Descriptive Complexity and Poynomial Time

A Tutorial

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Descriptive Complexity

Descriptive Complexity provides an alternative perspective on Computational Complexity.

Computational Complexity

- Measure use of resources (space, time, *etc.*) on a machine model of computation;
- Complexity of a language—i.e. a set of strings.

Descriptive Complexity

- Complexity of a class of structures—e.g. a collection of graphs.
- Measure the complexity of describing the collection in a formal logic, using resources such as variables, quantifiers, higher-order operators, *etc.*

There is a fascinating interplay between the views.

First-Order Logic

For a first-order sentence φ, we ask what is the *computational complexity* of the problem:

Given: a structure $\mathbb{A}$

Decide: if $\mathbb{A} \models \varphi$

In other words, how complex can the collection of finite models of φ be?

In order to talk of the complexity of a class of finite structures, we need to fix some way of representing finite structures as strings.

Encoding Structures

We use an alphabet $\Sigma = \{0, 1, \#, -\}$.

For a structure $\mathbb{A} = (A, R_1, \ldots, R_m, f_1, \ldots, f_l)$, fix a linear order $<$ on $A = \{a_1, \ldots, a_n\}$.

R_i (of arity k) is encoded by a string $[R_i]_<$ of 0s and 1s of length n^k.

f_i is encoded by a string $[f_i]_<$ of 0s, 1s and $-$s of length $n^k \log n$.

$$[\mathbb{A}]_< = \underbrace{1 \cdots 1}_{n} \#[R_1]_< \# \cdots \#[R_m]_< \#[f_1]_< \# \cdots \#[f_l]_<$$

The exact string obtained depends on the choice of order.

Naïve Algorithm

The straightforward algorithm proceeds recursively on the structure of φ:

- Atomic formulas by direct lookup.
- Boolean connectives are easy.
- If $\varphi \equiv \exists x\, \psi$ then for each $a \in \mathbb{A}$ check whether

$$(\mathbb{A}, c \mapsto a) \models \psi[c/x],$$

where c is a new constant symbol.

This runs n time $O(ln^m)$ and $O(m \log n)$ space, where m is the nesting depth of quantifiers in φ.

$$\mathrm{Mod}(\varphi) = \{\mathbb{A} \mid \mathbb{A} \models \varphi\}$$

is in *logarithmic space* and *polynomial time*.

Second-Order Logic

There are computationally easy properties that are not definable in first-order logic.

- There is no sentence φ of first-order logic such that $\mathbb{A} \models \varphi$ if, and only if, $|A|$ is even.
- There is no formula $\varphi(E, x, y)$ that defines the transitive closure of a binary relation E.

Consider second-order logic, extending first-order logic with *relational quantifiers* — $\exists X \varphi$

Examples

Evennness

This formula is true in a structure if, and only if, the size of the domain is even.

$$\exists B \exists S \quad \forall x \exists y B(x,y) \land \forall x \forall y \forall z B(x,y) \land B(x,z) \rightarrow y = z$$

$$\forall x \forall y \forall z B(x,z) \land B(y,z) \rightarrow x = y$$

$$\forall x \forall y S(x) \land B(x,y) \rightarrow \neg S(y)$$

$$\forall x \forall y \neg S(x) \land B(x,y) \rightarrow S(y)$$

Examples

Transitive Closure

This formula is true of a pair of elements a, b in a structure if, and only if, there is an E-path from a to b.

$$\begin{aligned}\exists P \quad & \forall x \forall y\, P(x,y) \rightarrow E(x,y) \\ & \exists x P(a,x) \wedge \exists x P(x,b) \wedge \neg \exists x P(x,a) \wedge \neg \exists x P(b,x) \\ & \forall x \forall y (P(x,y) \rightarrow \forall z (P(x,z) \rightarrow y = z)) \\ & \forall x \forall y (P(x,y) \rightarrow \forall z (P(z,x) \rightarrow y = z)) \\ & \forall x ((x \neq a \wedge \exists y P(x,y)) \rightarrow \exists z P(z,x)) \\ & \forall x ((x \neq b \wedge \exists y P(y,x)) \rightarrow \exists z P(x,z))\end{aligned}$$

Examples

3-Colourability

The following formula is true in a graph (V, E) if, and only if, it is 3-colourable.

$$
\begin{aligned}
\exists R \exists B \exists G \quad & \forall x(Rx \vee Bx \vee Gx) \wedge \\
& \forall x(\quad \neg(Rx \wedge Bx) \wedge \neg(Bx \wedge Gx) \wedge \neg(Rx \wedge Gx)) \wedge \\
& \forall x \forall y(Exy \rightarrow (\quad \neg(Rx \wedge Ry) \wedge \\
& \qquad\qquad\qquad\qquad \neg(Bx \wedge By) \wedge \\
& \qquad\qquad\qquad\qquad \neg(Gx \wedge Gy)))
\end{aligned}
$$

Fagin's Theorem

Theorem (Fagin)

A class $\mathcal{C}$ of finite structures is definable by a sentence of *existential second-order logic* if, and only if, it is decidable by a *nondeterminisitic machine* running in polynomial time.

$$\text{ESO} = \text{NP}$$

One direction is easy: Given $\mathbb{A}$ and $\exists P_1 \ldots \exists P_m \varphi$.

a nondeterministic machine can guess an interpretation for $P_1, \ldots, P_m$ and then verify φ.

Fagin's Theorem

Given a machine M and an integer k, there is an ESO sentence φ such that $\mathbb{A} \models \varphi$ if, and only if, M accepts $[\mathbb{A}]_<$, for some order $<$ in n^k steps.

$\exists <$ $\exists$State $\exists$Head $\exists$Tape

$<$ is a linear order $\land$

State$(t+1, s_1) \rightarrow$ State$(t, s) \lor \ldots$

$\land$State$(t+1, s_2) \rightarrow \ldots$

$\land$Tape$(t+1, p) \leftrightarrow$ Head$(t, p) \ldots$

$\land$Head$(t+1, h+1) \leftrightarrow \ldots$

$\land$Head$(t+1, h-1) \leftrightarrow \ldots$

} encoding transitions of M

$\land$at time 0 the tape contains a description of $\mathbb{A}$

$\land$State(max, s) for some accepting s

Fagin's Theorem

State, Tape and Head are $2k$-ary relations, that use the lexicographic order on k-tuples.

To state that Tape encodes the input structure:

$$\begin{array}{ll} \forall \mathbf{x} & \mathbf{x} < n \rightarrow \mathsf{Tape}(0, \mathbf{x}) \\ & \mathbf{x} < n^a \rightarrow (\mathsf{Tape}(0, \mathbf{x} + n) \leftrightarrow R_1(\mathbf{x}|_a)) \\ \dots & \end{array}$$

where,

$$\mathbf{x} < n^a \quad : \quad \bigwedge_{i \leq (k-a)} x_i = 0$$

Is there a logic for P?

The major open question in *Descriptive Complexity* (first asked by Chandra and Harel in 1982) is whether there is a logic $\mathcal{L}$ such that

> for any class of finite structures $\mathcal{C}$, $\mathcal{C}$ is definable by a sentence of $\mathcal{L}$ if, and only if, $\mathcal{C}$ is decidable by a deterministic machine running in polynomial time.

Formally, we require $\mathcal{L}$ to be a *recursively enumerable* set of sentences, with a computable map taking each sentence to a Turing machine M and a polynomial time bound p such that (M, p) accepts a *class of structures*.

(Gurevich 1988)

Enumerating Queries

For a given structure $\mathbb{A}$ with n elements, there may be as many as $n!$ distinct strings $[\mathbb{A}]_<$ encoding it.

Given $(M_0, p_0), \ldots, (M_i, p_i), \ldots$—an enumeration of polynomially-clocked Turing machines.

Can we enumerate a subsequence of those that compute graph properties, i.e. are *encoding invariant*, while including all such properties?

Inductive Definitions

Let $\varphi(R, x_1, \ldots, x_k)$ be a first-order formula in the vocabulary $\sigma \cup \{R\}$

Associate an operator Φ on a given structure $\mathbb{A}$:

$$\Phi(R^{\mathbb{A}}) = \{\mathbf{a} \mid (\mathbb{A}, R^{\mathbb{A}}, \mathbf{a}) \models \varphi(R, \mathbf{x})\}$$

We define the *increasing* sequence of relations on $\mathbb{A}$:

$$\Phi^0 = \emptyset$$

$$\Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

The *inflationary fixed point* of Φ is the limit of this sequence.

On a structure with n elements, the limit is reached after at most n^k stages.

IFP

The logic IFP is formed by closing first-order logic under the rule:

If φ is a formula of vocabulary $\sigma \cup \{R\}$ then $[\mathbf{ifp}_{R,\mathbf{x}}\varphi](\mathbf{t})$ is a formula of vocabulary σ.

The formula is read as:

the tuple $\mathbf{t}$ is in the inflationary fixed point of the operator defined by φ

LFP is the similar logic obtained using *least fixed points* of *monotone* operators defined by *positive* formulas.

LFP and IFP have the same expressive power **(Gurevich-Shelah; Kreutzer)**.

Transitive Closure

The formula

$$[\textbf{ifp}_{T,xy}(x = y \vee \exists z(E(x,z) \wedge T(z,y)))](u,v)$$

defines the *transitive closure* of the relation E

The expressive power of IFP properly extends that of first-order logic.

On structures which come equipped with a linear order IFP expresses exactly the properties that are in PTime.

(Immerman; Vardi)

Immerman-Vardi Theorem

$\exists <$ $\exists$State $\exists$Head $\exists$Tape

$<$ is a linear order $\wedge$

$$\left.\begin{array}{l}
\text{State}(t+1, s_1) \rightarrow \text{State}(t, s) \vee \ldots \\
\wedge\text{State}(t+1, s_2) \rightarrow \ldots \\
\wedge\text{Tape}(t+1, p) \leftrightarrow \text{Head}(t, p) \ldots \\
\wedge\text{Head}(t+1, h+1) \leftrightarrow \ldots \\
\wedge\text{Head}(t+1, h-1) \leftrightarrow \ldots
\end{array}\right\} \begin{array}{l}\text{encoding} \\ \text{transitions} \\ \text{of } M\end{array}$$

$\wedge$at time 0 the tape contains a description of $\mathbb{A}$

$\wedge$State(max, s) for some accepting s

With a deterministic machine, the relations State, Tape and Head can be define *inductively*.

IFP vs. Ptime

The order cannot be built up inductively.

It is an open question whether a *canonical* string representation of a structure can be constructed in polynomial-time.

> If it can, there is a logic for PTime.
>
> If not, then PTime $\neq$ NP.

All PTime classes of structures can be expressed by a sentence of IFP with $<$, which is invariant under the choice of order. The set of all such sentences is not *r.e.*

IFP by itself is too weak to express all properties in PTime.

Evenness is not definable in IFP.

Finite Variable Logic

We write L^k for the first order formulas using only the variables $x_1, \ldots, x_k$.

$$(\mathbb{A}, \mathbf{a}) \equiv^k (\mathbb{B}, \mathbf{b})$$

denotes that there is no formula φ of L^k such that $\mathbb{A} \models \varphi[\mathbf{a}]$ and $\mathbb{B} \not\models \varphi[\mathbf{b}]$

If $\varphi(R, \mathbf{x})$ has k variables all together, then each of the relations in the sequence:

$$\Phi^0 = \emptyset; \; \Phi^{m+1} = \Phi^m \cup \Phi(\Phi^m)$$

is definable in L^{2k}.

Proof by induction, using *substitution* and *renaming* of bound variables.

Pebble Game

The k-pebble game is played on two structures $\mathbb{A}$ and $\mathbb{B}$, by two players—*Spoiler* and *Duplicator*—using k pairs of pebbles $\{(a_1, b_1), \ldots, (a_k, b_k)\}$.

Spoiler moves by picking a pebble and placing it on an element (a_i on anelement of $\mathbb{A}$ or b_i on an element of $\mathbb{B}$).

Duplicator responds by picking the matching pebble and placing it on an element of the other structure

Spoiler wins at any stage if the partial map from $\mathbb{A}$ to $\mathbb{B}$ definedby the pebble pairs is not a partial isomorphism

If *Duplicator* has a winning strategy for q moves, then $\mathbb{A}$ and $\mathbb{B}$ agree on all sentences of L^k of quantifier rank at most q. **(Barwise)**

$\mathbb{A} \equiv^k \mathbb{B}$ if, for every q, *Duplicator* wins the q round, k pebble game on $\mathbb{A}$ and $\mathbb{B}$. Equivalently (on finite structures) *Duplicator* has a strategy to play forever.

Evenness

To show that *Evenness* is not definable in IFP, it suffices to show that:

for every k, there are structures $\mathbb{A}_k$ and $\mathbb{B}_k$ such that $\mathbb{A}_k$ has an even number of elements, $\mathbb{B}_k$ has an odd number of elements and

$$\mathbb{A} \equiv^k \mathbb{B}.$$

It is easily seen that *Duplicator* has a strategy to play forever when one structure is a set containing k elements (and no other relations) and the other structure has $k+1$ elements.

Fixed-point Logic with Counting

Immerman proposed IFP + C—the extension of IFP with a mechanism for *counting*

Two sorts of variables:

- $x_1, x_2, \ldots$ range over $|A|$—the domain of the structure;
- $\nu_1, \nu_2, \ldots$ which range over *numbers* in the range $0, \ldots, |A|$

If $\varphi(x)$ is a formula with free variable x, then $\nu = \#x\varphi$ denotes that ν is the number of elements of A that satisfy the formula φ.

We also have the order $\nu_1 < \nu_2$, which allows us (using recursion) to define arithmetic operations.

Counting Quantifiers

C^k is the logic obtained from *first-order logic* by allowing:

- allowing *counting quantifiers*: $\exists^i x\,\varphi$; and
- only the variables $x_1, \ldots . x_k$.

Every formula of C^k is equivalent to a formula of first-order logic, albeit one with more variables.

For every sentence φ of IFP $+$ C, there is a k such that if $\mathbb{A} \equiv^{C^k} \mathbb{B}$, then

$$\mathbb{A} \models \varphi \quad \text{if, and only if,} \quad \mathbb{B} \models \varphi.$$

Cai-Fürer-Immerman Graphs

There are polynomial-time decidable properties of graphs that are not definable in IFP $+$ C. **(Cai, Fürer, Immerman, 1992)**

More precisely, we can construct a sequence of pairs of graphs $G_k, H_k (k \in \omega)$ such that:

- $G_k \equiv^{C^k} H_k$ for all k.
- There is a polynomial time decidable class of graphs that includes all G_k and excludes all H_k.

Still, IFP $+$ C is a *natural* level of expressiveness within PTime.

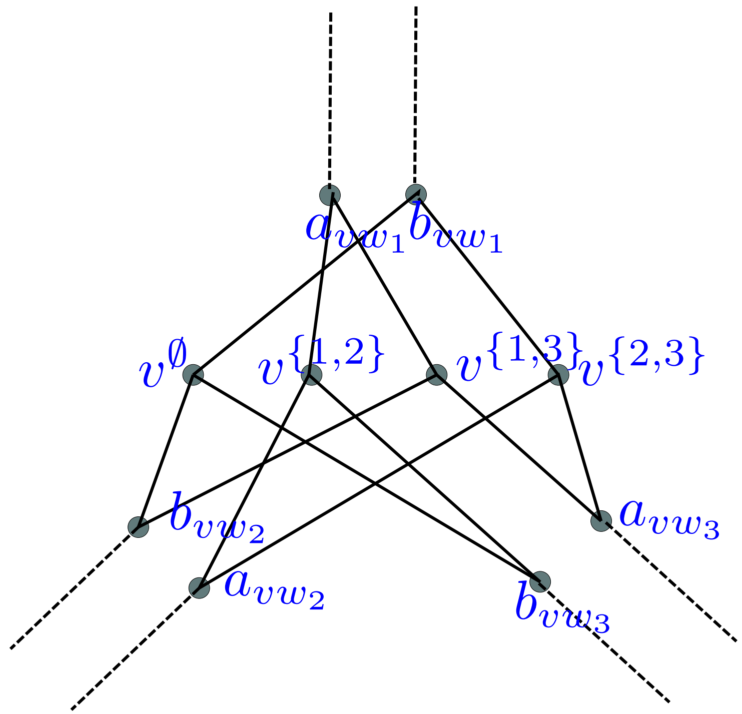
Constructing G_k and H_k

Given any graph G, we can define a graph X_G by replacing every edge with a pair of edges, and every vertex with a gadget.

The picture shows the gadget for a vertex v that is adjacent in G to vertices w_1, w_2 and w_3.

The vertex v^S is adjacent to $a_{vw_i} (i \in S)$ and $b_{vw_i} (i \notin S)$ and there is one vertex for all *even size* S.

The graph $\tilde{X}_G$ is like X_G except that at *one vertex* v, we include V^S for *odd size* S.



Properties

If G is *connected* and has *treewidth* at least k, then:

1. $X_G \not\cong \tilde{X}_G$; and
2. $X_G \equiv^{C^k} \tilde{X}_G$.

(1) allows us to construct a polynomial time property separating X_G and $\tilde{X}_G$.

(2) is proved by a game argument.

The original proof of **(Cai, Fürer, Immerman)** relied on the existence of balanced separators in G. The characterisation in terms of treewidth is from **(D., Richerby 07)**.

Bijection Games

$\equiv^{C^k}$ is characterised by a k-pebble *bijection game*. **(Hella 96)**.

The game is played on structures $\mathbb{A}$ and $\mathbb{B}$ with pebbles $a_1, \ldots, a_k$ on $\mathbb{A}$ and $b_1, \ldots, b_k$ on $\mathbb{B}$.

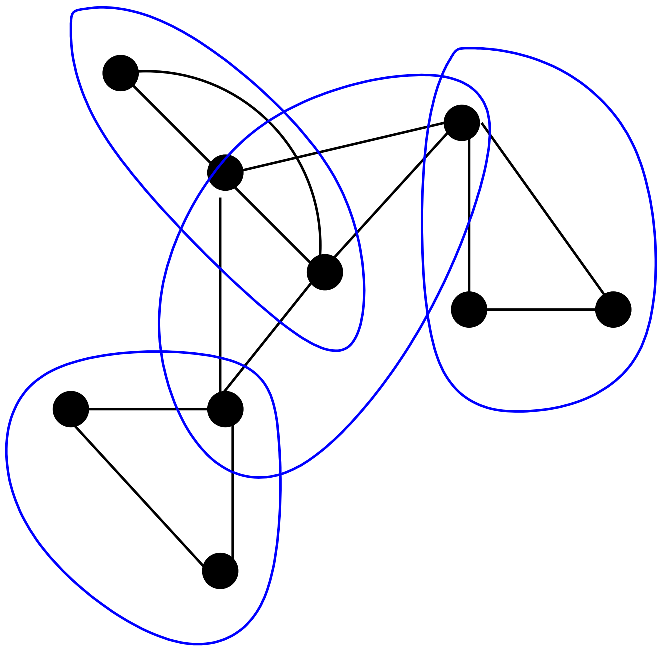
- *Spoiler* chooses a pair of pebbles a_i and b_i;
- *Duplicator* chooses a bijection $h : A \to B$ such that for pebbles a_j and $b_j (j \neq i)$, $h(a_j) = b_j$;
- *Spoiler* chooses $a \in A$ and places a_i on a and b_i on $h(a)$.

Duplicator loses if the partial map $a_i \mapsto b_i$ is not a partial isomorphism.
Duplicator has a strategy to play forever if, and only if, $\mathbb{A} \equiv^{C^k} \mathbb{B}$.

TreeWidth

The *treewidth* of a graph is a measure of how tree-like the graph is.

A graph has treewidth k if it can be covered by subgraphs of at most $k+1$ nodes in a tree-like fashion.



TreeWidth

Formal Definition:

For a graph $G = (V, E)$, a *tree decomposition* of G is a relation $D \subset V \times T$ with a tree T such that:

- for each $v \in V$, the set $\{t \mid (v, t) \in D\}$ forms a connected subtree of T; and
- for each edge $(u, v) \in E$, there is a $t \in T$ such that $(u, t), (v, t) \in D$.

The *treewidth* of G is the least k such that there is a tree T and a tree-decomposition $D \subset V \times T$ such that for each $t \in T$,

$$|\{v \in V \mid (v, t) \in D\}| \leq k + 1.$$

Cops and Robbers

A game played on an undirected graph $G = (V, E)$ between a player controlling k *cops* and another player in charge of a *robber*.

At any point, the cops are sitting on a set $X \subseteq V$ of the nodes and the robber on a node $r \in V$.

A move consists in the cop player removing some cops from $X' \subseteq X$ nodes and announcing a new position Y for them. The robber responds by moving along a path from r to some node s such that the path does not go through $X \setminus X'$.

The new position is $(X \setminus X') \cup Y$ and s. If a cop and the robber are on the same node, the robber is caught and the game ends.

Strategies and Decompositions

Theorem (Seymour and Thomas 93):
There is a winning strategy for the *cop player* with k cops on a graph G if, and only if, the tree-width of G is at most $k-1$.

It is not difficult to construct, from a tree decomposition of width k, a winning strategy for $k+1$ cops.

Somewhat more involved to show that a winning strategy yields a decomposition.

Cops, Robbers and Bijections

If G has treewidth k or more, than the *robber* has a winning strategy in the k-*cops and robbers* game played on G.

We use this to construct a winning strategy for Duplicator in the k-pebble bijection game on X_G and $\tilde{X}_G$.

- A bijection $h : X_G \to \tilde{X}_G$ is *good bar* v if it is an isomorphism everywhere except at the vertices v^S.
- If h is good bar v and there is a path from v to u, then there is a bijection h' that is good bar u such that h and h' differ only at vertices corresponding to the path from v to u.
- Duplicator plays bijections that are good bar v, where v is the robber position in G when the cop position is given by the currently pebbled elements.

Restricted Graph Classes

If we restrict the class of structures we consider, IFP $+$ C may be powerful enough to express all polynomial-time decidable properties.

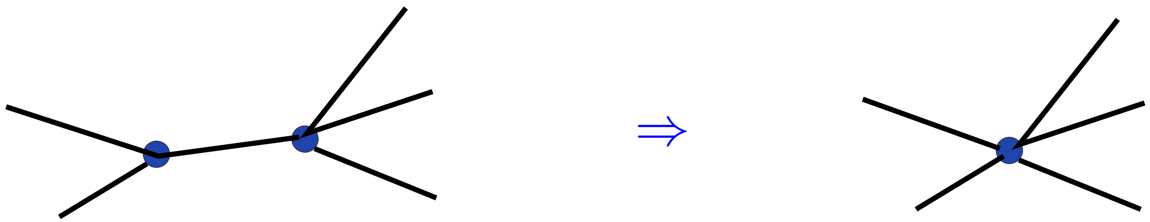
- IFP $+$ C captures PTime on *trees*. **(Immerman and Lander 1989)**.
- IFP $+$ C captures PTime on any class of graphs of *bounded treewidth*. **(Grohe and Mariño 1999)**.
- IFP $+$ C captures PTime on the class of *planar graphs*. **(Grohe 1998)**.

In each case, the proof proceeds by showing that for any G in the class, a *canonical, ordered* representaton of G can be interpreted in G using IFP $+$ C.

Graph Minors

We say that a graph G is a minor of graph H (written $G \prec H$) if G can be obtained from H by repeated applications of the operations:

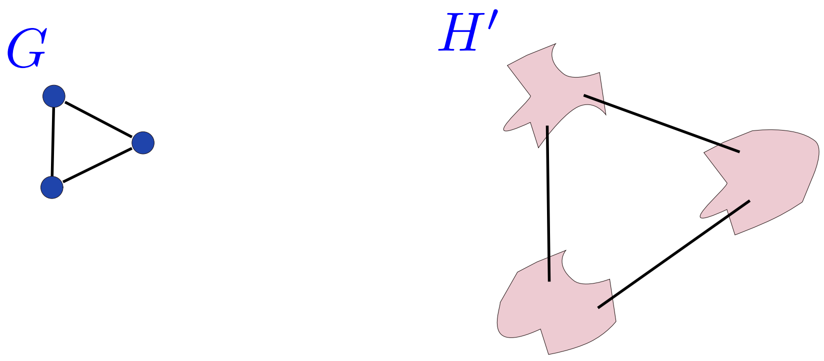
- *delete an edge*;
- *deete an vertex* (and all incident edges); and
- *contract an edge*



Graph Minors

Alternatively, $G = (V, E)$ is a minor of $H = (U, F)$, if there is a graph $H' = (U', F')$ with $U' \subseteq U$ and $F' \subseteq F$ and a surjective map $M : U' \to V$ such that

- for each $v \in V$, $M^{-1}(v)$ is a connected subgraph of H'; and
- for each edge $(u, v) \in E$, there is an edge in F' between some $x \in M^{-1}(u)$ and some $y \in M^{-1}(v)$.



Facts about Graph Minors

- G is planar if, and only if, $K_5 \not\prec G$ and $K_{3,3} \not\prec G$.
- If $G \subset H$ then $G \prec H$.
- The relation $\prec$ is transitive.
- If $G \prec H$, then $\mathrm{tw}(G) \leq \mathrm{tw}(H)$.
- If $\mathrm{tw}(G) < k - 1$, then $K_k \not\prec G$.

Say that a class of structures $\mathcal{C}$ *excludes* H *as a minor* if $H \not\prec G\mathbb{A}$ for all $\mathbb{A} \in \mathcal{C}$.

$\mathcal{C}$ has *excluded minors* if it excludes some H as a minor (equivalently, it excludes some K_k as a minor).

- $\mathcal{T}_k$ excludes K_{k+2} as a minor.

More Facts about Graph Minors

Theorem (Robertson-Seymour)

In any infinite collection $\{G_i \mid i \in \omega\}$ of graphs, there are i, j with $G_i \prec G_j$.

Corollary

For any class $\mathcal{C}$ *closed under minors*, there is a finite collection $\mathcal{F}$ of graphs such that $G \in \mathcal{C}$ *if, and only if,* $F \not\prec G$ for all $F \in \mathcal{F}$.

Theorem (Robertson-Seymour)

For any G there is an $O(n^3)$ algorithm for deciding, given H, whether $G \prec H$.

Corollary

Any class $\mathcal{C}$ closed under minors is decidable in *cubic time*.

Ptime on Minor-Closed Classes

Conjecture **(Grohe)**

IFP $+$ C captures PTime on every proper minor-closed class of graphs.

Theorem **(Grohe 2008)**

IFP $+$ C captures PTime on the class of graphs that exclude K_5 as a minor.

The Cai-Fürer-Immerman construction cannot be used to refute Grohe's conjecture.

> If $\mathcal{C}$—a class of graphs contains X_G and $\tilde{X}_G$ for graphs G of *unbounded treewidth*, then $\mathcal{C}$ does not exclude any graph as a minor.
>
> **(D., Richerby 2007)**

Logics with Choice

Extending IFP with a *choice* operator allows us to define all polynomial-time decidable classses.

This is akin to adding *order* to the logic. It also allows sentences whose interpretation is dependent on choices, and therfore not determined by the structure alone.

The sentences that are invariant under choices express all polynomial-time properties, but do not form an *r.e.* set.

Gire and Hoang considered a method of restricting the choice operator to ensure that the interpretation of sentences was invariant.

Non-deterministic Choice

Given two formulas $\varphi(R, X, \mathbf{x}); \psi(R, X, \mathbf{y})$ and a structure $\mathbb{A}$, we define the following sequence of pairs of relations.

$\Phi^0 = \emptyset \quad \Psi^0 = \emptyset$;

$\Phi^{i+1} = \Phi^i \cup \varphi^{\mathbb{A}}(\Phi^i/R, \Psi^i/X)$;

$\Psi^{i+1} = \{\mathbf{a}\}$ for some $\mathbf{a}$ such that $\mathbb{A} \models \psi(\Phi^i/R, \Psi^i/X)[\mathbf{a}]$.

The sequence Φ^i converges to a limit.

We say that the pair of formulas $\varphi; \psi$ is *choice-invariant* if the limit does not depend on the choice of the sequence Ψ^i.

The collection of choice-invariant formulas captures PTime, but is not an $r.e.$ set.

Symmetric Choice

Alter the definition of the sequence so that:

$\Phi^0 = \emptyset \quad \Psi^0 = \emptyset$;

$\Phi^{i+1} = \Phi^i \cup \varphi^{\mathbb{A}}(\Phi^i/R, \Psi^i/X)$;

$$\Psi^{i+1} = \begin{cases} \{\mathbf{a}\} \text{ for } \mathbb{A} \models \psi(\Phi^i/R, \Psi^i/X)[\mathbf{a}] & \\ \quad \text{if } \psi(\Phi^i/R, \Psi^i/X) \text{ defines an automorphism class of } \mathbb{A} & \\ \emptyset \quad \text{otherwise} & \end{cases}$$

Now, the limit of the sequence Φ^i is independent of the choices.

However, it is not clear that a pair of formulas can be evaluated in *polynomial time*.

The semantics involves an *automorphism test*.

Specified Symmetric Choice

The logic of *specified symmetric choice* (SSC-IFP) defines fixed points for triples $\varphi; \psi; \theta$ of formulas.

$\Psi^{i+1} = \{\mathbf{a}\}$ only if $\psi(\Phi^i/R, \Psi^i/X)$ defines an automorphism class *and* this is witnessed by θ (*i.e.* this formula defines, for each pair of tuples satisfying ψ, an automorphism mapping one to the other).

Any formula of SSC-IFP can be evaluated in polynomial time.

The Cai-Fürer-Immerman property can be expressed in SSC-IFP.

(Gire-Hoang 98)

Open Question: Is there a polynomial-time decidable property that cannot be expressed in SSC-IFP?

Choiceless Polynomial Time

Choiceless Polynomial Time ($\tilde{C}PT$) is a class of computational problems defined by **Blass, Gurevich and Shelah**.

It is based on a *machine model (Gurevich Abstract State Machines)* that works directly on a relational structure (rather than on a string representation).

The machine can access the collection of hereditarily finite sets over the universe of the structure.

$\tilde{C}PT$ is the polynomial time and space restriction of the machines.

$\tilde{C}PT$ is strictly more expressive than IFP, but still cannot express counting properties.

Consider $\tilde{C}PT(Card)$—the extension of $\tilde{C}PT$ with counting.

Does it express all properties in PTime?

Choiceless Polynomial Time

$\tilde{\text{C}}\text{PT}$ can express the property of **Cai, Fürer and Immerman**.

Any program of $\tilde{\text{C}}\text{PT}(\text{Card})$ that expresses the CFI property must use sets of *unbounded rank*.

$\text{IFP} + \text{C}$ can be translated to programs of $\tilde{\text{C}}\text{PT}(\text{Card})$ of bounded rank.

(D., Richerby and Rossman 2008)

Ongoing Research

Is there a PTime decidable class that cannot be expressed in SSC-IFP?

Is there PTime decidable class that is not in $\tilde{\mathsf{C}}\mathsf{PT}(\mathsf{Card})$?

How does the expressive power of SSC-IFP compare with that of $\tilde{\mathsf{C}}\mathsf{PT}(\mathsf{Card})$?

Are there other natural extensions of IFP that might capture PTime?