Heyting Algebras and Higher-Order Logic

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Heyting Day 2025

Heyting Day 2025:
Models of intuitionism and computability

celebrating Jaap van Oosten

Category Theory is crucial

e.g.: J v O, *Realizability: An Introduction to its Categorical Side* (Elsevier, 2008)

Aims

- Some observations about the role of category theory in logic.
- Recall a long-standing open problem connecting intuitionistic higher-order logic and category theory, in the hope that someone will settle it.
- (Try not to be too technical!)

Category Theory

Set Theory describes mathematical structures in terms of their elements (and functions on, and relations between, elements).

In contrast,

Category Theory describes mathematical structures entirely in terms of structure-preserving transformations: at base, the only way to give a kind of structure is to specify its transformations and how they compose.

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Here:

- Adjunctions in proof theory
- Topos models of intuitionistic higher-order logic

Adjointness in proof theory

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Simple special case for monotone function $f : (P, \leq_P) \to (Q, \leq_Q)$ between posets:

Given $y \in Q$, the right adjoint of f at $y \in Q$, if it exists, is the unique $gy \in P$ with

$$\text{for all } x \in P, \quad f(x) \leq_Q y \text{ iff } x \leq_P gy$$

If gy exists for all y we get $g : Q \to P$, the right adjoint of f.
It is uniquely determined by f and monotone.

[Left adjoints dually.]

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[Left adjoints dually.]

Lawvere and Lambek observed that many logical constructs can be characterised as left or right adjoints (substitution stable ones), so their existence is a property of models not extra structure.

Non-example: modalities are not usually characterisable as adjoints.

Example: $\forall$ is right adjoint to weakening

In Heyting's intuitionistic predicate logic

$$\varphi(\vec{y}) \vdash \psi(x, \vec{y}) \quad \text{iff} \quad \varphi(\vec{y}) \vdash \forall x\, \psi(x, \vec{y}) \qquad [x : X, \vec{y} : \vec{Y}]$$

intuitionistic entailment

variables x, $\vec{y}$ stand for unknown individuals of some sorts X, $\vec{Y}$

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The "f" in this case is the weakening function

$$(\{\text{formulas in } \vec{y}\}, \vdash) \hookrightarrow (\{\text{formulas in } x \text{ and } \vec{y}\}, \vdash)$$

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$$\varphi(\vec{q}) \vdash \psi(p, \vec{q}) \quad \text{iff} \quad \varphi(\vec{q}) \vdash ? \qquad [p, \vec{q} : \Omega]$$

variables $p, \vec{q}$ stand for unknown propositions (sort Ω)

The "f" in this case is the weakening function

$$(\{\text{formulas in } \vec{q}\}, \vdash) \hookrightarrow (\{\text{formulas in } p \text{ and } \vec{q}\}, \vdash)$$

built up using $\bot$, $\top$, $\wedge$, $\vee$, $\rightarrow$

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$$\varphi(\vec{q}) \vdash \psi(p, \vec{q}) \quad \text{iff} \quad \varphi(\vec{q}) \vdash A_p\psi(\vec{q}) \qquad [p, \vec{q} : \Omega]$$

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Surprising fact about intuitionistic propositional logic (IPL):

there exist IPL formulas $A_p\psi(\vec{q})$ for right adjoints to weakening
(and dually, formulas $E_p\psi(\vec{q})$ for left adjoints—existential quantifiers).

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(In <u>classical</u> PL, $A_p\psi(\vec{q}) = \psi(\top, \vec{q}) \wedge \psi(\bot, \vec{q})$ and $E_p\psi(\vec{q}) = \psi(\top, \vec{q}) \vee \psi(\bot, \vec{q})$)

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Theorem. Given any formula ψ built up using $\bot$, $\top$, $\wedge$, $\vee$ and $\to$ from variables $p, q, \ldots$, there is another such formula $A_p\psi$, not involving p, which is the value at ψ of the right adjoint to weakening; and dually for the left adjoint (existential quantifier) $E_p\psi$.

My original proof [JSL 57(1992)] was via proof theory (using the Vorob'ev-Hudelmaier-Dyckhoff version of Gentzen sequent calculus) and was constructive—A_p and E_p are computable functions.

TABLE 5. Definition of $E_p(\Delta)$ and $A_p(\Delta; \phi)$
(q denotes any propositional variable not equal to p).

	Δ matches	$\mathscr{E}_p(\Delta)$ contains
(E0)	$\Delta'\bot$	$\bot$
(E1)	$\Delta' q$	$E_p(\Delta') \wedge q$
(E2)	$\Delta'(\delta_1 \wedge \delta_2)$	$E_p(\Delta'\delta_1\delta_2)$
(E3)	$\Delta'(\delta_1 \vee \delta_2)$	$E_p(\Delta'\delta_1) \vee E_p(\Delta'\delta_2)$
(E4)	$\Delta'(q \to \delta)$	$q \to E_p(\Delta'\delta)$
(E5)	$\Delta' p(p \to \delta)$	$E_p(\Delta' p\delta)$
(E6)	$\Delta'((\delta_1 \wedge \delta_2) \to \delta_3)$	$E_p(\Delta'(\delta_1 \to (\delta_2 \to \delta_3)))$
(E7)	$\Delta'((\delta_1 \vee \delta_2) \to \delta_3)$	$E_p(\Delta'(\delta_1 \to \delta_3)(\delta_2 \to \delta_3))$
(E8)	$\Delta'((\delta_1 \to \delta_2) \to \delta_3)$	$[E_p(\Delta'(\delta_2 \to \delta_3)) \to A_p(\Delta'(\delta_2 \to \delta_3); \delta_1 \to \delta_2)] \to E_p(\Delta'\delta_3)$

	$\Delta; \phi$ matches	$\mathscr{A}_p(\Delta; \phi)$ contains
(A1)	$\Delta' q; \phi$	$A_p(\Delta'; \phi)$
(A2)	$\Delta'(\delta_1 \wedge \delta_2); \phi$	$A_p(\Delta'\delta_1\delta_2; \phi)$
(A3)	$\Delta'(\delta_1 \vee \delta_2); \phi$	$[E_p(\Delta'\delta_1) \to A_p(\Delta'\delta_1; \phi)] \wedge [E_p(\Delta'\delta_2) \to A_p(\Delta'\delta_2; \phi)]$
(A4)	$\Delta'(q \to \delta); \phi$	$q \wedge A_p(\Delta'\delta; \phi)$
(A5)	$\Delta' p(p \to \delta); \phi$	$A_p(\Delta' p\delta; \phi)$
(A6)	$\Delta'((\delta_1 \wedge \delta_2) \to \delta_3); \phi$	$A_p(\Delta'(\delta_1 \to (\delta_2 \to \delta_3)); \phi)$
(A7)	$\Delta'((\delta_1 \vee \delta_2) \to \delta_3); \phi$	$A_p(\Delta'(\delta_1 \to \delta_3)(\delta_2 \to \delta_3); \phi)$
(A8)	$\Delta'((\delta_1 \to \delta_2) \to \delta_3); \phi$	$[E_p(\Delta'(\delta_2 \to \delta_3)) \to A_p(\Delta'(\delta_2 \to \delta_3); \delta_1 \to \delta_2)] \wedge A_p(\Delta'\delta_3; \phi)$
(A9)	$\Delta; q$	q
(A10)	$\Delta' p; p$	$\top$
(A11)	$\Delta; \phi_1 \wedge \phi_2$	$A_p(\Delta; \phi_1) \wedge A_p(\Delta; \phi_2)$
(A12)	$\Delta; \phi_1 \vee \phi_2$	$A_p(\Delta; \phi_1) \vee A_p(\Delta; \phi_2)$
(A13)	$\Delta; \phi_1 \to \phi_2$	$E_p(\Delta\phi_1) \to A_p(\Delta\phi_1; \phi_2)$

$$E_p\varphi \triangleq E_p(\{\varphi\})$$
$$E_p(\Delta) \triangleq \bigwedge \mathcal{E}_p(\Delta)$$

$$A_p\varphi \triangleq A_p(\emptyset; \varphi)$$
$$A_p(\Delta; \varphi) \triangleq \bigvee \mathcal{A}_p(\Delta; \varphi)$$

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Recent improvement, formalization and verification of algorithms for A_p and E_p by Férée & van Gool [ACM Cert. Progs & Proofs Conference (2023)].

Non-constructive, model-theoretic proof by Ghilardi & Zawadowski [JSL 60(1995)] uses a duality between Heyting algebras and sheaves-with-game-theoretic-structure. Later simplified by van Gool & Reggio [Canadian Math. Bull. 62(2019)] using topological duality between Heyting algebras and Esakia spaces (roughly: finitely presentable extensions in HAs have adjoints because their duals are open maps).

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Craig interpolation property:

$$\varphi(p, q) \qquad\qquad \vdash \qquad\qquad \psi(q, r)$$

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$$\varphi(p,q) \quad\vdash\quad \underset{\text{exists}}{\underset{\uparrow}{\theta(q)}} \quad\vdash\quad \psi(q,r)$$

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Uniform interpolation property:

$$\varphi(p,q) \;\vdash\; E_p\varphi(q) \;\vdash\; \theta(q) \;\vdash\; A_r\psi(q) \;\vdash\; \psi(q,r)$$

Many modal logics satisfy uniform interpolation,
but some do not, e.g. S4 [Ghilardi & Zawadowski, Studia Log. 55(1995)].

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Originally I hoped that IPL would not (!), for the following reason…

Topos models of intuitionistic higher-order logic

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Intuitionistic Higher-Order Logic (IHOL) has

Sorts $\qquad X ::= \mathrm{N} \mid \mathrm{P}(X_1, \ldots, X_n)$

ground sorts
(e.g. for numbers)

power sort, whose varables stand for
subsets of $X_1 \times \cdots \times X_n$
(the sort Ω of propositions is $\mathrm{P}(\ )$, i.e. $n = 0$)

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$\bot, \top, \wedge, \vee, \leftrightarrow, \exists$ and $=$ are definable (impredicatively!)

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subject to the usual rules of intuitionistic logic plus

Comprehension $\quad \exists y : \mathrm{P}(\vec{X}).\, \forall \vec{x} : \vec{X}.\, (\vec{x} \in y \leftrightarrow \varphi(\vec{x}))$

(and Infinity axioms, if we have a ground sort for numbers)

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Simple, but expressive

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A topos is any category with

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gives adjoint characterisation of
cartesian product
injective functions / subsets
inverse image of subset along a function

gives adjoint characterisation of
powerset + comprehension

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1945-74 [Leray, Cartan, Weil, Serre, Godement $\rangle$ Grothendieck – sheaf toposes

1974-79 Lawvere & Tierney $\langle$ Freyd, Joyal, Lambek, Johnstone, ...] – elementary toposes
connection with IHOL

1979 [Kreisel, Gandy $\rangle$ Hyland – realizability toposes

Three examples

- **Classical truth**: $[\![\varphi]\!] \subseteq \{*\}$ (two subsets)
  $* \in [\![\varphi \to \psi]\!]$ iff either $* \in [\![\psi]\!]$, or $* \notin [\![\varphi]\!]$
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  $n \in [\![\varphi \to \psi]\!]$ iff for all $m \in [\![\varphi]\!]$, the value of the nth partial recursive function at m is defined and is in $[\![\psi]\!]$

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What kind of "truth values" does a topos have in general?

Heyting algebras

are bounded ($\top$, $\bot$) distributive lattices ($\wedge$, $\vee$) with relative pseudocomplements ($\rightarrow$).

They relate to Heyting's "sterile" formalization of Brouwer's intuitionism as Boolean algebras relate to classical logic.
In particular

the sentences of an IHOL theory modulo provability

or equivalently

the global elements $\mathcal{E}(1, \Omega)$ of the subobject classifier Ω of a topos $\mathcal{E}$

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Open question: is every Heyting algebra isomorphic to $\mathcal{E}(1, \Omega)$ for some topos $\mathcal{E}$?

Open question: $\forall\, H, \exists?\ \mathcal{E}$ with $H \cong \mathcal{E}(1, \Omega)$

It suffices to resolve the question in the case when H is a free Heyting algebra $\mathrm{F}[X]$ on a set X, i.e. $\{\text{sentences of IPL over } X\}$ mod $\vdash$.

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My failed strategy from 1980s for resolving the question in the negative:

Not hard to prove:

if $\mathrm{F}[X]$ is of the form $\mathcal{E}(1, \Omega)$, then
the quantifier structure in $\mathcal{E}$ (it's a model of IHOL) implies that IPL has uniform interpolants.

So to prove there is no such $\mathcal{E}$, it suffices to find an IPL formula φ for which $A_p\varphi$ or $E_p\varphi$ cannot exist.

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1992: oh no! $A_p\varphi$ and $E_p\varphi$ always exist.

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Dito Pataraia (d.2011) strategy from 2007 for resolving the question in the positive:

1. Formulate IHOL in terms of algebraic logic—notion of a "higher-order cylindric Heyting algebra" (hocha).

   Every hocha H has a Heyting algebra of sentences $H(1, \Omega)$

   Every hocha H generates a topos $\mathcal{E}$ (like tripos-to-topos construction of Hyland-Johnstone-P) with $\mathcal{E}(1, \Omega) \cong H(1, \Omega)$.

   [all OK]

2. Somehow use Ghilardi's construction of free Heyting algebras
   [C.R Math. Acad. Sci. Canada(14)1992] to construct a hocha H with $H(1, \Omega) \cong \mathrm{F}[X]$.

   But it's not clear how DP intended to do that :-(

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In the modern era, logical (type theoretical) account of higher-dimensional category theory ("it sucks" , DS) is proving essential for obtaining a homotopy theory of spaces that has computational content (cf. the cubical type theory of Coquand *et al*)

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Thank you for your attention!