Nominal Cubical model of type theory

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Nominal Cubical

model of type theory

[CCHM]
C. Cohen, T. Coquand,
S. Huber, and A. Mörtberg,
Cubical type theory: a constructive interpretation of the univalence axiom.
The CCHM model of Homotopy Type Theory can be reformulated using (some) nominal techniques.


This simplifies the description if some parts of the model and may lead to new models of univalence.
Plan

- Motivation: the univalence axiom [HoTT]
- Overview of the Cohen-Coquand-Huber-Mörtberg presheaf model of univalent type theory [CCHM, OP, B+]
- Toposes of $\mathbb{M}$-sets
- CCHM cubical sets as finitely supported $\mathbb{M}$-sets [Pit]
- Path objects
- Cofibrant propositions and fibrant families
- A univalent universe [CCHM]
Main sources


Univalence

In Martin-Löf Type Theory (MLTT), Voevodsky's univalence axiom is an extensionality property of types in a universe $\mathcal{U}$. 
Univalence

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given $X, Y : \mathcal{U}$, every $p : X =_\mathcal{U} Y$

type of identifications (proofs of equality) between $X$ and $Y$ in $\mathcal{U}$
In Martin-Löf Type Theory (MLTT), Voevodsky's univalence axiom is an extensionality property of types in a universe $\mathcal{U}$:

given $X, Y : \mathcal{U}$, every $p : X =_\mathcal{U} Y$ induces an isomorphism $X \cong Y$ (relative to $=$).

$p_* : X \to Y \quad p^* : Y \to X$

$\eta : (\text{id} =_{Y \to Y} p_* \circ p^*) \quad \varepsilon : (p^* \circ p_* =_{X \to X} \text{id})$

well-defined by just giving the case when $p \equiv \text{refl}$

(for which $p_* \equiv p^* \equiv \lambda x. x$ and $\eta \equiv \varepsilon \equiv \text{refl}$)
Univalence

In Martin-Löf Type Theory (MLTT), Voevodsky's univalence axiom is an extensionality property of types in a universe \( \mathcal{U} \):

given \( X, Y : \mathcal{U} \), every \( p : X \simeq \mathcal{U} Y \) induces an isomorphism \( X \simeq Y \) (relative to \( \simeq \)).

\( \mathcal{U} \) is univalent if there is a proof of “all isomorphisms \( X \simeq Y \) in \( \mathcal{U} \) are induced by some \( p : X \simeq \mathcal{U} Y \).”

(Notation: \( \text{UTT} \equiv \text{MLTT} + \text{univalence} \).)

Licata, Shulman et al: the above is logically equivalent to, but a bit simpler than Voevodsky’s original definition.
Univalence

In Martin-Löf Type Theory (MLTT), Voevodsky's univalence axiom is an extensionality property of types in a universe $\mathcal{U}$:

given $X, Y : \mathcal{U}$, every $p : X \equiv_{\mathcal{U}} Y$ induces an isomorphism $X \cong Y$ (relative to $\equiv$).

$\mathcal{U}$ is univalent if there is a proof of "all isomorphisms $X \cong Y$ in $\mathcal{U}$ are induced by some $p : X \equiv_{\mathcal{U}} Y$".
(Notation: UTT $\equiv$ MLTT + univalence.)

N.B. univalence is inconsistent with extensional type theory (ETT).

ETT satisfies: if $p : x \equiv_A y$, then $x \equiv y$ and $p \equiv \text{refl}$
**Univalence**

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N.B. univalence is inconsistent with extensional type theory (ETT). Need a source of models of "intensional" identification types.
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(Notation: \( \text{UTT} \equiv \text{MLTT} + \text{univalence.} \))

N.B. univalence is inconsistent with extensional type theory (ETT).

Need a source of models of “intensional” identification types.

Homotopy Type Theory to the rescue: elements \( p : x \equiv_{A} y \) are analogous to paths \( p \) from point \( x \) to point \( y \) in a space \( A \) with \( \text{refl} : x \equiv_{A} x \) corresponding to a constant path [Awodey-Warren, Voevodsky,...]
In Martin-Löf Type Theory (MLTT), Voevodsky's univalence axiom is an extensionality property of types in a universe \( \mathcal{U} \):

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(Notation: UTT \( \equiv \) MLTT + univalence.)

All (?) existing models with non-truncated univalent universes stem in some way from:

- Kan simplicial sets in classical set theory [Voevodsky et al]
- uniform-Kan cubical sets in constructive set theory [CCHM]

(We need more, and simpler, examples!)
Univalence

In Martin-Löf Type Theory (MLTT), Voevodsky's univalence axiom is an extensionality property of types in a universe \( \mathcal{U} \):

given \( X, Y : \mathcal{U} \), every \( p : X =_{\mathcal{U}} Y \) induces an isomorphism \( X \cong Y \) (relative to \( = \)).

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Overview of the CCHM model

Uses Dybjer’s **Category with Families (CwF)** for the semantics of MLTT.

Brief recap here – see [Hof] for details.
Category with Families (CwF)

A CwF is given by

- category $\mathcal{C}$ with a terminal object $1$
  
  [objects $\Gamma, \Delta, \ldots \in \mathcal{C}$ model typing contexts;
   morphisms $\gamma \in \mathcal{C}(\Delta, \Gamma)$ model simultaneous substitutions mapping variables to terms (context morphisms);
   $1$ denotes the empty context]
A CwF is given by

- category \( \mathcal{C} \) with a terminal object \( 1 \)
- for each \( \Gamma \in \mathcal{C} \), a set \( \mathcal{C}(\Gamma) \) of families over \( \Gamma \)
- and for each \( \gamma \in \mathcal{C}(\Delta, \Gamma) \) a re-indexing function \( \_\gamma : \mathcal{C}(\Gamma) \to \mathcal{C}(\Delta) \), functorial in \( \gamma \)
A CwF is given by

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- for each $\Gamma \in \mathcal{C}$, a set $\mathcal{C}(\Gamma)$ of families over $\Gamma$
- and for each $\gamma \in \mathcal{C}(\Delta, \Gamma)$ a re-indexing function $\_\gamma : \mathcal{C}(\Gamma) \to \mathcal{C}(\Delta)$, functorial in $\gamma$

[families model types-in-context; re-indexing models substitution of terms for variables in types]
Category with Families (CwF)

A CwF is given by

- category $\mathcal{C}$ with a terminal object $1$
- for each $\Gamma \in \mathcal{C}$, a set $\mathcal{C}(\Gamma)$ of families over $\Gamma$
- for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{C}(\Gamma)$, a set $\mathcal{C}(\Gamma \vdash A)$ of elements of the family $A$ over $\Gamma$
- and for each $\gamma \in \mathcal{C}(\Delta, \Gamma)$ a re-indexing function $\_\gamma : \mathcal{C}(\Gamma \vdash A) \to \mathcal{C}(\Delta \vdash A[\gamma])$, (dependently) functorial in $\gamma$
A CwF is given by

- category $\mathcal{C}$ with a terminal object $1$
- for each $\Gamma \in \mathcal{C}$, a set $\mathcal{C}(\Gamma)$ of families over $\Gamma$
- for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{C}(\Gamma)$, a set $\mathcal{C}(\Gamma \vdash A)$ of elements of the family $A$ over $\Gamma$
- and for each $\gamma \in \mathcal{C}(\Delta, \Gamma)$ a re-indexing function
  $$\_ [\gamma] : \mathcal{C}(\Gamma \vdash A) \to \mathcal{C}(\Delta \vdash A[\gamma])$$
  (dependently) functorial in $\gamma$

[elements model terms-in-context of a given type; re-indexing models substitution of terms for variables in terms]
Category with Families (CwF)

A CwF is given by

- category $\mathcal{C}$ with a terminal object $\mathbf{1}$
- for each $\Gamma \in \mathcal{C}$, a set $\mathcal{C}(\Gamma)$ of families over $\Gamma$
- for each $\Gamma \in \mathcal{C}$ and $A \in \mathcal{C}(\Gamma)$, a set $\mathcal{C}(\Gamma \vdash A)$ of elements of the family $A$ over $\Gamma$
- comprehension structure...
Category with Families (CwF)

A CwF is given by... plus a comprehension structure:

[modelling the basic properties of the judgements of MLTT, independent of any particular type-forming constructs]
Category with Families (CwF)

A CwF is given by... plus a comprehension structure:

for each $\Gamma \in C$ and $A \in C(\Gamma)$, an object $\Gamma.A \in C$,
projection morphism $p_A \in C(\Gamma.A, \Gamma)$, generic element
$q_A \in C(\Gamma.A \vdash A[p])$ and pairing operation

$\langle \gamma, a \rangle \in C(\Delta, \Gamma.A)$

\[
\begin{align*}
\gamma \in C(\Delta, \Gamma) & \quad a \in C(\Delta \vdash A[\gamma]) \\
\langle \gamma, a \rangle & \in C(\Delta, \Gamma.A)
\end{align*}
\]

satisfying

\[
\begin{align*}
p_A \circ \langle \gamma, a \rangle & = \gamma \\
q_A[\langle \gamma, a \rangle] & = a \\
\langle \gamma, a \rangle \circ \delta & = \langle \gamma \circ \delta, a[\delta] \rangle \\
\langle p_A, q_A \rangle & = \text{id}_{\Gamma.A}
\end{align*}
\]
Overview of the CCHM model

Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod $\Gamma$, $\mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma$.

[These are models of ETT, with the identification type for $A \to \Gamma$ given by the diagonal $A \Delta \to A \times_\Gamma A$.]
Overview of the CCHM model

Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod $\Gamma$, $\mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma$. For CCHM we take $\mathcal{E} \equiv \text{Set}^{\text{C}^{\text{op}}}$ where $\text{C}$ is the small category of free, finitely generated De Morgan algebras (more on those later).
Overview of the CCHM model

- Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod $\Gamma$, $\mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma$.

- Using an interval $0, 1 : 1 \Rightarrow I$ and a subobject of cofibrant propositions $F \hookrightarrow \Omega$ in the topos $\mathcal{E}$, one defines a notion of fibration structure $\alpha \in \text{Fib}(A)$ on families $A \in \mathcal{E}(\Gamma)$, giving a new CwF $\mathcal{F}$ (based on $\mathcal{E}$) with $\mathcal{F}(\Gamma) \equiv \sum_{A \in \mathcal{E}(\Gamma)} \text{Fib}(A)$ and $\mathcal{F}(\Gamma \vdash (A, \alpha)) \equiv \mathcal{E}(\Gamma \vdash A)$. 

Overview of the CCHM model

- Every topos \( \mathcal{E} \) has an associated CwF so that families over \( \Gamma \in \mathcal{E} \) equivalent to morphisms with cod \( \Gamma \), \( \mathcal{E}(\Gamma) \simeq \mathcal{E}/\Gamma \).

- Using an interval \( 0, 1 : 1 \Rightarrow \mathbb{I} \) and a subobject of cofibrant propositions \( F \hookrightarrow \Omega \) in the topos \( \mathcal{E} \), one defines a notion of fibration structure \( \alpha \in \text{Fib}(A) \) on families \( A \in \mathcal{E}(\Gamma) \), giving a new CwF \( \mathcal{F} \) (based on \( \mathcal{E} \)) with \( \mathcal{F}(\Gamma) \equiv \sum_{A \in \mathcal{E}(\Gamma)} \text{Fib}(A) \) and \( \mathcal{F}(\Gamma \vdash (A, \alpha)) \equiv \mathcal{E}(\Gamma \vdash A) \).

- Working in the internal ETT of a topos \( \mathcal{E} \), [OP] identifies axioms on \( 0, 1 : 1 \Rightarrow \mathbb{I} \) and \( F \hookrightarrow \Omega \) that ensure we get a model of intensional MLTT:
  - fibrations are closed under \( \mathcal{E} \)'s \( \Pi, \Sigma, W, \ldots \)
  - e.g. have \( \text{Fib}(A) \to \text{Fib}(B) \to \text{Fib}(\Pi A B) \)
  - path objects \( \mathbb{I} \to \Gamma \) yield (propositional, non-truncated) identification types in \( \mathcal{F} \)
Overview of the CCHM model

- Every topos $\mathcal{E}$ has an associated CwF so that families over $\Gamma \in \mathcal{E}$ equivalent to morphisms with cod $\Gamma$, $\mathcal{E}(\Gamma) \sim \mathcal{E}/\Gamma$.

- Using an interval $0, 1 : 1 \Rightarrow \mathbb{I}$ and a subobject of cofibrant propositions $F \hookrightarrow \Omega$ in the topos $\mathcal{E}$, one defines a notion of fibration structure $\alpha \in \text{Fib}(A)$ on families $A \in \mathcal{E}(\Gamma)$, giving a new CwF $\mathcal{F}$ (based on $\mathcal{E}$) with $\mathcal{F}(\Gamma) \equiv \sum_{A \in \mathcal{E}(\Gamma)} \text{Fib}(A)$ and $\mathcal{F}(\Gamma \vdash (A, \alpha)) \equiv \mathcal{E}(\Gamma \vdash A)$.

- Working in the internal ETT of a topos $\mathcal{E}$, [OP] identifies axioms on $0, 1 : 1 \Rightarrow \mathbb{I}$ and $F \hookrightarrow \Omega$ that ensure we get a model of intensional MLTT.

- When $\mathcal{E} = \text{Set}^{\text{Cop}}$ with $\mathcal{C}$ the category of free finitely generated De Morgan algebras, [CCHM] show that Hofmann-Streicher universe construction in $\mathcal{E}$ can be extended so that $\mathcal{F}$ is a model of UTT.
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- Using an interval $0, 1 : 1 \Rightarrow \mathbb{I}$ and a subobject of cofibrant propositions $F \triangleright \Omega$ in the topos $\mathcal{E}$, one defines a notion of fibration structure $\alpha \in \text{Fib}(A)$ on families $A \in \mathcal{E}(\Gamma)$, giving a new CwF $F$ (based on $\mathcal{E}$) with $F(\Gamma) \equiv \sum_{A \in \mathcal{E}(\Gamma)} \text{Fib}(A)$ and $F(\Gamma \vdash (A, \alpha)) \equiv \mathcal{E}(\Gamma \vdash A)$.

The details are complicated!

Here I give an equivalent, “nominal” formulation of $\text{Set}^{\text{Cop}}$ as a topos of finitely supported $\mathbb{M}$-sets that may enable a simpler treatment.

Path types in the new formulation look like name abstraction sets from the theory of nominal sets.

- When $\mathcal{E} = \text{Set}^{\text{Cop}}$ with $\mathcal{C}$ the category of free finitely generated De Morgan algebras, [CCHM] show that Hofmann-Streicher universe construction in $\mathcal{E}$ can be extended so that $\mathcal{F}$ is a model of UTT.
Plan

- Motivation: the univalence axiom [HoTT]
- Overview of the Cohen-Coquand-Huber-Mörtberg presheaf model of univalent type theory [CCHM, OP]
- Toposes of $\mathbb{M}$-sets
- CCHM cubical sets as finitely supported $\mathbb{M}$-sets [Pit]
- Paths objects
- Cofibrant propositions and fibrant families
- A univalent universe [CCHM]
CwF of $\mathbb{M}$-sets, $\text{Set}^\mathbb{M}$

Fix a monoid $(\mathbb{M}, \_ \circ \_, \text{id})$.

\[
m \circ (m' \circ m'') = (m \circ m') \circ m''
\]
\[
\text{id} \circ m = m
\]
\[
m \circ \text{id} = m
\]

(w.l.o.g. $\mathbb{M}$ is a set of endofunctions)
CwF of $\mathbb{M}$-sets, $\text{Set}^{\mathbb{M}}$

Objects $\Gamma \in \text{Set}^{\mathbb{M}}$ are sets equipped with an $\mathbb{M}$-action

\[ m \in \mathbb{M}, x \in \Gamma \mapsto m \cdot x \in \Gamma \]

\[ m' \cdot (m \cdot x) = (m' \circ m) \cdot x \]

\[ \text{id} \cdot x = x \]
CwF of $\mathbb{M}$-sets, $\text{Set}^\mathbb{M}$

Objects $\Gamma \in \text{Set}^\mathbb{M}$ are sets equipped with an $\mathbb{M}$-action.

Morphisms $\gamma \in \text{Set}^\mathbb{M}(\Delta, \Gamma)$ are functions preserving the $\mathbb{M}$-action

$$m \cdot (\gamma x) = \gamma(m \cdot x)$$
CwF of $\mathbb{M}$-sets, $\text{Set}^\mathbb{M}$

Objects $\Gamma \in \text{Set}^\mathbb{M}$ are sets equipped with an $\mathbb{M}$-action.

Morphisms $\gamma \in \text{Set}^\mathbb{M}(\Delta, \Gamma)$ are functions preserving the $\mathbb{M}$-action.

Families $A \in \text{Set}^\mathbb{M}(\Gamma)$ are families of sets $(A \times \in \text{Set} \mid x \in \Gamma)$ equipped with a dependently-typed $\mathbb{M}$-action

$$m \in \mathbb{M}, a \in A \times \mapsto m \cdot a \in A(m \cdot x) \quad (x \in \Gamma)$$

$$m' \cdot (m \cdot a) = (m' \circ m) \cdot a$$

$$\text{id} \cdot a = a$$
CwF of $\mathbf{M}$-sets, $\mathbf{Set}^{\mathbf{M}}$

Objects $\Gamma \in \mathbf{Set}^{\mathbf{M}}$ are sets equipped with an $\mathbf{M}$-action.

Morphisms $\gamma \in \mathbf{Set}^{\mathbf{M}}(\Delta, \Gamma)$ are functions preserving the $\mathbf{M}$-action.

Families $A \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$ are families of sets $(A_x \in \mathbf{Set} \mid x \in \Gamma)$ equipped with a dependently-typed $\mathbf{M}$-action.

Elements $\alpha \in \mathbf{Set}^{\mathbf{M}}(\Gamma \vdash A)$ are dependent functions $\alpha \in \prod_{x \in \Gamma} A_x$ preserving the $\mathbf{M}$-action.

\[ m \cdot (\alpha x) = \alpha(m \cdot x) \]
CwF of $\mathbb{M}$-sets, $\text{Set}^\mathbb{M}$

Comprehension structure:

$$\Gamma.A \equiv \sum_{x \in \Gamma} A x$$
$$m \cdot (x, a) \equiv (m \cdot x, m \cdot a)$$
$$p_A(x, a) \equiv x$$
$$q_A(x, a) \equiv a$$
$$\langle \gamma, \alpha \rangle y \equiv (\gamma y, \alpha y)$$
CwF of $\mathbb{M}$-sets, $\text{Set}^\mathbb{M}$

$\Sigma$-types [Hof, Definition 3.15]:

given $\Gamma \in \text{Set}^\mathbb{M}$, $A \in \text{Set}^\mathbb{M}(\Gamma)$ and $B \in \text{Set}^\mathbb{M}(\Gamma.A)$, we get

$$\Sigma A B \in \text{Set}^\mathbb{M}(\Gamma)$$

with

$$(\Sigma A B) x \equiv \sum_{a \in A} x B(x, a)$$

$$m \cdot (a, b) \equiv (m \cdot a, m \cdot b)$$

etc
CwF of $\mathbb{M}$-sets, $\mathsf{Set}^{\mathbb{M}}$

**Π-types** [Hof, Definition 3.18]:

Given $\Gamma \in \mathsf{Set}^{\mathbb{M}}$, $A \in \mathsf{Set}^{\mathbb{M}}(\Gamma)$ and $B \in \mathsf{Set}^{\mathbb{M}}(\Gamma.A)$, we get

$$\Pi A B \in \mathsf{Set}^{\mathbb{M}}(\Gamma)$$

where for each $x \in \Gamma$, $(\Pi A B) x$ is the set

$$\{ f \in \Pi_{m \in \mathbb{M}} \Pi_{a \in A(m.x)} B(m \cdot x, a) \mid (\forall m, m', a) \ m' \cdot (f m a) = f(m' \circ m)(m' \cdot a) \}$$

with $\mathbb{M}$-action given by $(m' \cdot f) m a \equiv f(m \circ m') a$. 

Etc.
Topos structure of $\text{Set}^\mathbb{M}$

Limits (& colimits) are created by the forgetful functor $U : \text{Set}^\mathbb{M} \to \text{Set}$.

Subobjects of $\Gamma \in \text{Set}^\mathbb{M}$ correspond to subsets of $U\Gamma \in \text{Set}$ that are closed under the $\mathbb{M}$-action.

Subobject classifier:

$$\Omega \equiv \{ \varphi \subseteq \mathbb{M} \mid (\forall m, m') \ m \in \varphi \Rightarrow m' \circ m \in \varphi \}$$

$$m \cdot \varphi \equiv \{ m' \in \mathbb{M} \mid m' \circ m \in \varphi \}$$

so $m' \in m \cdot \varphi \iff m' \circ m \in \varphi$
Topos structure of $\text{Set}^\text{M}$

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Subobject classifier:

$$\Omega \equiv \{\varphi \subseteq \text{M} \mid (\forall m, m') \; m \in \varphi \Rightarrow m' \circ m \in \varphi\}$$
$$m \cdot \varphi \equiv \{m' \in \text{M} \mid m' \circ m \in \varphi\}$$

Truth $\top \in \text{Set}^\text{M}(1, \Omega)$ is $\top(0) \equiv \text{M}$

Classifier of $S \rightrightarrows \Gamma$ is $\chi_S \in \text{Set}^\text{M}(\Gamma, \Omega)$ where

$$\chi_S x \equiv \{m \in \text{M} \mid m \cdot x \in S\}$$
The CCHM monoid

From now on we take \( M \) to be the monoid of finitary endomorphisms of the free De Morgan algebra \( \mathbb{II} \) on a countably infinite set \( J \)
The CCHM monoid

From now on we take \( M \) to be the monoid of finitary endomorphisms of the free De Morgan algebra \( I \) on a countably infinite set \( J \).

A distributive lattice \((D, \vee, \wedge, 0, 1)\) equipped with a function
\[ d \mapsto \mathbf{1} - d \]
which is involutive
\[ \mathbf{1} - (\mathbf{1} - d) = d \]
and satisfies De Morgan’s Law
\[ \mathbf{1} - (d_1 \vee d_2) = (\mathbf{1} - d_1) \wedge (\mathbf{1} - d_2) \]
The CCHM monoid

From now on we take $\mathbb{M}$ to be the monoid of finitary endomorphisms of the free De Morgan algebra $\mathbb{I}$ on a countably infinite set $\mathbb{I}$.

We call elements of $\mathbb{I}$ cartesian directions and write them as $i, j, k, \ldots$.
The CCHM monoid

From now on we take \( M \) to be the monoid of finitary endomorphisms of the free De Morgan algebra \( I \) on a countably infinite set \( J \).

Elements of \( I \) are equivalence classes for the equational theory of De Morgan algebra of ‘De Morgan polynomials’

\[
d ::= i \mid 0 \mid 1 \mid d \lor d \mid d \land d \mid 1 - d \quad (i \in J)
\]
The CCHM monoid

From now on we take \( M \) to be the monoid of finitary endomorphisms of the free De Morgan algebra \( \mathbb{I} \) on a countably infinite set \( I \). Elements of \( \mathbb{I} \) are De Morgan algebra homomorphisms \( m : \mathbb{I} \rightarrow \mathbb{I} \) for which \( \text{dom}(m) \equiv \{ i \in I \mid m(i) \neq i \} \) is finite. (Since \( \mathbb{I} \) is the free De Morgan algebra on \( I \), \( m \) is uniquely determined as a function by its restriction to the finite set \( \text{dom}(m) \).

Notation: \((d/i) \in M\) is the homomorphism \( m \) with \( \text{dom}(m) = \{i\} \) and \( m(i) = d \).
Let $\Gamma \in \text{Set}^M$ and $x \in \Gamma$

A finite set of directions $I \subseteq \text{fin} \ J$ supports $x$ if for all $m, m' \in M$

$$((\forall i \in I) \ m_i = m'_i) \implies m \cdot x = m' \cdot x$$

(If $M$ is a group (has inverses), this is equivalent to the usual nominal sets notion of finite support.)
Finite support property

Let $\Gamma \in \text{Set}^\mathbb{M}$ and $x \in \Gamma$

A finite set of directions $I \subseteq_{\text{fin}} \mathcal{I}$ supports $x$ if for all $m, m' \in \mathbb{M}$

$((\forall i \in I) \ m_i = m'_i) \Rightarrow m \cdot x = m' \cdot x$

Lemma. $I \subseteq_{\text{fin}} \mathcal{I}$ supports $x$ iff

$((\forall i \in I) \ i \not\in I) \Rightarrow (0/i) \cdot x = x$

(iff $((\forall i \in I) \ i \not\in I) \Rightarrow (1/i) \cdot x = x$)
**Finite support property**

Let $\Gamma \in \text{Set}^M$ and $x \in \Gamma$

A finite set of directions $I \subseteq_{\text{fin}} \mathcal{I}$ supports $x$ if for all $m, m' \in \mathbb{M}$

$$((\forall i \in I) \ m_i = m'_i) \Rightarrow m \cdot x = m' \cdot x$$

**Lemma.** $I \subseteq_{\text{fin}} \mathcal{I}$ supports $x$ iff

$$\left( \forall i \in \mathcal{I} \right) \ i \notin I \Rightarrow (0/i) \cdot x = x$$

The **interval**: $\mathbb{M}$ acts on $\mathbb{I}$ via function application: $m \cdot d \equiv m d$. With respect to this action, each $d \in \mathcal{I}$ is supported by the finite set $I$ of directions occurring in some De Morgan polynomial representing $d$, since if $i \notin I$, then $(0/i)d = d$. 
De Morgan sets

The category $\text{Dms}$ of De Morgan sets is the full subcategory of $\text{Set}^\text{M}$ consisting of those $\text{M}$-sets $\Gamma$ such that every $x \in \Gamma$ possesses a finite support.
The category $\mathbf{Dms}$ of De Morgan sets is the full subcategory of $\mathbf{Set}^\mathbf{M}$ consisting of those $\mathbf{M}$-sets $\Gamma$ such that every $x \in \Gamma$ possesses a finite support.

$\mathbf{Dms}$ is closed under taking finite limits and the inclusion $\mathbf{Dms} \hookrightarrow \mathbf{Set}^\mathbf{M}$ has a right adjoint, given by

$$\Gamma \mapsto \Gamma_{fs} \equiv \{x \in \Gamma \mid x \text{ has a finite support}\}$$
De Morgan sets

The category $\mathbf{Dms}$ of De Morgan sets is the full subcategory of $\mathbf{Set}^M$ consisting of those $M$-sets $\Gamma$ such that every $x \in \Gamma$ possesses a finite support.

$\mathbf{Dms}$ is closed under taking finite limits and the inclusion $\mathbf{Dms} \hookrightarrow \mathbf{Set}^M$ has a right adjoint, given by

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Hence $\textbf{Dms}$ is a topos. In fact:

**Theorem.** (Orton, AMP) $\textbf{Dms}$ is equivalent to the presheaf topos $\textbf{Set}^{\text{C}^{\text{op}}}$ used in [CCHM].

($\text{C}^{\text{op}}$ is the category of free, finitely generated De Morgan algebras and homomorphisms.)
One can view De Morgan sets $\Gamma \in \mathbf{Dms}$ as sets whose elements depend implicitly on finitely many (named) dimensions $i, j, k, \ldots$
One can view De Morgan sets $\Gamma \in D_{m,s}$ as sets whose elements depend implicitly on finitely many (named) dimensions $i, j, k, \ldots$.

\[ x \in \Gamma \] supported by \{i, j, k\}

\[ x \rightarrow j \]
One can view De Morgan sets $\Gamma \in \mathbb{Dms}$ as sets whose elements depend implicitly on finitely many (named) dimensions $i, j, k, \ldots$, with the dependency described by the $\mathbb{M}$-action on $\Gamma$.

$$k \xrightarrow{(0/i) \cdot (0/j) \cdot x} \xrightarrow{(1/i) \cdot x} j \xrightarrow{(1/i) \cdot (0/j) \cdot (0/k) \cdot x} i$$
One can view De Morgan sets $\Gamma \in Dms$ as sets whose elements depend implicitly on finitely many (named) dimensions $i, j, k, \ldots$, with the dependency described by the $M$-action on $\Gamma$

in the [CCHM] version using $\text{Set}^{\text{Cop}}$, dependency is explicit $\rightsquigarrow$ “weakening hell”
Other toposes of interest for modelling Homotopy Type Theory can be presented (usefully?) as categories of finitely supported $M$-sets for various monoids $M$.

E.g. other variations on the notion of “cubical set”

**Theorem.** [Pit] The presheaf category on Grothendieck’s “smallest test category” (non-trivial bipointed finite sets)$^{\text{op}}$ is equivalent to the category of finitely supported $M$-sets where $M$ is the monoid of endofunctions on $\{\bot\} \cup \mathbb{Z} \cup \{\top\}$ preserving $\bot$ and $\top$. 

Other toposes of interest for modelling Homotopy Type Theory can be presented (usefully?) as categories of finitely supported $M$-sets for various monoids $M$.

E.g. other variations on the notion of “cubical set” but also simplicial sets:

**Theorem.** (Faber) The presheaf topos $\text{Set}^{\Delta^{\text{op}}}$ of simplicial sets is equivalent to the category of finitely supported $M$-sets where $M$ is the monoid of order-preserving endofunctions on

$$\{\bot \leq \cdots \leq -2 \leq -1 \leq 0 \leq 1 \leq 2 \leq \cdots \leq \top\}$$

preserving $\bot$ and $\top$. 