

LAX DESCENT FOR ESSENTIAL SURJECTIONS

GENERAL AIM: Find and use "exactness properties" of doctrines in Categorical Logic.

PARTICULAR EXAMPLE: Lax descent in Geometric Logic

SLIDE (1)

Given $f: \mathcal{F} \rightarrow \mathcal{E}$ in $\mathbf{GTOP}$ (= 2-cat. of Grothendieck toposes), form

$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{p_1} & \mathcal{F} \\ p_0 \downarrow & \lambda \Rightarrow & \downarrow f \\ \mathcal{F} & \xrightarrow{f} & \mathcal{E} \end{array}$$

comma $\square$

and

$$\begin{array}{ccc} \mathcal{F}_2 & \xrightarrow{p_2} & \mathcal{F}_1 \\ p_{01} \downarrow & \varphi_1 \cong & \downarrow p_0 \\ \mathcal{F}_1 & \xrightarrow{p_1} & \mathcal{F} \end{array} ;$$

bipullback $\square$

plus associated

diagonal $d: \mathcal{F} \rightarrow \mathcal{F}_1$, $\delta_0: p_0 d \cong 1_{\mathcal{F}}$, $\delta_1: p_1 d \cong 1_{\mathcal{F}}$; $f \delta_1 \circ \lambda d = f \delta_0$;

projection $p_{02}: \mathcal{F}_2 \rightarrow \mathcal{F}_1$, $\varphi_0: p_0 p_{02} \cong p_0 p_{01}$, $\varphi_2: p_1 p_{02} \cong p_1 p_{01}$; $f \varphi_2^{-1} \lambda p_{02} f \varphi_1 \lambda p_{01} f \varphi_0 = \lambda p_{02}$

SLIDE (2)

Get $\mathcal{C}_f: \mathcal{F}_2 \rightrightarrows \mathcal{F}_1 \rightrightarrows \mathcal{F}$ the lax kernel of f

[a pseudocategory object in $\mathbf{GTOP}$]

Category of descent data $\text{Des}(\mathcal{C}_f)$ $\left\{ \begin{array}{l} \bullet \text{ objects } (Y, a): p_0^* Y \xrightarrow{a} p_1^* Y \text{ st} \\ (\delta_1)_Y \cdot d^* a \cdot (\delta_0)_Y^{-1} = 1_Y \text{ (unit condition)} \\ (\varphi_2)_Y^{-1} p_{12}^* a (\varphi_1)_Y p_{01}^* a (\varphi_0)_Y = p_{02}^* a \text{ (cocycle condition)} \\ \bullet \text{ arrows } \eta: (Y, a) \rightarrow (Y', a'): \eta: Y \rightarrow Y' \text{ st} \\ a' \cdot p_0^* \eta = p_1^* \eta \cdot a \end{array} \right.$

Comparison functor $K: \mathcal{E} \rightarrow \text{Des}(\mathcal{C}_f)$
 $x \mapsto (x, \lambda_x)$

Given : $\mathcal{F} \xrightarrow{f} \mathcal{E}$ geometric morphism between Grothendieck toposes

form :
$$\begin{array}{ccc} \mathcal{F}_1 & \xrightarrow{p_1} & \mathcal{F} \\ p_0 \downarrow & \xRightarrow{\lambda} & \downarrow f \\ \mathcal{F} & \xrightarrow{f} & \mathcal{E} \end{array}$$
 comma square in GTOP

plus :
$$\begin{array}{ccc} \mathcal{F}_2 & \xrightarrow{p_{12}} & \mathcal{F}_1 \\ p_{01} \downarrow & \xRightarrow{\varphi_2} & \downarrow p_0 \\ \mathcal{F}_1 & \xrightarrow{p_1} & \mathcal{F} \end{array}$$
 pullback square in GTOP

together with
3rd projection $p_{02}: \mathcal{F}_2 \rightarrow \mathcal{F}_1$ + $\left\{ \begin{array}{l} \varphi_2: p_1 p_{02} \cong p_1 p_{12} \\ \varphi_0: p_0 p_{02} \cong p_0 p_{01} \end{array} \right.$ so that

& diagonal $d: \mathcal{F} \rightarrow \mathcal{F}_1$ + $\left\{ \begin{array}{l} \delta_0: p_0 d \cong 1 \\ \delta_1: p_1 d \cong 1 \end{array} \right.$ so that

$$\begin{array}{ccc} & f\varphi_0 & \\ & \cong \downarrow & \lambda p_{01} \\ \lambda p_{02} \downarrow & \cong \downarrow & \cong \downarrow f\varphi_1 \\ & \cong \downarrow & \lambda p_{12} \\ & f\varphi_2 & \text{commutes} \end{array}$$

$$\begin{array}{ccc} & f\delta_0 & \\ & \cong \downarrow & \\ \lambda d \downarrow & \cong \downarrow & \\ & f\delta_1 & \text{commutes.} \end{array}$$

$$\mathbb{C}_f : \mathcal{F}_2 \rightrightarrows_{p_{ij}} \mathcal{F}_1 \rightrightarrows_{p_i} \mathcal{F} \quad \text{pseudo category object in } \mathbf{Grp}$$

Category of descent data for $\mathbb{C}_f$: $\text{Des}(\mathbb{C}_f)$

- objects (Y, a) where $Y \in \mathcal{F}$, $a : p_0^* Y \rightarrow p_1^* Y$ in $\mathcal{F}_1$
 satisfying $(\delta_1)_Y \circ d^*(a) \circ (\delta_0)_Y^{-1} = 1_Y$ (counit condition)
 and $(\varphi_2)_Y^{-1} \circ p_{12}^*(a) \circ (\varphi_1)_Y \circ p_{01}^*(a) \circ (\varphi_0)_Y = p_{02}^*(a)$ (cocycle condition)
- arrows $y : (Y, a) \rightarrow (Y', a')$ where $y : Y \rightarrow Y'$ in $\mathcal{F}$
 satisfies $a' \circ p_0^*(y) = p_1^*(y) \circ a$

Comparison functor : $K : \mathcal{E} \rightarrow \text{Des}(\mathbb{C}_f)$

given on objects by : $X \mapsto (X, \lambda_X)$

(“Lax Descent in geometric logic”)

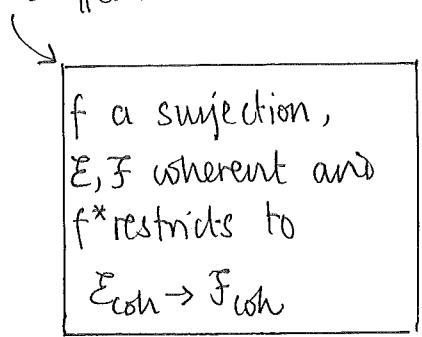
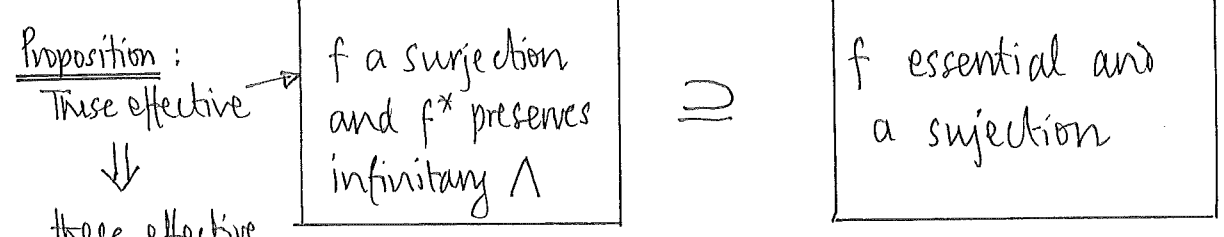
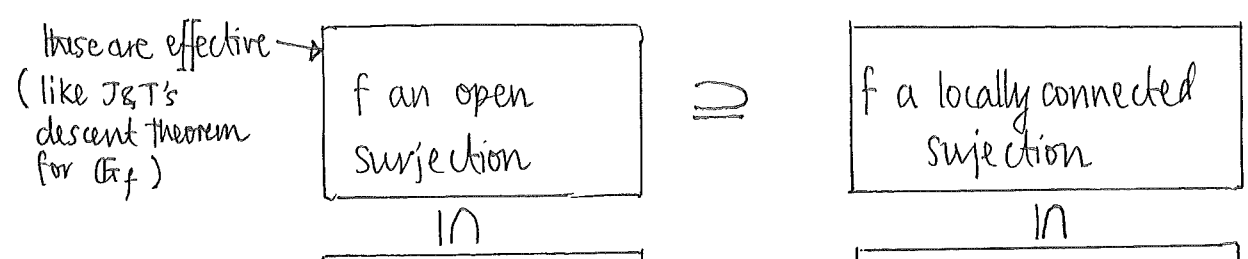
QUESTION: For which f is $K: E \rightarrow \text{Des}(C_f)$ an equivalence?

[Also have “kernel of f ” C_f , a pseudogroupoid in GTOP , plus $K: E \rightarrow \text{Des}(C_f)$. Easy to see that

K for C_f full & faithful $\Rightarrow K$ for C_f full & faithful

But ess. surjective $\nRightarrow$ ess. surjective.]

Definition: For brevity, say f is effective if $K: E \rightarrow \text{Des}(C_f)$ is an equivalence.



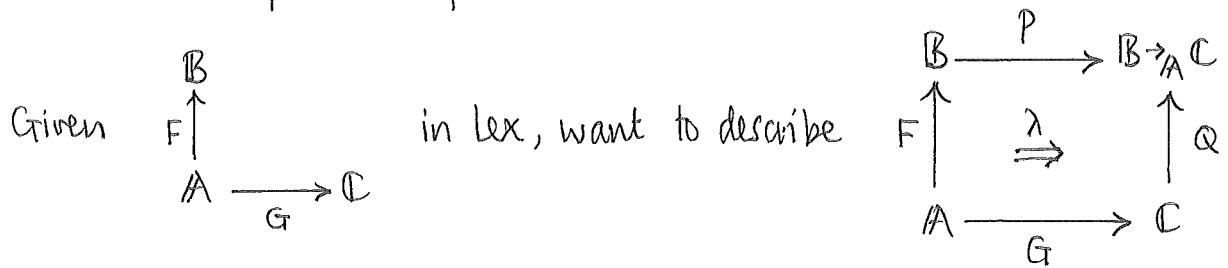
effectivity of these is solely a property of pretoposes (if it holds).

Theorem: These are effective

Will indicate the proof of this.

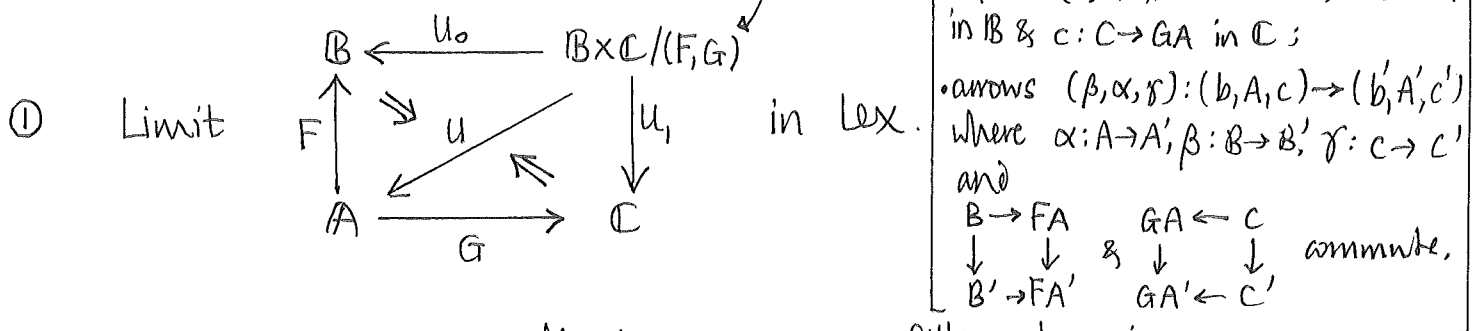
EXPLICIT CONSTRUCTION OF COCOMMA SQUARES IN LEX

Lex = 2-category of small categories which have and functors which preserve finite limits.

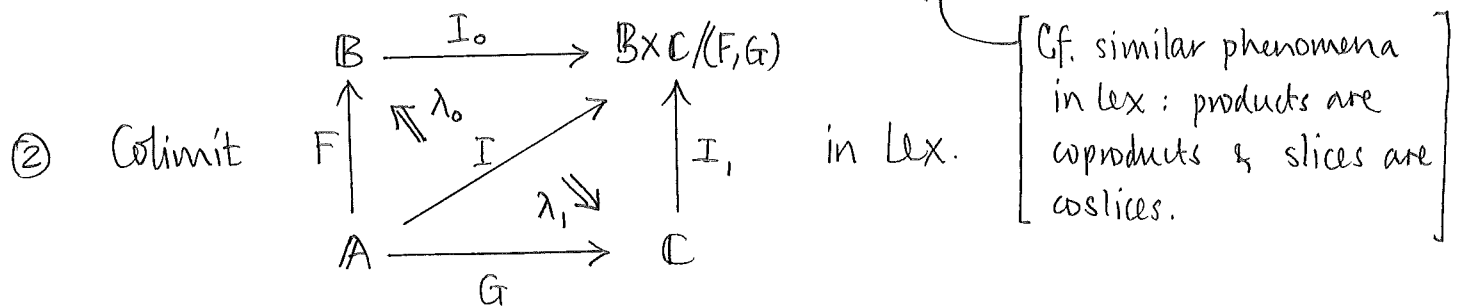


cocomma $\square$.

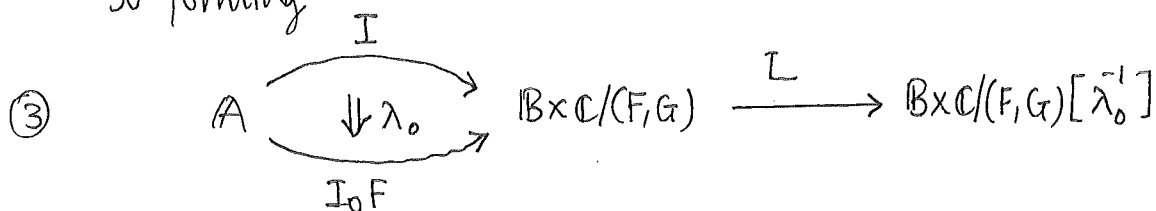
Construction :



u_0, u, u_1 have right adjoints I_0, I, I_1 , fitting to give



So forming



coinverter diagram in Lex,

then necessarily

(4)

$$\begin{array}{ccc}
 B & \xrightarrow{LI_0} & B \times C / (F, G) [\lambda_0^{-1}] \\
 \uparrow F & \xRightarrow{L\lambda_1 \circ (L\lambda_0)^{-1}} & \uparrow LI_1 \\
 A & \xrightarrow{G} & C
 \end{array}$$

is a cocomma square in Lex.

is a comonad square in Lex.

⑤ Claim that in ③ can take L to be $B \times C / (F, G) \xrightarrow{P_\Sigma} B \times C / (F, G) [\Sigma^{-1}]$ where Σ is the right calculus of fractions consisting of morphisms

$$[b, A, c] \xrightarrow{(\beta, \alpha, \gamma)} (b', A', c')$$

where

$$\begin{array}{ccc}
 B & \xrightarrow[\cong]{\beta} & B' \\
 b \downarrow & & \downarrow b' \\
 FA & \xrightarrow{Fox} & FA'
 \end{array}$$

β is an iso in $\mathcal{B}$

and

$$\begin{array}{ccc} GA & \xrightarrow{G\alpha} & GA' \\ \uparrow c & & \uparrow c' \\ C & \xrightarrow{\gamma} & C' \end{array}$$

is a pullback $\square$ in $\mathcal{C}$.

[Proof of claim: on the one hand, components of $\lambda_0 \subseteq \Sigma$;
conversely, if $(\beta, \alpha, \gamma) \in \Sigma$ then $\exists$ diagram

$$\begin{array}{ccccc}
 & & \xrightarrow{(\beta, \alpha, \gamma)} & & \\
 & \downarrow & & \downarrow & \\
 (\lambda_0)_A & \xleftarrow{\quad} & & \xrightarrow{\quad} & (\lambda_0)_{A'} \\
 & \downarrow & \xrightarrow{\quad} & \downarrow & \\
 & & \xrightarrow{\quad} & &
 \end{array}$$

from which it follows that $L: B \times C / (F, G) \rightarrow B \times C / (F, G) [\bar{\lambda}_0^{-1}]$ inverts all the morphisms in Σ .]

Thus have given an explicit construction of $B \rightarrow_A C$
as an object of Cat rather than of Lex.

CONSEQUENCES OF THE CONSTRUCTION OF $B \rightarrow_A C$

1. Every object X of $B \rightarrow_A C$ can be presented as a pullback of the form

$$\begin{array}{ccc} X & \xrightarrow{\quad} & QC \\ \downarrow \lrcorner & & \downarrow Q_C \\ PB & \xrightarrow[P_b]{} PFA \xrightarrow[\lambda_A]{} & QGA \end{array}$$

where $A \in \mathcal{A}$, $b: B \rightarrow FA$ in B & $c: C \rightarrow GA$ in C .

2. [Interpolation] For $B \in B$, $C \in C$, every arrow $k: PB \rightarrow QC$ in $B \rightarrow_A C$ factors as

$$k = PB \xrightarrow{P_b} PFA \xrightarrow{\lambda_A} QGA \xrightarrow{Q_C} QC$$

for some $A \in \mathcal{A}$, $b: B \rightarrow FA$ in B & $c: GA \rightarrow C$ in C .

3. Theorem Suppose that $\begin{array}{ccc} B & \xrightarrow{P} & B \rightarrow_A C \\ F \uparrow & \xRightarrow{\lambda} & \uparrow Q \\ A & \xrightarrow{G} & C \end{array}$ is a cocomma $\square$ in Lex

and that F , as a functor, has a left adjoint $F_! : B \rightarrow A$.

Then so does Q : $Q_! : (B \rightarrow_A C) \rightarrow C$; moreover $Q_!$ is

C -indexed, i.e.

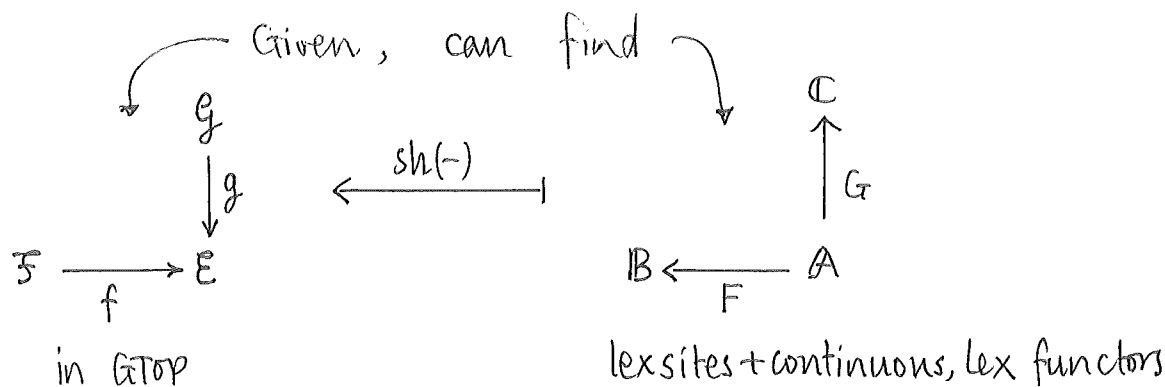
$$\begin{array}{ccc} X' \xrightarrow{\xi} X \\ f' \downarrow \lrcorner \quad \downarrow f \\ QC' \xrightarrow[Q_\gamma]{} QC \\ \text{pullback in } B \rightarrow_A C \end{array} \Rightarrow \begin{array}{ccc} Q_! X' \xrightarrow{Q_! \xi} Q_! X \\ \bar{f}' \downarrow \lrcorner \quad \downarrow \bar{f} \\ C' \xrightarrow[\gamma]{} C \\ \text{pullback in } C \end{array}$$

Also the cocomma square satisfies a BCC, viz. the canonical natural transformation $Q_! P \rightarrow G F_!$ is an isomorphism.

□

[Proof: use the fact that (not nec. lex) functors $B \rightarrow_A C \rightarrow \bullet$ (such as $Q_!$) correspond to functors $B \times C / (F, G) \rightarrow \bullet$ inverting Σ .]

COMMA SQUARES IN G_{TOP} VIA LEXSITES



Construct $B \xrightarrow{P} (B \rightarrow_A \mathcal{C}) \xleftarrow{Q} \mathcal{C}$ in lex and put the least Groth.
topology on $B \rightarrow_A \mathcal{C}$ which makes P & Q continuous. Then

$$\text{sh}(B \rightarrow_A \mathcal{C}) \simeq \text{comma topos} \quad \begin{array}{ccc} \mathcal{F} & \xrightarrow{g} & \mathcal{G} \\ p \downarrow & \Rightarrow & \downarrow q \\ \mathcal{F} & \xrightarrow{f} & \mathcal{E} \end{array}$$

Question: constructing $B \rightarrow_A \mathcal{C}$ as above, under what conditions
 is the topology on it subcanonical? (given that those on B & $\mathcal{C}$ are)?

4. Lemma Suppose F has a left adjoint $F_! : B \rightarrow A$ which is continuous. ~~Then~~ Suppose also the topologies on B & $\mathcal{C}$ are subcanonical. Then:
- (i) the topology on $B \rightarrow_A \mathcal{C}$ (as above) is subcanonical;
 - (ii) $Q_!$ (given by Theorem 3) is continuous. □

5. Lemma Suppose $Q : \mathcal{C} \rightarrow \mathcal{D}$ in lex is continuous for subcanonical topologies on $\mathcal{C}$ and $\mathcal{D}$. If Q has a continuous, $\mathcal{C}$ -indexed left adjoint, then the geometric morphism $\text{sh } \mathcal{D} \rightarrow \text{sh } \mathcal{C}$ induced by Q is locally connected. □

6. Theorem let
$$\begin{array}{ccc} \mathcal{F} \rightarrow_{\mathcal{E}} \mathcal{G} & \xrightarrow{q} & \mathcal{G} \\ p \downarrow & \Rightarrow & \downarrow g \\ \mathcal{F} & \xrightarrow{f} & \mathcal{E} \end{array}$$
 be a comma square in $\mathbf{GTOP}$.

Suppose that f is essential. Then q is locally connected and the canonical natural transformation $q_! p^* \rightarrow g^* f_!$ is an iso.

□

[Proof: combine Theorem 3 with Lemmas 4 & 5.]

Compare Theorem 6 with Tierney's result about pullbacks of locally connected geometric morphisms. Using both these results one gets that for an essential geometric morphism

$f: \mathcal{F} \rightarrow \mathcal{E}$

$$\text{Des}(\mathcal{C}_f) \simeq (f^* f_! \text{-algebras in } \mathcal{F}) [\simeq (f^* f_* \text{-coalgebras in } \mathcal{F})]$$

and hence:

7. Theorem [Lax descent for essential surjective geometric morphisms]

When $f: \mathcal{F} \rightarrow \mathcal{E}$ is essential and surjective then

$K: \mathcal{E} \rightarrow \text{Des}(\mathcal{C}_f)$ is an equivalence.

□

[Precisely, this follows from ~~requires~~ f^*, p_1^* & p_{12}^* have left adjoints $f_!, (p_1)_!,$ & $(p_{12})_!$ so that $(p_1)_! p_0^* \rightarrow f^* f_!$ & $(p_{12})_! p_{01}^* \rightarrow p_0^* (p_1)_!$ are iso's.]