Camb. Cat. Conf.
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LAX DESCENT FOR ESSENTIAL SURJECTIONS

GENERAL AIM: Find and use "exactness properties" of doctrines in Categorical Logic.

PARTICULAR EXAMPLE: <u>Lax descent in Geometric Logic</u>

<u>SLIDE</u> 1

Given $f : \mathcal{F} \to \mathcal{E}$ in GTOP (= 2-cat. of Grothendieck toposes),
form

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{E} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{F}_2 \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, phantom, "\overset{\varphi_1}{\cong}"] & \mathcal{F}_1 \arrow[d, "p_0"] \\ \mathcal{F}_1 \arrow[r, "p_1"'] & \mathcal{F} \end{tikzcd} \quad ;$$

comma $\square$ bipullback $\square$

plus associated

diagonal $d : \mathcal{F} \to \mathcal{F}_1$, $\delta_0 : p_0 d \cong 1_{\mathcal{F}}$, $\delta_1 : p_1 d \cong 1_{\mathcal{F}}$; $f\delta_1 \circ \lambda d = f\delta_0$;

projection $p_{02} : \mathcal{F}_2 \to \mathcal{F}_1$, $\varphi_0 : p_0 p_{02} \cong p_0 p_{01}$, $\varphi_2 : p_1 p_{02} \cong p_1 p_{12}$; $f\varphi_2^{-1} \lambda p_{12} f\varphi_1 \lambda p_{01} f\varphi_0 = \lambda p_{02}$

<u>SLIDE</u> 2

Get $\mathbb{C}_f : \begin{tikzcd} \mathcal{F}_2 \arrow[r, shift left=2, "p_{ij}"] \arrow[r] \arrow[r, shift right=2] & \mathcal{F}_1 \arrow[r, shift left=1, "p_i"] \arrow[r, shift right=1] & \mathcal{F} \arrow[l, bend left=60, "d"] \end{tikzcd}$ the <u>lax kernel of f</u>

[a pseudocategory object in GTOP]

<u>Category of descent data</u> $\mathrm{Des}(\mathbb{C}_f)$
- objects $(Y, a) : p_0^* Y \xrightarrow{a} p_1^* Y$ s.t.
  $(\delta_1)_Y \cdot d^* a \cdot (\delta_0)_Y^{-1} = 1_Y$ (unit condition)
  $(\varphi_2)_Y^{-1} \, p_{12}^* a \, (\varphi_1)_Y \, p_{01}^* a \, (\varphi_0)_Y = p_{02}^* a$ (cocycle condition)
- arrows $\eta : (Y, a) \to (Y', a') : \eta : Y \to Y'$ st
  $a' \cdot p_0^* \eta = p_1^* \eta \cdot a$

<u>Comparison functor</u> $K : \mathcal{E} \to \mathrm{Des}(\mathbb{C}_f)$

$X \longmapsto (X, \lambda_X)$

Given : $\mathcal{F} \xrightarrow{f} \mathcal{E}$ geometric morphism between Grothendieck toposes

form :
$$\begin{tikzcd}
\mathcal{F}_1 \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & \mathcal{F} \arrow[d, "f"] \\
\mathcal{F} \arrow[r, "f"'] & \mathcal{E}
\end{tikzcd}$$
comma square in GTOP

plus :
$$\begin{tikzcd}
\mathcal{F}_2 \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] \arrow[dr, phantom, "\overset{\varphi_1}{\cong}"] & \mathcal{F}_1 \arrow[d, "p_0"] \\
\mathcal{F}_1 \arrow[r, "p_1"'] & \mathcal{F}
\end{tikzcd}$$
pullback square in GTOP

together with 3rd projection $p_{02} : \mathcal{F}_2 \to \mathcal{F}_1$ + $\begin{cases} \varphi_2 : p_1 p_{02} \cong p_1 p_{12} \\ \varphi_0 : p_0 p_{02} \cong p_0 p_{01} \end{cases}$ so that

$$\begin{tikzcd}
\bullet \arrow[r, phantom, "\overset{f\varphi_0}{\cong}"] \arrow[ddd, "\lambda p_{02}"'] & \bullet \arrow[d, "\lambda p_{01}"] \\
 & \bullet \arrow[d, phantom, "\wr\| \, f\varphi_1"] \\
 & \bullet \arrow[d, "\lambda p_{12}"] \\
\bullet \arrow[r, phantom, "\underset{f\varphi_2}{\cong}"] & \bullet
\end{tikzcd}$$
commutes

& diagonal $d : \mathcal{F} \to \mathcal{F}_1$ + $\begin{cases} \delta_0 : p_0 d \cong 1 \\ \delta_1 : p_1 d \cong 1 \end{cases}$ so that

$$\begin{tikzcd}
\bullet \arrow[r, phantom, "\overset{f\delta_0}{\cong}"] \arrow[d, "\lambda d"'] & \bullet \arrow[d, "1"] \\
\bullet \arrow[r, phantom, "\overset{f\delta_1}{\cong}"] & \bullet
\end{tikzcd}$$
commutes.

$$\begin{tikzcd} \mathbb{C}_f : \mathcal{F}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2, "p_{ij}"'] & \mathcal{F}_1 \arrow[r, shift left] \arrow[r, shift right, "p_i"'] & \mathcal{F} \arrow[l, bend left=60, "d"] \end{tikzcd}$$

pseudo category object in GTOP

<u>Category of descent data for</u> $\mathbb{C}_f$: $\mathrm{Des}(\mathbb{C}_f)$

- objects (Y, a) where $Y \in \mathcal{F}$, $a : p_0^* Y \to p_1^* Y$ in $\mathcal{F}_2$

  satisfying $(\delta_1)_Y \circ d^*(a) \circ (\delta_0)_Y^{-1} = 1_Y$ (counit condition)

  and $(\varphi_2)_Y^{-1} \circ p_{12}^*(a) \circ (\varphi_1)_Y \circ p_{01}^*(a) \circ (\varphi_0)_Y = p_{02}^*(a)$

  (cocycle condition)

- arrows $y : (Y, a) \to (Y', a')$ where $y : Y \to Y'$ in $\mathcal{F}$

  satisfies $a' \circ p_0^*(y) = p_1^*(y) \circ a$

<u>Comparison functor</u> : $K : \mathcal{E} \longrightarrow \mathrm{Des}(\mathbb{C}_f)$

given on objects by : $X \longmapsto (X, \lambda_X)$

("Lax Descent in geometric logic")

QUESTION: For which f is $K : \mathcal{E} \to \mathrm{Des}(\mathbb{C}_f)$ an equivalence?

[Also have "bikernel of f" $\mathbb{G}_f$, a pseudogroupoid in GTOP, plus $K : \mathcal{E} \to \mathrm{Des}(\mathbb{G}_f)$. Easy to see that

K for $\mathbb{G}_f$ full & faithful $\Rightarrow$ K for $\mathbb{C}_f$ full & faithful

But " " " ess. surjective $\overset{?}{\not\Rightarrow}$ " " " ess. surjective.]

Defintion: For brevity, say f is effective if $K : \mathcal{E} \to \mathrm{Des}(\mathbb{C}_f)$ is an equivalence.

These are effective (like J&T's descent theorem for $\mathbb{G}_f$)

$\boxed{\text{$f$ an open surjection}} \supseteq \boxed{\text{f a locally connected surjection}}$

$\cup\!\mid$ $\qquad\qquad\qquad\qquad$ $\cup\!\mid$

$\boxed{\text{$f$ a surjection and f^* preserves infinitary $\wedge$}} \supseteq \boxed{\text{f essential and a surjection}}$

Proposition: These effective $\Downarrow$ these effective

$\boxed{\text{$f$ a surjection, $\mathcal{E}, \mathcal{F}$ coherent and f^* restricts to $\mathcal{E}_{coh} \to \mathcal{F}_{coh}$}}$

effectivity of these is solely a property of pretoposes (if it holds).

Theorem: These are effective

Will indicate the proof of this.

EXPLICIT CONSTRUCTION OF COCOMMA SQUARES IN LEX

Lex = 2-category of small categories which have and functors which preserve finite limits.

Given
$$\begin{tikzcd} & \mathbb{B} \\ \mathbb{A} \arrow[r, "G"'] \arrow[u, "F"] & \mathbb{C} \end{tikzcd}$$
in lex, want to describe
$$\begin{tikzcd} \mathbb{B} \arrow[r, "P"] & \mathbb{B} \to_{\mathbb{A}} \mathbb{C} \\ \mathbb{A} \arrow[u, "F"] \arrow[r, "G"'] \arrow[ur, phantom, "\overset{\lambda}{\Rightarrow}"] & \mathbb{C} \arrow[u, "Q"'] \end{tikzcd}$$
cocomma $\square$.

Construction :

① Limit
$$\begin{tikzcd} \mathbb{B} \arrow[dr, phantom, "\Searrow" near start] & \mathbb{B}\times\mathbb{C}/(F,G) \arrow[l, "u_0"'] \arrow[d, "u_1"] \arrow[dl, "u" description] \\ \mathbb{A} \arrow[u, "F"] \arrow[r, "G"'] \arrow[ur, phantom, "\Nwarrow" near end] & \mathbb{C} \end{tikzcd}$$
in Lex.

[• objects (b,A,c), where $A \in \mathbb{A}$, $b : B \to FA$ in $\mathbb{B}$ & $c : C \to GA$ in $\mathbb{C}$;
• arrows $(\beta,\alpha,\gamma) : (b,A,c) \to (b',A',c')$ where $\alpha : A \to A'$, $\beta : B \to B'$, $\gamma : C \to C'$ and
$$\begin{array}{ccc} B \to FA & & GA \leftarrow C \\ \downarrow \quad \downarrow & \& & \downarrow \quad \downarrow \\ B' \to FA' & & GA' \leftarrow C' \end{array}$$
commute.]

u_0, u, u_1 have right adjoints I_0, I, I_1, fitting to give

② Colimit
$$\begin{tikzcd} \mathbb{B} \arrow[r, "I_0"] \arrow[dr, phantom, "\Nwarrow\lambda_0" near start] & \mathbb{B}\times\mathbb{C}/(F,G) \\ \mathbb{A} \arrow[u, "F"] \arrow[r, "G"'] \arrow[ur, "I" description] \arrow[ur, phantom, "\lambda_1\Searrow" near end] & \mathbb{C} \arrow[u, "I_1"'] \end{tikzcd}$$
in Lex.

[Cf. similar phenomena in lex : products are coproducts & slices are coslices.]

So forming

③
$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "I"] \arrow[r, bend right, "I_0F"'] \arrow[r, phantom, "\Downarrow\lambda_0"] & \mathbb{B}\times\mathbb{C}/(F,G) \arrow[r, "L"] & \mathbb{B}\times\mathbb{C}/(F,G)[\lambda_0^{-1}] \end{tikzcd}$$

coinverter diagram in Lex,

then necessarily

④
$$\begin{tikzcd} \mathbb{B} \arrow[r, "LI_0"] & \mathbb{B}\times\mathbb{C}/(F,G)[\lambda_0^{-1}] \\ \mathbb{A} \arrow[u, "F"] \arrow[r, "G"'] \arrow[ur, phantom, "\xRightarrow{L\lambda_1\circ(L\lambda_0)^{-1}}"] & \mathbb{C} \arrow[u, "LI_1"'] \end{tikzcd}$$
is a cocomma square in Lex.

⑤ <u>Claim</u> that in ③ can take L to be $\mathbb{B}\times\mathbb{C}/(F,G) \xrightarrow{P_\Sigma} \mathbb{B}\times\mathbb{C}/(F,G)[\Sigma^{-1}]$ where Σ is the <u>right calculus of fractions</u> consisting of morphisms

$$(b,A,c) \xrightarrow{(\beta,\alpha,\gamma)} (b',A',c')$$

where
$$\begin{tikzcd} B \arrow[r, "\beta"] \arrow[r, phantom, "\cong"' near start, shift right=2] \arrow[d, "b"'] & B' \arrow[d, "b'"] \\ FA \arrow[r, "F\alpha"] & FA' \end{tikzcd}$$
β is an iso in $\mathbb{B}$

and
$$\begin{tikzcd} GA \arrow[r, "G\alpha"] & GA' \\ C \arrow[u, "c"] \arrow[r, "\gamma"'] \arrow[ur, phantom, "\lrcorner" very near start] & C' \arrow[u, "c'"'] \end{tikzcd}$$
is a pullback □ in $\mathbb{C}$.

[<u>Proof</u> of claim: one the one hand, components of $\lambda_0 \subseteq \Sigma$; conversely, if $(\beta,\alpha,\gamma)\in\Sigma$ then $\exists$ diagram

$$\begin{tikzcd} & \bullet \arrow[r, "(\beta,\alpha,\gamma)"] \arrow[d] & \bullet \arrow[d] & \\ \bullet \arrow[d, "(\lambda_0)_A"'] & \bullet \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \bullet \arrow[d, "(\lambda_0)_{A'}"] \\ \bullet & \bullet \arrow[l] \arrow[r, "\cong"] & \bullet \arrow[r] & \bullet \end{tikzcd}$$
,

from which it follows that $L: \mathbb{B}\times\mathbb{C}/(F,G) \to \mathbb{B}\times\mathbb{C}/(F,G)[\lambda_0^{-1}]$ inverts all the morphisms in Σ.]

<u>Thus have given an explicit construction of $\mathbb{B}\Rightarrow_{\mathbb{A}}\mathbb{C}$</u>
<u>as an object of Cat rather than of Lex.</u>

CONSEQUENCES OF THE CONSTRUCTION OF $\mathbb{B} \to_{\mathbb{A}} \mathbb{C}$

1. Every object X of $\mathbb{B} \to_{\mathbb{A}} \mathbb{C}$ can be presented as a pullback of the form

$$\begin{tikzcd} X \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & QC \arrow[d, "Qc"] \\ PB \arrow[r, "Pb"'] & PFA \arrow[r, "\lambda_A"'] & QGA \end{tikzcd}$$

where $A \in \mathbb{A}$, $b : B \to FA$ in $\mathbb{B}$ & $c : C \to GA$ in $\mathbb{C}$.

2. [Interpolation] For $B \in \mathbb{B}$, $C \in \mathbb{C}$, every arrow $k : PB \to QC$ in $\mathbb{B} \to_{\mathbb{A}} \mathbb{C}$ factors as

$$k = PB \xrightarrow{Pb} PFA \xrightarrow{\lambda_A} QGA \xrightarrow{Qc} QC$$

for some $A \in \mathbb{A}$, $b : B \to FA$ in $\mathbb{B}$ & $c : GA \to C$ in $\mathbb{C}$.

3. Theorem Suppose that $\begin{tikzcd} \mathbb{B} \arrow[r, "P"] & \mathbb{B} \to_{\mathbb{A}} \mathbb{C} \\ \mathbb{A} \arrow[u, "F"] \arrow[r, "G"'] \arrow[ur, phantom, "\overset{\lambda}{\Rightarrow}"] & \mathbb{C} \arrow[u, "Q"'] \end{tikzcd}$ is a cocomma $\square$ in Lex

and that F, as a functor, has a left adjoint $F_! : \mathbb{B} \to \mathbb{A}$. Then so does Q : $Q_! : (\mathbb{B} \to_{\mathbb{A}} \mathbb{C}) \to \mathbb{C}$; moreover $Q_!$ is $\mathbb{C}$-indexed, i.e.

$$\begin{tikzcd} X' \arrow[r, "\xi"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ QC' \arrow[r, "Q\gamma"'] & QC \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} Q_!X' \arrow[r, "Q_!\xi"] \arrow[d, "\bar{f}'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Q_!X \arrow[d, "\bar{f}"] \\ C' \arrow[r, "\gamma"'] & C \end{tikzcd}$$

pullback in $\mathbb{B} \to_{\mathbb{A}} \mathbb{C}$ $\qquad$ pullback in $\mathbb{C}$.

Also the cocomma square satisfies a BCC, viz. the canonical natural transformation $Q_! P \to G F_!$ is an isomorphism.

$\square$

[Proof : use the fact that (not nec. lex) functors $\mathbb{B} \to_{\mathbb{A}} \mathbb{C} \to \bullet$ (such as $Q_!$) correspond to functors $\mathbb{B} \times \mathbb{C}/(F, G) \to \bullet$ inverting Σ.]

COMMA SQUARES IN GTOP VIA LEXSITES

Given, can find

$$\begin{tikzcd} & \mathcal{G} \arrow[d, "g"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{E} \end{tikzcd} \qquad \xleftarrow{\ sh(-)\ } \qquad \begin{tikzcd} & \mathbb{C} \\ \mathbb{B} & \mathbb{A} \arrow[l, "F"] \arrow[u, "G"'] \end{tikzcd}$$

in GTOP $\qquad\qquad\qquad$ lexsites + continuous, lex functors

Construct $\mathbb{B} \xrightarrow{P} (\mathbb{B} \to_{\mathbb{A}} \mathbb{C}) \xleftarrow{Q} \mathbb{C}$ in lex and put <u>the least Groth. topology on</u> $\mathbb{B} \to_{\mathbb{A}} \mathbb{C}$ <u>which makes</u> P & Q <u>continuous</u>. Then

$$sh(\mathbb{B} \to_{\mathbb{A}} \mathbb{C}) \simeq \text{comma topos} \quad \begin{tikzcd} \mathcal{F} \to_{\mathcal{E}} \mathcal{G} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{E} \end{tikzcd}$$

<u>Question</u>: constructing $\mathbb{B} \to_{\mathbb{A}} \mathbb{C}$ as above, under what conditions is the topology on it <u>subcanonical</u> (given that those on $\mathbb{B}$ & $\mathbb{C}$ are)?

4. <u>Lemma</u> Suppose F has a left adjoint $F_! : \mathbb{B} \to \mathbb{A}$ which is continuous. ~~Then~~, Suppose also the topologies on $\mathbb{B}$ & $\mathbb{C}$ are subcanonical. Then:

(i) the topology on $\mathbb{B} \to_{\mathbb{A}} \mathbb{C}$ (as above) is subcanonical;

(ii) $Q_!$ (given by Theorem 3) is continuous.

$\square$

5. <u>Lemma</u> Suppose $Q : \mathbb{C} \to \mathbb{D}$ in lex is continuous for subcanonical topologies on $\mathbb{C}$ and $\mathbb{D}$. If Q has a continuous, $\mathbb{C}$-indexed left adjoint, then the geometric morphism $sh\mathbb{D} \to sh\mathbb{C}$ induced by Q is locally connected.

$\square$

6. <u>Theorem</u> Let

$$\begin{tikzcd} \mathcal{F} \downarrow_{\mathcal{E}} \mathcal{G} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{G} \arrow[d, "g"] \\ \mathcal{F} \arrow[r, "f"'] & \mathcal{E} \end{tikzcd}$$

be a comma square in GTOP.

Suppose that f is essential. Then q is locally connected and the canonical natural transformation $q_! p^* \to g^* f_!$ is an iso.

$\square$

[Proof: combine Theorem 3 with Lemmas 4 & 5.]

Compare Theorem 6 with Tierney's result about pullbacks of locally connected geometric morphisms. Using <u>both</u> these results one gets that for an essential geometric morphism $f : \mathcal{F} \to \mathcal{E}$

$$\boxed{\mathrm{Des}(\mathbb{C}_f) \simeq (f^* f_!\text{-algebras in } \mathcal{F}) \ [\simeq (f^* f_*\text{-coalgebras in } \mathcal{F})]}$$

and hence:

7. <u>Theorem</u> [<u>Lax descent for essential surjective geometric morphisms</u>]

When $f : \mathcal{F} \to \mathcal{E}$ is essential and surjective then $K : \mathcal{E} \to \mathrm{Des}(\mathbb{C}_f)$ is an equivalence.

$\square$

[Precisely, this ~~requires~~ follows from: $\begin{cases} f^*, p_1^* \text{ \& } p_{12}^* \text{ have left adjoints } f_!, (p_1)_! \text{ \& } (p_{12})_! \text{ so that} \\ (p_1)_! p_0^* \to f^* f_! \text{ \& } (p_{12})_! p_{01}^* \to p_0^* (p_1)_! \text{ are iso's.} \end{cases}$]