Setoids in Intensional Type Theory

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Discussions with Marcelo a year ago

David Berry and Marcelo [2] have formalized Marcelo's [4] in intensional type theory, but it's NBE only for simple types.

Thierry Coquand [3] has a (deceptively) simple NBE for categories-with-families-with-universes, formalized within constructive set theory.

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Can one adapt Coquand's [3] to get an intensional proof of NBE for intensional Type Theory (ITT) via universes of setoids in ITT (safe Agda)?

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why bother with weak meta-logics?

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I have a new take on this

Conventional wisdom

If one wants extensionality within intensional type theory use
“the setoid model of type theory”

(but be warned, it’s hell)

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Classical notion of "setoid = set + equivalence relation" can be transferred from set theory to type theory in several different ways. The equivalence relation could be valued in a universe of types (Palmgren [6]), an impredicative universe of propositions (Hofmann [5]), or a predicative universe of proof irrelevant ("strict") propositions (Altenkirch *et al* [1]).

[1] Thorsten Altenkirch, Simon Boulier, Ambrus Kaposi, Christian Sattler and Filippo Sestini, *Constructing a universe for the setoid model*, FoSSaCS 2021.

[5] Martin Hofmann, *Extensional Concepts in Intensional Type Theory*, PhD Thesis, Edinburgh, 1995.

[6] Erik Palmgren, *From type theory to setoids and back*, MSCS 32(2022)1283–1312.

Setoids in safe Agda

A *setoid* is a type $A : \texttt{Set}$ equipped with $(A \ni \_\sim\_) : A \to A \to \texttt{Set}$

think of $(A \ni x \sim y) : \texttt{Set}$ as the type of proofs that A considers x and y to be equal

and functions

$$\forall x \to (A \ni x \sim x)$$

$$\forall x\ y \to (A \ni x \sim y) \to (A \ni y \sim x)$$

$$\forall x\ y\ z \to (A \ni x \sim y) \to (A \ni y \sim z) \to (A \ni x \sim z)$$

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$$\forall x \to (A \ni x \sim x)$$

$$\forall x\, y \to (A \ni x \sim y) \to (A \ni y \sim x)$$

$$\forall x\, y\, z \to (A \ni x \sim y) \to (A \ni y \sim z) \to (A \ni x \sim z)$$

$f : A \to A'$ is a *morphism of setoids* if it comes with a function

$$\forall x\, y \to (A \ni x \sim y) \to (A' \ni f\, x \sim f\, y)$$

No groupoid laws or coherence equalities!

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Guiding Principle : types that are used as propositions (like $A \ni x \sim y$) should never have their elements compared for intensional identity

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If one wants extensionality within intensional type theory use
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Extensional Type Theory; but modelling universes has proved to be problematic [1, 6].

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Extensional equality types

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathtt{Eq}\, A\, a\, b\ \mathtt{type}}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathtt{refl}\, A\, a : \mathtt{Eq}\, A\, a\, a}$$

$$\frac{\Gamma \vdash p : \mathtt{Eq}\, A\, a\, b}{\Gamma \vdash a = b : A}$$

$$\frac{\Gamma \vdash p : \mathtt{Eq}\, A\, a\, b}{\Gamma \vdash p = \mathtt{refl}\, A\, a : \mathtt{Eq}\, A\, a\, a}$$

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It's common to describe the various *categorical structures* built from setoids that could be used to give a semantics of (some) type theory, but no-one defines within ITT *the actual semantics functions* (even Palmgren [6])

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Displayed setoid = ?

The crucial question (for modelling dependent types):

Given a setoid A, what structure is needed to make $B : A \to \mathtt{Set}$ into a well-behaved displayed setoid (*aka* family of setoids)?

Answers in the literature:

Palmgren [6] does not follow our Guiding Principle

Altenkirch *et al* [1] avoid the problem by valuing $\_\sim\_$ in $\mathtt{Prop}$, but that takes one out of safe Agda

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The crucial question (for modelling dependent types):

Given a setoid A, what structure is needed to make $B : A \to \mathtt{Set}$ into a well-behaved displayed setoid (*aka* family of setoids)?

Certainly need setoid structure on $\Sigma\, A\, B$ making $\mathtt{fst}$ a morphism:

$$\begin{array}{ccc} & \Sigma\, A\, B \\ \mathtt{fst} & \downarrow \\ & A \end{array} \qquad \begin{array}{rcl} \Sigma\, A\, B \ni (x, y) & \overset{?}{\sim} & (x', y') \\ & \downarrow & \\ A \ni x & \sim & x' \end{array}$$

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Certainly need setoid structure on $\Sigma\, A\, B$ making $\mathtt{fst}$ a morphism:

$$\begin{array}{ccc} \Sigma\, A\, B & & (A \ni x \sim x') \times (B \ni x, y \approx x', y') \\ \mathtt{fst} \downarrow & & \downarrow \\ A & & A \ni x \sim x' \end{array}$$

cartesian product, not dependent product, so that proofs of $B \ni x, y \approx x', y'$ do not depend upon proofs of $A \ni x \sim x'$

Heterogeneous equivalence relation

$$(B \ni \_, \_ \approx \_, \_) : (x : A)(y : B\,x)(x' : A)(y' : B\,x') \to \mathtt{Set}$$

structure ensuring $(A \ni x \sim x') \times (B \ni x, y \approx x', y')$ is an equiv. rel. for $\Sigma\,A\,B$:

structure ensuring $B\,x$ has congruence property (transport) for any $A \ni x \sim x'$:

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structure ensuring $(A \ni x \sim x') \times (B \ni x, y \approx x', y')$ is an equiv. rel. for $\Sigma\, A\, B$:

$$\forall x\, y \to (B \ni x, y \approx x, y)$$

$$\forall x\, y\, x'\, y' \to (A \ni x \sim x') \to (B \ni x, y \approx x', y') \to (B \ni x', y' \approx x, y)$$

$$\forall x\, y\, x'\, y'\, x''\, x'' \to (A \ni x \sim x') \to (A \ni x' \sim x'') \to$$
$$(B \ni x, y \approx x', y') \to (B \ni x', y' \approx x'', y'') \to (B \ni x, y \approx x'', y'')$$

structure ensuring $B\, x$ has congruence property (transport) for any $A \ni x \sim x'$:

$$\mathsf{coe} : \forall \{x\, x'\} \to (A \ni x \sim x') \to B\, x \to B\, x'$$

$$\forall x\, y\, x' \to (e : A \ni x \sim x') \to (B \ni x, y \approx x', \mathsf{coe}\, e\, y)$$

Displayed setoids

Displayed setoid over a setoid A is
$B : A \to \mathtt{Set}$ + heterogeneous equiv. rel.

Element of a displayed setoid $B : A \to \mathtt{Set}$ is
a dependent function $b : (x : A) \to B\,x$ equipped with
$\forall x\,x' \to (A \ni x \sim x') \to (B \ni x, b\,x \approx x', b\,x')$

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Setoids in intensional type theory, their morphisms, displays and elements
can be organised into a setoid-enriched category-with-families [definition omitted]
with lots of good properties for modelling type theories;
but there is a problem…

Setoid universes

The problem: the collection of all setoids (of some size) can be made into a setoid in several ways (intensional identity, isomorphism, …), but none, it seems, can be endowed with a generic displayed setoid giving rise to a full model of type-theory-with-universes.

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The solution: decide in advance what constructs the universe should contain and then make an *inductive* definition of codes for setoids and simultaneously *recursive* definition of the setoid denoted by a code.

Altenkirch *et al* [1] and Palmgren [6] both do this, but in very different ways. I adapt and extend Altenkirch's approach.

Setoid universes of types in safe Agda

inductive

$\mathcal{U}_n : \texttt{Set}$

recursive

$\mathcal{E}\ell_n : \mathcal{U}_n \to \texttt{Set}$

$\mathcal{U}_n \ni \_\sim\_ : \mathcal{U}_n \to \mathcal{U}_n \to \texttt{Set}$

$\mathcal{E}\ell_n \ni \_,\_\approx\_,\_ : (X : \mathcal{U}_n)(x : \mathcal{E}\ell_n\, X)(X' : \mathcal{U}_n)(x' : \mathcal{E}\ell_n\, X') \to \mathit{Set}$

(where $n : \mathbb{N}$, the type of natural numbers)

Setoid universes of types in safe Agda

Universe types: $\mathtt{U}_n : \mathcal{U}_{n+1} \quad \mathcal{E}\ell_{n+1}(\mathtt{U}_n) = \mathcal{U}_n$

$$\mathcal{U}_{n+1} \ni \mathtt{U}_n \sim \mathtt{U}_n = 1 \qquad \mathcal{E}\ell_{n+1} \ni \mathtt{U}_n, X \approx \mathtt{U}_n, X' = (\mathcal{U}_n \ni X \sim X')$$
$$\mathcal{U}_{n+1} \ni \mathtt{U}_n \sim \_ = \emptyset \qquad \mathcal{E}\ell_{n+1} \ni \mathtt{U}_n, \_ \approx \_, \_ = \emptyset$$
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Natural number type: $\mathtt{Nat} : \mathcal{U}_0 \quad \mathcal{E}\ell_0(\mathtt{Nat}) = \mathbb{N}$

$$\mathcal{U}_0 \ni \mathtt{Nat} \sim \mathtt{Nat} = 1 \qquad \mathcal{E}\ell_0 \ni \mathtt{Nat}, n \approx \mathtt{Nat}, n' = (n \equiv n')$$
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Π-types: [description omitted; see [1]]

Setoid universes of types in safe Agda

Equality types:

$$\frac{\mathtt{Eq}_n : (X : \mathcal{U}_n) \quad (x\ x' : \mathcal{E}\ell_n\, X)}{\mathcal{U}_n} \qquad \mathcal{E}\ell_n(\mathtt{Eq}_n\, X\, x\, x') = (\mathcal{E}\ell_n \ni X, x \approx X, x')$$

$$\begin{aligned}
\mathcal{U}_n \ni (\mathtt{Eq}_n\, X\, x\, x') \sim (\mathtt{Eq}_n\, Y\, y\, y') &= (\mathcal{U}_n \ni X \sim Y) \times (\mathcal{E}\ell_n \ni X, x \approx Y, y) \times \\
& (\mathcal{E}\ell_n \ni X, x' \approx Y, y') \\
\mathcal{U}_n \ni (\mathtt{Eq}_n\, X\, x\, x') \sim _ &= \emptyset \\
\mathcal{U}_n \ni _ \sim (\mathtt{Eq}_n\, X\, x\, x') &= \emptyset
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}\ell_n \ni (\mathtt{Eq}_n\, _\, _\, _), _ \approx (\mathtt{Eq}_n\, _\, _\, _), _ &= 1 \\
\mathcal{E}\ell_n \ni (\mathtt{Eq}_n\, _\, _\, _), _ \approx _, _ &= \emptyset \\
\mathcal{E}\ell_n \ni _, _ \approx (\mathtt{Eq}_n\, _\, _\, _), _ &= \emptyset
\end{aligned}$$

Models *extensional* identity types.

Theorem. This notion of displayed setoids and setoid universes gives a semantics for Extensional Type Theory with a countable hierarchy of universes closed under Π, natural numbers and equality types.

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Proof. Everything is done in safe Agda:

- the inductive definition of the ETT syntax and type system ($\vdash_{\mathsf{ETT}}$)
(using a library for well-scoped locally nameless representation of syntax [`https://github.com/amp12/WSLN`])

- the inductive-recursive setoid universe structure

- the setoid semantics of ETT types and terms and proof of soundness. □

The semantics uses the Hofmann-Streicher approach, but there is no reasonable notion of "partial element", so *graphs* of semantics functions for types and terms are defined and then proved single-valued and total.

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Proof. Everything is done in safe Agda:

- the inductive definition of the ETT syntax and (undecidable!) type system ($\vdash_{\mathsf{ETT}}$)

Corollary. (Relative consistency.) In safe Agda there is a function

$$(\vdash_{\mathsf{ETT}} \mathtt{zero} = \mathtt{suc}\,\mathtt{zero} : \mathtt{Nat}) \to \emptyset$$

Proof. Soundness gives $(\vdash_{\mathsf{ETT}} \mathtt{zero} = \mathtt{suc}\,\mathtt{zero} : \mathtt{Nat}) \to [\![\vdash \mathtt{zero} : \mathtt{Nat}]\!] \equiv [\![\vdash \mathtt{suc}\,\mathtt{zero} : \mathtt{Nat}]\!]$.

By definition of the semantics, $[\![\vdash \mathtt{zero} : \mathtt{Nat}]\!] \equiv 0$ and $[\![\vdash \mathtt{suc}\,\mathtt{zero} : \mathtt{Nat}]\!] \equiv 1$

and $0, 1 : \mathbb{N}$ satisfy $\neg(0 \equiv 1)$. □

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Can one adapt Coquand's [3] to get an intensional proof of NBE for intensional Type Theory (ITT) via universes of setoids in ITT (safe Agda)?

To do.

Happy 60th Birthday Marcelo!

The Agda development described in this talk is at
[https://github.com/amp12/IntensionalSetoids]

References

[1] Thorsten Altenkirch, Simon Boulier, Ambrus Kaposi, Christian Sattler and Filippo Sestini, *Constructing a universe for the setoid model*, FoSSaCS 2021.

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