences

(\( \alpha \)) \quad \forall x. P \equiv \forall x'. P[x'/x] \quad \text{if} \ x' \notin \text{fn}(P)

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are quite general properties of a “spatial” notion of scope restriction.

**Aim:** a construct in nominal sets satisfying \((\alpha)-(\sigma)\) that allows us to discuss scope extrusion \((\varepsilon)\) in a syntax-independent way.
Nominal restriction structure, \((R, \rho)\)

is given by a nominal set \(R\) and an equivariant function \(\rho : [A]R \to R\)

satisfying

\[
\begin{align*}
\rho[a]r &= r \\
\rho[a]\rho[a']r &= \rho[a']\rho[a]r \\
a \not\equiv r &\implies \rho[a]r = r
\end{align*}
\]
Nominal restriction structure, \((R, \rho)\)

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\rho : [\Delta]R \to R
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\text{a } \# \text{ r } & \implies \rho[a]r = r \\
\rho[a]\rho[a']r & = \rho[a']\rho[a]r
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\]

\((\alpha)\) \(\nu x. P \equiv \nu x'. P[x'/x]\) if \(x' \notin \text{fn}(P)\)
Nominal restriction structure, $(R, \rho)$

is given by a nominal set $R$ and an equivariant function $\rho : [A]R \to R$

satisfying

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\begin{align*}
    a \neq r & \implies \rho[a]r = r \\
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$(\gamma) \ \forall x. P \equiv P \text{ if } x \notin \text{fn}(P)$
Nominal restriction structure, \((R, \rho)\)

is given by a nominal set \(R\) and an equivariant function \(\rho : [A]R \to R\) satisfying

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\begin{align*}
\alpha \neq r & \Rightarrow \rho[\alpha]r = r \\
\rho[\alpha]\rho[\alpha']r = \rho[\alpha']\rho[\alpha]r
\end{align*}
\]

\((\sigma) \ \nu x. \nu x'. P \equiv \nu x'. \nu x. P\)
Nominal restriction structure, \((R, \rho)\)

is given by a nominal set \(R\) and an equivariant function \(\rho : [\Lambda]R \to R\)
satisfying

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\begin{align*}
\text{Theorem: every nominal set } X \text{ possesses a freely generated restriction structure:} \\
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Nominal restriction structure, \((R, \rho)\)

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**Theorem:** Every nominal set \(X\) possesses a freely generated restriction structure:

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & \nu X \\
& \searrow & \\
& & R
\end{array}
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\((\nu X, \rho_X)\)
Nominal restriction structure, \((R, \rho)\)

is given by a nominal set \(R\) and an equivariant function
\[ \rho : [\Delta]R \to R, \]

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  a \# r & \Rightarrow \rho[a]r = r \\
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**Theorem:** every nominal set \(X\) possesses a freely generated restriction structure:

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\begin{array}{ccc}
  X & \xrightarrow{\eta_X} & \nu X \\
  \downarrow \forall & & \downarrow \exists! \\
  R & & (R, \rho)
\end{array}
\]

preserving restriction

\[
\begin{array}{ccc}
  \nu X & \xrightarrow{\rho_X} & (\nu X, \rho_X)
\end{array}
\]
Construction of $\nu X$

$$\nu X \triangleq (X \times P_{\text{fin}}A)/\sim$$
Construction of $\nu X$

$$\nu X \triangleq (X \times P_{\text{fin}} \mathbb{A}) / \sim$$

finite sets $\overline{a}$ of atoms
Construction of $\nu X$

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with equivalence relation $(x, \overline{a}) \sim (x', \overline{a}')$ given by:

$$\text{supp}(x) - \overline{a} = \text{supp}(x') - \overline{a}' \quad \text{and} \quad \pi \cdot x = x'$$

for some $\pi \in G$ with $\pi \# \text{supp}(x) - \overline{a}$
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Construction of \( \nu X \)

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for some \( \pi \in \mathbb{G} \) with \( \pi \neq \text{supp}(x) - \bar{a} \)
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and $\pi \cdot x = x'$ for some $\pi \in \mathbb{G}$ with $\pi \# \text{supp}(x) - \overline{a}$
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Write $x \backslash \bar{a}$ for the $\sim$-equivalence class of $(x, \bar{a})$. 

Construction of $\nu X$

$$\nu X \triangleq (X \times P_{\text{fin}} A) / \sim$$

with equivalence relation $(x, \bar{a}) \sim (x', \bar{a}')$ given by:

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$G$-action is:

$$\pi \cdot (x \backslash \bar{a}) \triangleq (\pi \cdot x) \backslash \{\pi(a) \mid a \in \bar{a}\}$$

(and it follows that $\text{supp}(x \backslash \bar{a}) = \text{supp}(x) - \bar{a}$).
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Restriction operation $\rho_X : [A] \nu X \rightarrow \nu X$ is:

$$
\rho_X[a](x \backslash \bar{a}) \triangleq x \backslash (\{a\} \cup \bar{a})
$$
Construction of $\nu X$

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Insertion operation $\eta_X : X \rightarrow \nu X$ is:

$$\eta_X(x) \triangleq x \backslash \emptyset$$
Abstraction from restriction

The equivalence relation used to construct $\nu X$ seems quite similar to that used to construct $[A]X$. What is the relationship?
Abstraction from restriction

The equivalence relation used to construct $\nu X$ seems quite similar to that used to construct $[A] X$. What is the relationship?

Lemma: The subset of $\nu(A \times X)$ given by

$$\{(a, x) \setminus \{a\} \mid a \in A \& x \in X\}$$

is isomorphic to $[A] X$ via the correspondence

$$(a, x) \setminus \{a\} \leftrightarrow [a] x$$
Scope extrusion

For $\pi$-calculus, symmetry of $(-)\parallel (-)$ plus

$$((\varepsilon) \ (\nu x. \ P)|P') \equiv \nu x. (P|P') \text{ if } x \notin \text{fn}(P')$$

says operation $(-)\parallel (-)$ on $\{\pi\text{-processes}\}/\equiv$ is a

bimorphism of restriction structures:
equivariant function $(-)\parallel (-): R_1 \times R_2 \to R_3$
satisfying

$$\begin{cases} 
(\rho_1[a]r)|r' = \rho_3[a](r|r') & \text{if } a \not\in r' \\
r|(\rho_2[a]r') = \rho_3[a](r|r') & \text{if } a \not\in r 
\end{cases}$$
Scope extrusion

For $\pi$-calculus, symmetry of $(-)\|(-)$ plus

$$(\varepsilon) \; (\nu x. \; P)|P' \equiv \nu x. (P|P') \quad \text{if} \; x \notin \text{fn}(P')$$

says operation $(-)\|(-)$ on $\{\pi\text{-processes}\}/\equiv$ is a

bimorphism of restriction structures:

equivariant function $(-)\|(-): R_1 \times R_2 \rightarrow R_3$

satisfying

\[
\begin{align*}
(r_1[a]r)|r' &= r_3[a](r|r') \quad \text{if} \; a \neq r' \\
(r|(r_2[a]r')) &= r_3[a](r|r') \quad \text{if} \; a \neq r
\end{align*}
\]

Theorem: Given $R_1$ & $R_2$, there is a universal bimorphism $R_1 \times R_2 \rightarrow R_1 \uplus R_2$
Scope extrusion

For $\pi$-calculus, symmetry of \((\texttt{--}) \mid (\texttt{--})\) plus

\[(\varepsilon) \quad (\nu x. \, P) \mid P' \equiv \nu x. \, (P \mid P') \quad \text{if} \quad x \notin \text{fn}(P')\]

says operation \((\texttt{--}) \mid (\texttt{--})\) on \(\{\pi\text{-processes}\} / \equiv\) is a

<table>
<thead>
<tr>
<th>bimorphism of restriction structures:</th>
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<tr>
<td>equivariant function ((\texttt{--}) \mid (\texttt{--}) : R_1 \times R_2 \to R_3)</td>
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</table>
| satisfying \(\left\{ \begin{array}{ll}
(\rho_1[a]r) \mid r' = \rho_3[a](r \mid r') & \text{if} \quad a \neq r' \\
 r \mid (\rho_2[a]r') = \rho_3[a](r \mid r') & \text{if} \quad a \neq r
\end{array} \right.\) |

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\[\forall \text{ bimorphisms} \quad R_3\]

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Scope extrusion

For $\pi$-calculus, symmetry of $(-)|(-)$ plus

$$(\varepsilon) \quad (\nu x. P)|P' \equiv \nu x. (P|P') \quad \text{if} \quad x \notin fn(P')$$

says operation $(-)|(-)$ on $\{\pi$-processes$\}/\equiv$ is a

bimorphism of restriction structures:

equivariant function $(-)|(-) : R_1 \times R_2 \rightarrow R_3$

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$$\begin{cases}
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Theorem: Given $R_1$ & $R_2$, there is a universal bimorphism $R_1 \times R_2 \rightarrow R_1 \uplus R_2$

$\forall$ bimorphisms $\exists$! morphism $R_3$
Is there a (useful!) initial algebra semantics for “spatial” calculi/logics (e.g. $\pi$, ambients, TQL, ...) using restriction structures in nominal sets?

- Use $[A](-)$ for domain of name-binding operators.
- Use $- \circ -$ for domain of binary, spatial operators.
- Use $- \times -$ for domain of binary, non-spatial operators.

(Cf. presheaf semantics of $\pi$-calculus: Stark (LICS’96), Fiore-Moggi-Sangiorgi (LICS’96).)
Equations + freshness constraints = ?

In $\mathcal{Nset}$ we can model equational theories conditioned by freshness, e.g.:

$$a' \neq x \vdash R(a, x) = R(a', (a a') \cdot x)$$

$$\vdash R(a, R(a' x)) = R(a', R(a, x))$$

$$a \neq x \vdash R(a, x) = x$$
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In $\mathcal{Nset}$ we can model equational theories conditioned by freshness, e.g.:

$$a' \# x \vdash R(a, x) = R(a', (a \ a') \cdot x)$$
$$\vdash R(a, R(a'x)) = R(a', R(a, x))$$
$$a \# x \vdash R(a, x) = x$$

What is the categorical logic of this kind of theory? Is there a notion of classifying category (internal to $\mathcal{Nset}$) for this kind of theory within $\mathcal{Nset}$?

(Cf. Urban, Gabbay & AMP, Nominal Unification, TCS to appear.)
Dynamic allocation monads

\[ X \mapsto (\nu X, \rho_X) \] reflects nominal sets into nominal restriction structures. So \( \nu(-) \) is a monad on \( Nset \).

In fact the explicit construction of \( \nu X \) shows that \( \nu(-) \) corresponds to one of Moggi’s dynamic allocation monads on a functor-category, used for semantics of name creation.
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Abramsky-Ghica-Murawski-Ong-Stark (LICS’04) make use of nominal sets and \( \nu(-) \) to give a fully abstract game semantics for the Pitts-Stark \( \nu \)-calculus.
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What about old-fashioned denotational semantics, using domains?
Dynamic allocation monads

$X \mapsto (\nu X, \rho_X)$ reflects nominal sets into nominal restriction structures. So $\nu(-)$ is a monad on $Nset$.

In fact the explicit construction of $\nu X$ shows that $\nu(-)$ corresponds to one of Moggi’s dynamic allocation monads on a functor-category, used for semantics of name creation.

Abramsky-Ghica-Murawski-Ong-Stark (LICS’04) make use of nominal sets and $\nu(-)$ to give a fully abstract game semantics for the Pitts-Stark $\nu$-calculus.

What about old-fashioned denotational semantics, using domains? I’m glad you asked...
Domain theory in $N_{set}$
\(\omega\)-Cpos in \(N_{\text{set}}\)

Conventional domain theory in \(N_{\text{set}}\), up to and including the construction of recursively defined domains, is no harder than in \(\text{Set}\).

N.B. an internal \(\omega\)-chain is just an increasing sequence \(d_0 \subseteq d_1 \subseteq \cdots\) for which there is a single finite set of atoms supporting all the \(d_n\). So \(\omega\)-cpos in \(N_{\text{set}}\) may be incomplete externally—e.g. \((P_{\text{fin}}(A), \subseteq)\).
**ω-Cpos in Nset**

**Problem:**
Unlike $[A]D$, the free restriction structure $\nu D$ is not always an $\omega$-cpo in $Nset$ even if $D$ is.
(Partly explains why dynamic allocation monads on functor categories have had limited application.)
**ω-Cpos in \( \mathcal{Nset} \)**

**Problem:**  
Unlike \([A]D\), the free restriction structure \( \nu D \) is not always an \( \omega \)-cpo in \( \mathcal{Nset} \) even if \( D \) is.  
(Partly explains why dynamic allocation monads on functor categories have had limited application.)

**Solution:**  
Continuous function domain \( D \rightarrow \{ \perp, \top \} \) possesses a restriction structure; hence we can use the continuation monad \((\_ \rightarrow \{ \perp, \top \}) \rightarrow \{ \perp, \top \}\) to give a denotational semantics of dynamic allocation + fixpoint recursion.

See: Shinwell-Pitts, “On a Monadic Semantics for Freshness” (APPSEM’04).
Conclusion

Nominal sets provide a model of

- restriction & anonymity (via permutation-invariance)
- name-abstraction & implicit dependence on parameters (via the notion of “support”)
- function-abstraction & explicit dependence on parameters (via exponentials)
- dynamic allocation of fresh names (via the notion of support again)

that is pretty, pretty simple, and pretty rich in interesting properties.

We are only at the beginning of the computational consequences of taking this model seriously.
Thanks
James Cheney (Cornell), Jamie Gabbay (INRIA),
Mark Shinwell & Christian Urban (Cambridge).

Further info
www.cl.cam.ac.uk/users/amp12/freshml/