Constructive Initial Algebra Semantics

Andrew Pitts
joint work with Shaun Steenkamp

UNIVERSITY OF CAMBRIDGE

OWLS
21 April 2021

Constructive Initial Algebra Semantics

constructive logic

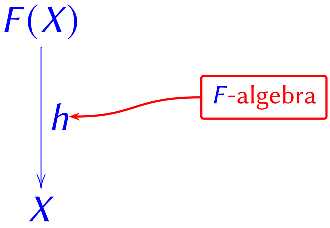
category theory
for computer science

Andrew Pitts
joint work with Shaun Steenkamp

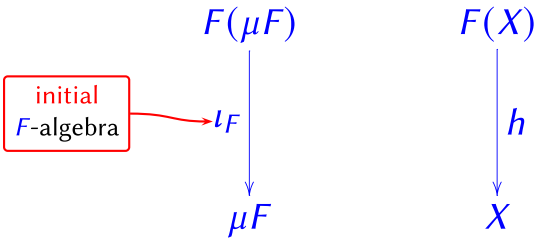
UNIVERSITY OF CAMBRIDGE

OWLS
21 April 2021

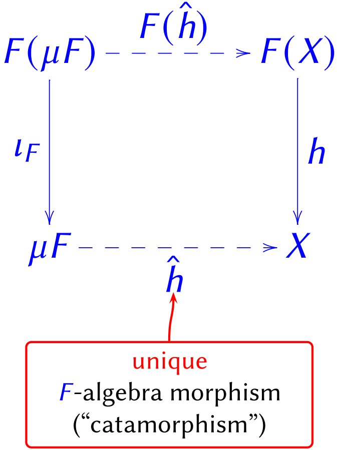
Initial algebra of an endofunctor $F : \mathbf{C} \to \mathbf{C}$



Initial algebra of an endofunctor $F : \mathbf{C} \to \mathbf{C}$



Initial algebra of an endofunctor $F : \mathbf{C} \to \mathbf{C}$



Initial algebra of an endofunctor $F : \mathbf{C} \to \mathbf{C}$

$$F(\mu F) \xrightarrow[\cong]{\iota_F} \mu F$$

For various choices of $\mathbf{C}$ and F
μF is used in the semantics of various kinds of inductive (or dually, coinductive) structures and associated (co)recursion schemes

For example...

W-types

Each "container" $(A \in \mathbf{Set}, B \in \mathbf{Set}^A)$
determines a polynomial endofunctor $F_{A,B} : \mathbf{Set} \longrightarrow \mathbf{Set}$

$$F_{A,B}(X) = \sum_{a \in A} X^{B\,a} = \{(a, f) \mid a \in A \wedge f \in X^{B\,a}\}$$

W-types

Each "container" $(A \in \mathbf{Set}, B \in \mathbf{Set}^A)$
determines a polynomial endofunctor $F_{A,B} : \mathbf{Set} \longrightarrow \mathbf{Set}$

$$F_{A,B}(X) = \sum_{a \in A} X^{B\,a} = \{(a, f) \mid a \in A \wedge f \in X^{B\,a}\}$$

$\mu F_{A,B}$ is the set $\boxed{W_{A,B}}$ of well-founded algebraic terms generated from a signature of operation symbols sup_a $(a \in A)$, each with (*possibly infinitary*) arity $B\,a$:

$$t \in W_{A,B} ::= \mathsf{sup}_a f \quad \text{where} \quad f \in (W_{A,B})^{B\,a}$$

W-types

Each "container" $(A \in \mathbf{Set}, B \in \mathbf{Set}^A)$
determines a polynomial endofunctor $F_{A,B} : \mathbf{Set} \longrightarrow \mathbf{Set}$

$$F_{A,B}(X) = \sum_{a \in A} X^{B\,a} = \{(a, f) \mid a \in A \wedge f \in X^{B\,a}\}$$

$\mu F_{A,B}$ is the set $\boxed{W_{A,B}}$ of well-founded algebraic terms generated from a signature of operation symbols sup_a $(a \in A)$, each with (*possibly infinitary*) arity $B\,a$:

$$t \in W_{A,B} ::= \mathsf{sup}_a f \quad \text{where} \quad f \in (W_{A,B})^{B\,a}$$

Given $h : (\sum_{a \in A} X^{B\,a}) \to X$, then $\hat{h} : W_{A,B} \to X$ is uniquely determined by the recursion equation

$$\boxed{\hat{h}(\mathsf{sup}_a f) = h(a, \lambda b.\, \hat{h}(f\ b))}$$

Classical construction of μF

Assume $\mathbf{C}$ has colimits of shape $(\alpha, <)$ for any ordinal α, and hence in particular an initial object 0.

Iterate $F : \mathbf{C} \to \mathbf{C}$ transfinitely, starting at 0

$$0 \xrightarrow{i_0} F0 \xrightarrow{i_1} F^2 0 \xrightarrow{i_2} \cdots \to F^{\alpha} 0 \xrightarrow{i_\alpha} F^{\alpha^+} 0 \to \cdots$$

$$F^{\alpha}0 = \begin{cases} 0 & \text{if } \alpha = 0 \\ F(F^{\beta}0) & \text{if } \alpha = \beta^+ \text{ is a successor ordinal} \\ \mathrm{colim}_{\beta<\lambda} F^{\beta}0 & \text{if } \alpha = \lambda \text{ is a limit ordinal} \end{cases}$$

$$i_\alpha = \begin{cases} \text{unique, by initiality of } 0 & \text{if } \alpha = 0 \\ F(i_\beta) & \text{if } \alpha = \beta^+ \\ \text{use univ. prop. of } \mathrm{colim}_{\beta<\lambda} & \text{if } \alpha = \lambda \end{cases}$$

Classical construction of μF

Assume $\mathbf{C}$ has colimits of shape $(\alpha, <)$ for any ordinal α, and hence in particular an initial object 0.

Iterate $F : \mathbf{C} \to \mathbf{C}$ transfinitely, starting at 0

$$0 \xrightarrow{i_0} F0 \xrightarrow{i_1} F^2 0 \xrightarrow{i_2} \cdots \to F^\alpha 0 \xrightarrow{i_\alpha} F^{\alpha^+} 0 \to \cdots$$

Theorem [Adamek, 1974] If F preserves colimits of shape $(\kappa, <)$ for some limit ordinal κ (that is, i_κ is an isomorphism), then it has initial algebra

$$\mu F = F^\kappa 0 = \mathrm{colim}_{\alpha < \kappa} F^\alpha 0$$

(with algebra structure given by $F(F^\kappa 0) = F^{\kappa^+} \xrightarrow[\cong]{(i_\kappa)^{-1}} F^\kappa 0$)

Classical construction of μF

Assume $\mathbf{C}$ has colimits of shape $(\alpha, <)$ for any ordinal α, and hence in particular an initial object 0.

Law of Excluded Middle (LEM)
$\forall p.\ p \vee \neg p$
is needed for the usual theory of ordinal numbers

…ting at 0

$$\cdots \to F^{\alpha}0 \xrightarrow{i_\alpha} F^{\alpha^+}0 \longrightarrow \cdots$$

$$F^{\alpha}0 = \begin{cases} 0 & \text{if } \alpha = 0 \\ F(F^{\beta}0) & \text{if } \alpha = \beta^+ \text{ is a successor ordinal} \\ \mathrm{colim}_{\beta<\lambda} F^{\beta}0 & \text{if } \alpha = \lambda \text{ is a limit ordinal} \end{cases}$$

$$i_\alpha = \begin{cases} \text{unique, by initiality of } 0 & \text{if } \alpha = 0 \\ F(i_\beta) & \text{if } \alpha = \beta^+ \\ \text{use univ. prop. of } \mathrm{colim}_{\beta<\lambda} & \text{if } \alpha = \lambda \end{cases}$$

Classical construction of μF

Assume $\mathbf{C}$ has colimits of shape $(\alpha, <)$ for any ordinal α,

Without some form of choice principle
there won't be many such F

$$0 \xrightarrow{i_0} F0 \xrightarrow{i_1} F^2 0 \xrightarrow{i_2} \cdots \longrightarrow F^\alpha 0 \xrightarrow{i_\alpha} F^{\alpha^+} 0 \longrightarrow \cdots$$

Theorem [Adamek, 1974] If F preserves colimits of shape $(\kappa, <)$ for some limit ordinal κ (that is, i_κ is an isomorphism), then it has initial algebra

$$\mu F = F^\kappa 0 = \mathrm{colim}_{\alpha < \kappa} F^\alpha 0$$

(with algebra structure given by $F(F^\kappa 0) = F^{\kappa^+} \xrightarrow[\cong]{(i_\kappa)^{-1}} F^\kappa 0$)

$\mathbf{C} = \mathbf{Set}$ (ZFC sets) **case of the theorem:**
$F : \mathbf{Set} \to \mathbf{Set}$ has an initial algebra if it preserves κ colimits for some limit ordinal κ.

Fact: for a polynomial endofunctor $F_{A,B}(X) = \sum_{a \in A} X^{B\,a}$,
if we take κ big enough so that

any $B\,a$-indexed family of ordinals $<\kappa$
has an upper bound $<\kappa$

then $F_{A,B} : \mathbf{Set} \to \mathbf{Set}$ preserves κ colimits.

$\mathbf{C} = \mathbf{Set}$ (ZFC sets) case of the theorem:
$F : \mathbf{Set} \to \mathbf{Set}$ has an initial algebra if it preserves κ colimits for some limit ordinal κ.

Fact: for a polynomial endofunctor $F_{A,B}(X) = \sum_{a \in A} X^{B\,a}$, if we take κ big enough so that

any $B\,a$-indexed family of ordinals $<\kappa$
has an upper bound $<\kappa$

then $F_{A,B} : \mathbf{Set} \to \mathbf{Set}$ preserves κ colimits.

Axiom of Choice (AC) can be used to prove this fact

Colimits in **Set** are given by quotients: $\mathrm{colim}_{\alpha<\kappa} D_\alpha = (\sum_{\alpha<\kappa} D_\alpha) /{\sim}$

To get from a function $f \in (\mathrm{colim}_{\alpha<\kappa} F^{\alpha}_{A,B} 0)^{B\,a}$

to an element of $\mathrm{colim}_{\alpha<\kappa}((F^{\alpha}_{A,B} 0)^{B\,a})$ when $B\,a$ is infinite,

we can use AC to pick a representative in the $\sim$-equivalence class $f(b)$ for each $b \in B\,a$; and then…

Constructive Logic – why bother?

- philosophy
- computational content

Constructive Logic – why bother?

- philosophy
- computational content
- utility: internal languages of many varieties of category have a constructive nature

Elementary Toposes

Simple definition

topos = category with finite limits, exponentials and a generic monomorphism

Elementary Toposes

Simple definition

topos = category with finite limits, exponentials and a generic monomorphism

but rich in properties, because there is a Lawvere-style category↔theory correspondence between them and theories in

intuitionistic higher-order logic with extensionality

or better still (if we restrict attention to toposes with a natural number object and universes)

extensional Martin-Löf Type Theory with an impredicative universe of propositions

Constructive Logic – why bother?

- philosophy
- computational content
- utility: internal languages of many varieties of category have a constructive nature

For example: use of internal type theory of a topos to describe the construction of the Coquand-Cohen-Huber-Mörtberg model of univalent foundations within a presheaf topos of cubical sets.

[I. Orton-AMP, *Axioms for Modelling Cubical Type Theory in a Topos*, LMCS 14(2018)]

[D. R. Licata, I. Orton, AMP and B. Spitters, *Internal Universes in Models of Homotopy Type Theory*, FSCD 2018]

Constructive Logic – why bother?

- philosophy
- computational content
- utility: internal languages of many varieties of category have a constructive nature

Is there a version of Adamek's theorem that works in any topos with NNO and universes?

Constructive Adamek - step 1

Avoid zero/successor/limit case distinction in

$$F^0 0 = 0$$
$$F^{\alpha^+} 0 = F(F^\alpha 0)$$
$$F^\lambda 0 = \mathrm{colim}_{\alpha<\lambda} F^\alpha$$

by using an "inflationary" iteration instead

$$\mu_\alpha F = \mathrm{colim}_{\beta<\alpha} F(\mu_\beta F)$$

Constructive Adamek - step 1

Avoid zero/successor/limit case distinction in

$$F^0 0 = 0$$
$$F^{\alpha^+} 0 = F(F^\alpha 0)$$
$$F^\lambda 0 = \mathrm{colim}_{\alpha<\lambda} F^\alpha$$

by using an "inflationary" iteration instead

$$\mu_\alpha F = \mathrm{colim}_{\beta<\alpha} F(\mu_\beta F)$$

and replace use of ordinals by the elements of any

Definition. A size is a set κ equipped with a binary relation $<$ which is transitive, *directed*

$$\exists \alpha \in \kappa \wedge \forall \alpha, \alpha' \in \kappa . \exists \beta \in \kappa .\ \alpha < \beta \wedge \alpha' < \beta$$

and *well-founded*

$$\forall S \subseteq \kappa . (\forall \alpha . (\forall \beta < \alpha .\ \beta \in S) \Rightarrow \alpha \in S) \Rightarrow \forall \alpha .\ \alpha \in S$$

(sizes play the role of limit ordinals in the constructive theory)

Constructive Adamek - step 1

Avoid zero/successor/limit case distinction in

$$F^0 0 = 0$$
$$F^{\alpha^+} 0 = F(F^\alpha 0)$$
$$F^\lambda 0 = \mathrm{colim}_{\alpha<\lambda} F^\alpha$$

by using an "inflationary" iteration instead

Lemma. Constructively, assuming $\mathbf{C}$ has small colimits, given any endofunctor $F : \mathbf{C} \to \mathbf{C}$ and size $(\kappa, <)$, there are objects $\mu_\alpha F \in \mathbf{C}$ for each $\alpha \in \kappa$ satisfying

$$\mu_\alpha F = \mathrm{colim}_{\beta<\alpha} F(\mu_\beta F)$$

(just need transitivity and well-foundedness of $<$, but not directedness, to construct $(\mu_\alpha F \mid \alpha \in \kappa)$)

Constructive Adamek - step 1

Avoid zero/successor/limit case distinction in

$$F^0 0 = 0$$
$$F^{\alpha^+} 0 = F(F^\alpha 0)$$
$$F^\lambda 0 = \mathrm{colim}_{\alpha<\lambda} F^\alpha$$

by using an "inflationary" iteration instead

Lemma. Constructively, assuming $\mathbf{C}$ has small colimits, given any endofunctor $F : \mathbf{C} \to \mathbf{C}$ and size $(\kappa, <)$, there are objects $\mu_\alpha F \in \mathbf{C}$ for each $\alpha \in \kappa$ satisfying

$$\mu_\alpha F = \mathrm{colim}_{\beta<\alpha} F(\mu_\beta F)$$

Theorem. Constructively, if $\mathbf{C}$ has small colimits and $F : \mathbf{C} \to \mathbf{C}$ preserves colimits of size $(\kappa, <)$, then it has initial algebra $\mu F = \mathrm{colim}_{\alpha\in\kappa} \mu_\alpha F$.

(proof uses directedness of $<$, which implies in particular that for each $\alpha \in \kappa$ there is $\alpha^+ \in \kappa$ with $\alpha < \alpha^+$)

Constructive Adamek - step 2

Theorem. Constructively, if $\mathbf{C}$ has small colimits and $F : \mathbf{C} \to \mathbf{C}$ preserves colimits of some size κ, then it has initial algebra given by taking the colimit of the κ-indexed inflationary iteration of F.

Are there (m)any such κ and F?

$\mathbf{C} = \mathbf{Set}$ (ZFC sets) **case of the theorem:**
$F : \mathbf{Set} \to \mathbf{Set}$ has an initial algebra if it preserves κ colimits for some limit ordinal κ.

Fact: for a polynomial endofunctor $F_{A,B}(X) = \sum_{a \in A} X^{B\,a}$, if we take κ big enough so that

any $B\,a$-indexed family of ordinals $<\kappa$
has an upper bound $<\kappa$

then $F_{A,B} : \mathbf{Set} \to \mathbf{Set}$ preserves κ colimits.

Suggests that for each signature $\Sigma = (A, B)$
we need to be able to find a size $(\kappa, <)$
that has $B\,a$-indexed $<$-upper bounds, for all $a \in A$

Recall that a size is a set κ with a transitive, directed and well-founded binary relation $<$

Given a signature $\Sigma = (A : \mathbf{Set}, B : \mathbf{Set}^A)$, say that a size $(\kappa, <)$ is Σ-filtered if

> for all $a \in A$, every $B\,a$-indexed family $(f\,b \in \kappa \mid b \in B\,a)$ has a $<$-upper bound in κ.

Theorem. For every Σ, there is a Σ-filtered size.

Proof uses W-types endowed with Paul Taylor's "plump" order…

Plump order

Recall that $W_{A,B} = \{\mathsf{sup}_a f \mid a \in A \wedge f \in (W_{A,B})^{B\,a}\}$ is the set of well-founded algebraic terms generated from a signature $(A \in \mathbf{Set}, B \in \mathbf{Set}^A)$ of operation symbols sup_a of arity $B\,a$ (for each $a \in A$)

The plump well-order (Paul Taylor, JSL 1996) $<$ on $W_{A,B}$ is mutually inductively defined with a pre-order $\leq$

$$\frac{\exists b.\ t \leq f\ b}{t < \mathsf{sup}_a f} \qquad \frac{\forall b.\ f\ b < t}{\mathsf{sup}_a f \leq t}$$

Plump order

Recall that $W_{A,B} = \{\sup_a f \mid a \in A \wedge f \in (W_{A,B})^{B\,a}\}$ is the set of well-founded algebraic terms generated from a signature $(A \in \mathbf{Set}, B \in \mathbf{Set}^A)$ of operation symbols $\sup_a$ of arity $B\,a$ (for each $a \in A$)

The plump well-order (Paul Taylor, JSL 1996) $<$ on $W_{A,B}$ is mutually inductively defined with a pre-order $\leq$

$$\frac{\exists b.\ t \leq f\ b}{t < \sup_a f} \qquad \frac{\forall b.\ f\ b < t}{\sup_a f \leq t}$$

$<$ is always transitive and well-founded

and we have $f\ b < \sup_a f$ (because $\leq$ is provably reflexive)

so in particular, if there are $a_0, a_2 \in A$ with $B\,a_0 = \emptyset$ and $B\,a_2 = 1 + 1$, then $<$ is directed – in which case $(W_{A,B}\,, <)$ is a size.

Recall that a size is a set κ with a transitive, directed and well-founded binary relation $<$

Given a signature $\Sigma = (A : \mathbf{Set}, B : \mathbf{Set}^A)$, say that a size $(\kappa, <)$ is Σ-filtered if

> for all $a \in A$, every $B\,a$-indexed family $(f\,b \in \kappa \mid b \in B\,a)$ has a $<$-upper bound in κ.

Theorem. For every Σ, there is a Σ-filtered size.

Proof uses W-types endowed with Paul Taylor's "plump" order.

Given $\Sigma = (A, B)$, we can take $\kappa = W_{A',B'}$ with its plump order, where

$$
\begin{aligned}
A' &\triangleq A \uplus \{0, 2\} \\
B'a &\triangleq B\,a \\
B'0 &\triangleq \emptyset \\
B'2 &\triangleq 1 + 1
\end{aligned}
$$

Constructive Adamek - step 2

Theorem. Constructively, if $\mathbf{C}$ has small colimits and $F : \mathbf{C} \to \mathbf{C}$ preserves colimits of some size κ, then it has initial algebra given by taking the colimit of the κ-indexed inflationary iteration of F.

Are there (m)any such κ and F?

Definition. A functor $F : \mathbf{C} \to \mathbf{D}$ between cocomplete categories is sized if it preserves colimits of Σ-filtered sizes, for some signature Σ.

Constructive Adamek - step 2

Theorem. Constructively, if $\mathbf{C}$ has small colimits and $F : \mathbf{C} \to \mathbf{C}$ preserves colimits of some size κ, then it has initial algebra given by taking the colimit of the κ-indexed inflationary iteration of F.

Are there (m)any such κ and F?

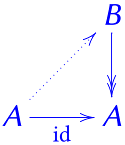
Definition. A functor $F : \mathbf{C} \to \mathbf{D}$ between cocomplete categories is sized if it preserves colimits of Σ-filtered sizes, for some signature Σ.

Some constructively valid closure properties for sized functors:

- constant functors
- quotients (coequalizers of pairs of natural transformations)
- $\sum$ (I-indexed coproducts, for any $I \in \mathbf{Set}$)
- What about $\prod$? (I-indexed products for any $I \in \mathbf{Set}$)

WISC

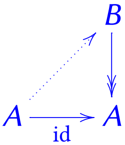
Assuming AC, for all $A \in \mathbf{Set}$



any surjection to A splits

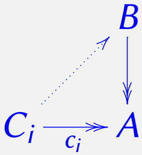
WISC

Assuming AC, for all $A \in \mathbf{Set}$



any surjection to A splits

WISC axiom [van den Berg, Moerdijk, Palmgren, Streicher]
weakens AC to merely assume that for each A there is a Set of surjections ("Covers") $\left\{ C_i \xrightarrow[c_i]{} A \mid i \in I \right\}$
which is Weakly Initial



WISC

ZFC **Set** satisfies WISC

If any elementary topos $\mathcal{E}$ satisfies WISC, so do toposes of (pre)sheaves and realizability toposes built from $\mathcal{E}$

[B. van den Berg & I. Moerdijk, J. Math. Logic, 2014]

But there are toposes not satisfying WISC

[D.M. Roberts, Studia Logica, 2015]

WISC

ZFC **Set** satisfies WISC

If any elementary topos $\mathcal{E}$ satisfies WISC, so do toposes of (pre)sheaves and realizability toposes built from $\mathcal{E}$

[B. van den Berg & I. Moerdijk, J. Math. Logic, 2014]

But there are toposes not satisfying WISC

[D.M. Roberts, Studia Logica, 2015]

Following a suggestion of Andrew Swan, we proved:

Theorem. In any elementary topos $\mathcal{E}$ with NNO and universes $\mathbf{Set}_n$ satisfying WISC, if $I \in \mathbf{Set}_n$ and $(F_i : \mathbf{Set}_n \to \mathbf{Set}_n \mid i \in I)$ are sized functors, then so is their product $\prod_{i \in I} F_i : \mathbf{Set}_n \to \mathbf{Set}_n$.

Constructive Adamek - step 2

As a corollary we get that in any topos
with NNO and universes $\mathcal{E}_n$ satisfying WISC,
we can construct initial algebras for
Gylterud's symmetric containers

$$F_{\mathbf{G},B}(X) \triangleq \mathsf{colim}_{g \in \mathbf{G}}\, X^{B\,g}$$

where $\mathbf{G}$ is a groupoid in $\mathcal{E}_n$ and $B : \mathbf{G}^{\mathrm{op}} \to \mathcal{E}_n$ a functor

Examples include infinitary multisets and infinite unordered branching trees

- constant functors
- quotients (coequalizers of pairs of natural transformations)
- $\sum$ (I-indexed coproducts, for any $I \in \mathbf{Set}$)
- $\prod$ (I-indexed products for any $I \in \mathbf{Set}$) provided WISC holds

Conclusions/Questions

- Plumply ordered W-types + WISC seem a useful constructive substitute for classical ordinal numbers, for some purposes.

  For further applications of the method see
  [M.P. Fiore, AMP & S.C. Steenkamp,
  Quotients, Inductive Types and Quotient Inductive Types,
  `arXiv:2101.02994`]

Conclusions/Questions

- Plumply ordered W-types + WISC seem a useful constructive substitute for classical ordinal numbers, for some purposes.

  How much of the classical theory of accessible categories survives this kind of constructivisation?

Conclusions/Questions

- Plumply ordered W-types + WISC seem a useful constructive substitute for classical ordinal numbers, for some purposes.

  How much of the classical theory of accessible categories survives this kind of constructivisation?

- The use of "inflationary" iteration ($\mu_\alpha F = \mathsf{colim}_{\beta<\alpha} F(\mu_\beta F)$) was suggested to me by the way sized types are used in Agda.

Conclusions/Questions

- Plumply ordered W-types + WISC seem a useful constructive substitute for classical ordinal numbers, for some purposes.

  How much of the classical theory of accessible categories survives this kind of constructivisation?

- The use of "inflationary" iteration ($\mu_\alpha F = \mathsf{colim}_{\beta<\alpha} F(\mu_\beta F)$) was suggested to me by the way sized types are used in Agda.

  Agda's sized types (although currently logically inconsistent!) are especially useful in connection with coinductively defined record types. Can the techniques described here be usefully applied to get constructive results about final coalgebras (via $\nu_\alpha F = \lim_{\beta<\alpha} F(\nu_\beta F)$)?

Conclusions/Questions

- Plumply ordered W-types + WISC seem a useful constructive substitute for classical ordinal numbers, for some purposes.

  How much of the classical theory of accessible categories survives this kind of constructivisation?

- The use of "inflationary" iteration ($\mu_\alpha F = \mathrm{colim}_{\beta<\alpha} F(\mu_\beta F)$) was suggested to me by the way sized types are used in Agda.

  Agda's sized types (although currently logically inconsistent!) are especially useful in connection with coinductively defined record types. Can the techniques described here be usefully applied to get constructive results about final coalgebras (via $\nu_\alpha F = \lim_{\beta<\alpha} F(\nu_\beta F)$)?

- Is the use of WISC really necessary? (cf. Agda's sized types)

Conclusions/Questions

- Plumply ordered W-types + WISC seem a useful constructive substitute for classical ordinal numbers, for some purposes.

  How much of the classical theory of accessible categories survives this kind of constructivisation?

- The use of "inflationary" iteration ($\mu_\alpha F = \mathrm{colim}_{\beta<\alpha} F(\mu_\beta F)$) was suggested to me by the way sized types are used in Agda.

  Agda's sized types (although currently logically inconsistent!) are especially useful in connection with coinductively defined record types. Can the techniques described here be usefully applied to get constructive results about final coalgebras (via $\nu_\alpha F = \lim_{\beta<\alpha} F(\nu_\beta F)$)?

- Is the use of WISC really necessary? (cf. Agda's sized types)

Thank you for your attention!