Ten Exercises on Nominal Sets

for Andrew Pitts' lectures
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1. Prove that the rules defining $=_\alpha$ in Figure 1 [p 11] do indeed define an equivalence relation.

2. Show that every atom-permutation is equal to a finite composition of transpositions. [p 12]

3. Prove that the atom-permutation action on terms defined in Example 7(ii) [p 13] preserves $=_\alpha$. (See Theorem 12 for the general reason why this is so.)

4. Prove the claim in Example 7(iii) [p 13] that the smallest support of an α-term is its finite set of free atoms. [Hint: use the method sketched in Example 7(ii), but replacing $=$ with $=_\alpha$ and $atm(-)$ with $fa(-)$.]

5. Show that in the product $X_1 \times X_2$ of two nominal sets X_1 and X_2 [p 14], support satisfies: $supp((x_1, x_2)) = supp(x_1) \cup supp(x_2)$.

6. Let X and X' be nominal sets. We call a function $f : X \to X'$ *equivariant* if it satisfies $f(\pi \cdot x) = \pi \cdot (f\,x)$ for all $x \in X$ and $\pi \in Perm$. Show that an element $f \in (X \to_{\mathrm{fs}} X')$ of the nominal set of finitely supported functions [p 14] satisfies $supp(f) = \emptyset$ iff f is an equivariant function.

7. Let $(a_n \mid n \in \mathbb{N})$ be an enumeration of the countably infinite set $\mathbb{A}$ of atoms. Is the function $n \mapsto a_n$ finitely supported? Is the set $\{a_{2n} \mid n \in \mathbb{N}\}$ a finitely supported subset of $\mathbb{A}$ regarded as a nominal set in the usual way (Example 7(i))?

8. Show that for every finitely supported subset S of the nominal set $\mathbb{A}$ of atoms, either S is finite or $\mathbb{A} - S$ is finite.

9. Let X be a nominal set. We call a subset $S \subseteq X$ *equivariant* if it satisfies $\pi \cdot x \in S$ for all $\pi \in Perm$ and all $x \in S$. Show that an element $S \in$

$P_{\mathrm{fs}}(X)$ of the nominal set of finitely supported subsets of X [p 14] satisfies $supp(S) = \emptyset$ iff S is an equivariant subset.

10. Let X and Y be nominal sets.

    Show that the following are equivariant subsets (cf. Exercise 9):

    (i) Truth: $X \in P_{\mathrm{fs}}(X)$.

    (ii) Equality: $\{(x, x') \in X \times X \mid x = x'\} \in P_{\mathrm{fs}}(X \times X)$.

    (iii) Membership: $\{(x, S) \in X \times P_{\mathrm{fs}}(X) \mid x \in S\} \in P_{\mathrm{fs}}(X \times P_{\mathrm{fs}}(X))$.

    Show that the following are equivariant functions (Exercise 6):

    (iv) Conjunction: $(-) \cap (-) \in P_{\mathrm{fs}}(X) \times P_{\mathrm{fs}}(X) \to_{\mathrm{fs}} P_{\mathrm{fs}}(X)$.

    (v) Negation: $\neg \in P_{\mathrm{fs}}(X) \to_{\mathrm{fs}} P_{\mathrm{fs}}(X)$, where $\neg S \triangleq \{x \in X \mid x \notin S\}$.

    (vi) Universal quantification: $\bigcap \in P_{\mathrm{fs}}(P_{\mathrm{fs}}(X)) \to_{\mathrm{fs}} P_{\mathrm{fs}}(X)$, where $\bigcap \mathcal{S} \triangleq \{x \in X \mid (\forall S \in \mathcal{S})\ x \in S\}$.

    (vii) Substitution: $f^* \in P_{\mathrm{fs}}(Y) \to_{\mathrm{fs}} P_{\mathrm{fs}}(X)$, where $f \in X \to_{\mathrm{fs}} Y$ is an equivariant function and $f^* S \triangleq \{x \in X \mid f(x) \in S\}$.

    (viii) Classification: $\chi \in P_{\mathrm{fs}}(X \times Y) \to_{\mathrm{fs}} (X \to_{\mathrm{fs}} P_{\mathrm{fs}}(Y))$, where $\chi\, S \triangleq \lambda x \in X.\{y \in Y \mid (x, y) \in S\}$.