Model-Checking First-Order Logic

Automata and Locality

Anuj Dawar

University of Cambridge

GAMES/CSL, Lausanne, 11 September 2007

The Model-Checking Problem

We are interested in the computational complexity of the following decision problem:

> Given: a first-order formula φ and a structure $\mathbb{A}$
>
> Decide: if $\mathbb{A} \models \varphi$

Or, what is the complexity of the *satisfaction relation* for first-order logic?

For the rest of the talk,

> We assume that $\mathbb{A}$ is finite and given explicitly in the input.
>
> We generally write l for the length of φ and n for the size of $\mathbb{A}$.
>
> We assume $\mathbb{A}$ is a *directed, coloured graph*—i.e., a structure interpreting one binary relation E, some unary relations and some constants. We write $G\mathbb{A}$ for the underlying undirected graph.

Naïve Algorithm

The straightforward algorithm proceeds recursively on the structure of φ:

- Atomic formulas by direct lookup.
- Boolean connectives are easy.
- If $\varphi \equiv \exists x\, \psi$ then for each $a \in \mathbb{A}$ check whether

$$(\mathbb{A}, c \mapsto a) \models \psi[c/x],$$

  where c is a new constant symbol.

This shows that the model-checking problem can be solved in time $O(ln^m)$ and $O(m \log n)$ space, where m is the nesting depth of quantifiers in φ (or by a more careful accounting, the number of distinct variables occurring in φ).

Complexity

This shows that the model checking problem is in PSpace and for a fixed sentence φ, the problem of deciding membership in the class

$$\mathrm{Mod}(\varphi) = \{\mathbb{A} \mid \mathbb{A} \models \varphi\}$$

is in *logarithmic space* and *polynomial time*.

QBF—satisfiability of quantified Boolean formulas can be easily reduced to the model checking problem with $\mathbb{A}$ a fixed two-element structure.

Thus, the problem is PSpace-complete, even for fixed $\mathbb{A}$.

Is FO contained in an initial segment of PTime?

Question posed in the title of a paper by **Stolboushkin and Taitslin (CSL 1994)**.

> Is there a fixed c such that for every first-order φ, $\mathrm{Mod}(\varphi)$ is decidable in time $O(n^c)$?

If PTime $=$ PSpace, then the answer is yes, as the satisfaction relation is then itself decidable in time $O(n^c)$ and this bounds the time for all formulas φ.

> Thus, though we expect the answer is no, this would be difficult to prove.

A more uniform version of their question is:

> Is there a constant c and a computable function f so that the satisfaction relation for first-order logic is decidable in time $O(f(l)n^c)$?

Fixed Parameter Tractability

If $\mathrm{Mod}(\varphi)$ is decidable in time $O(n^c)$ and the constants involved are bounded by some computable function of l, then the model-checking problem is *fixed-parameter tractable* (FPT) with the formula length as parameter.

The parameterised model-checking problem is AW$[\star]$-complete.

The parameterised model-checking problem restricted to Π_t formulas is hard for the class $W[t]$.

Thus, the whole edifice of parameterized intractability would collapse.

Restricted Classes of Structures

One way to get a handle on the complexity of first-order model checking is to consider restricted classes of structures.

> Given: a first-order formula φ and a structure $\mathbb{A} \in \mathcal{C}$
> Decide: if $\mathbb{A} \models \varphi$

For many classes $\mathcal{C}$, this problem has been shown to be FPT.

1. Every first-order (or even MSO) definable class of strings is a regular language and so decidable in linear time.
2. $\mathcal{T}_k$—the class of structures of tree-width at most k.
   Courcelle (1990) shows that every MSO definable property is decidable in linear time on this class.

Restricted Classes of Structures

3. $\mathcal{D}_k$—the class of structures of *degree* bounded by k.

   Seese (1996) shows that every FO definable property is decidable in linear time.

4. LTW$_t$—the class of structures of *local tree-width* bounded by a function t.

   Frick and Grohe (2001) show that every FO definable property is decidable in quadratic time.

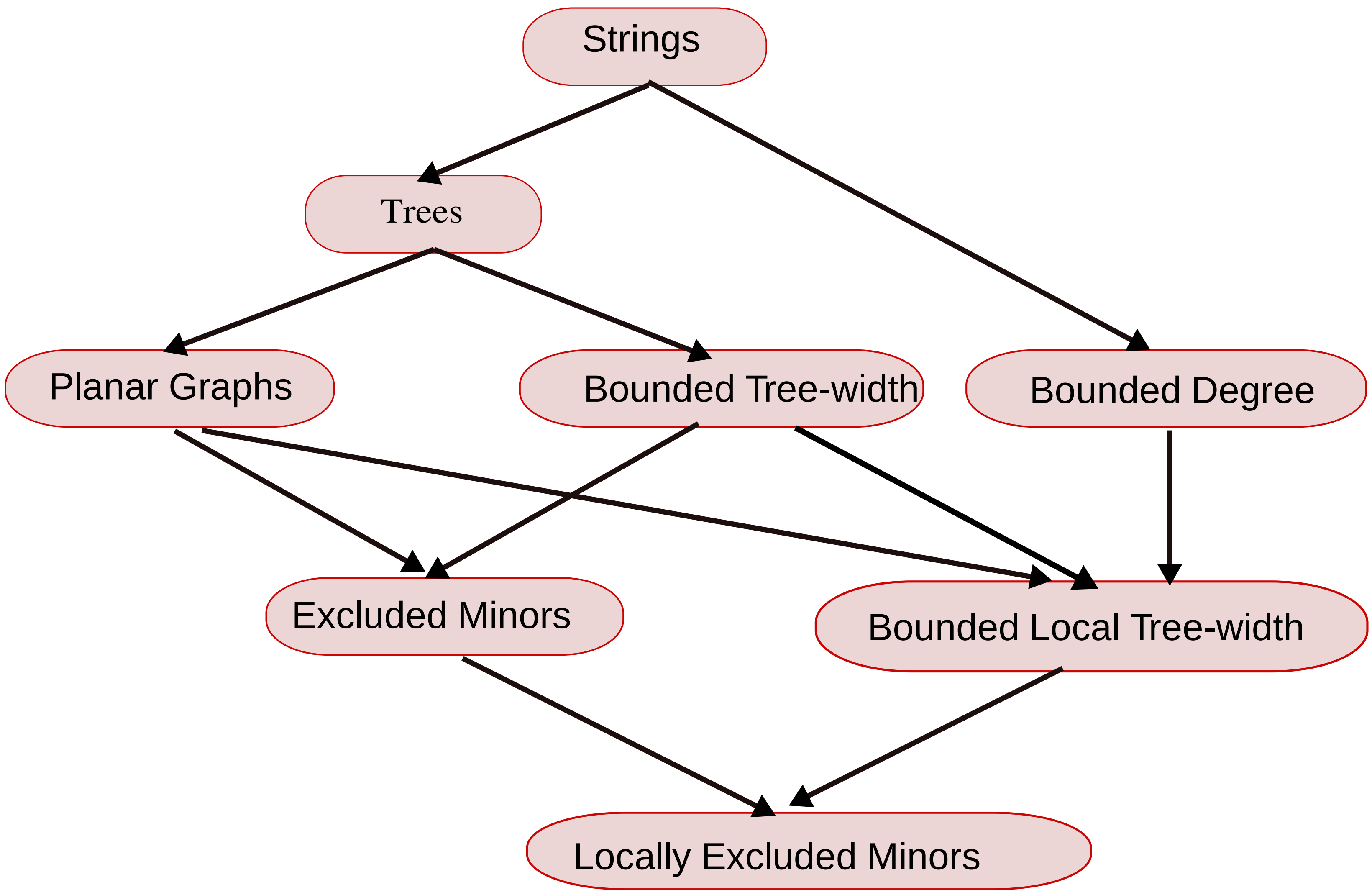
5. $\mathcal{M}_k$—the class of structures *excluding* K_k *as a minor*.

   Flum and Grohe (2001) show that every FO definable property is decidable in time $O(n^5)$.

6. LEM$_t$—the class of structures with *locally excluded minors* given by t.

   D., Grohe and Kreutzer (2007) show that every FO definable property is decidable in time $O(n^6)$.

Map of Restrictions



Automata and Locality

The methods of proof for the results are combinations of two general techniques:

- Methods of *automata* or *decompositions*; and
- Methods based on the *locality* of first-order logic.

In the rest of this talk, we first review these two methods using the results on

strings;

graphs of *bounded tree-width*; and

graphs of *bounded degree*.

We then show how the methods combine in the other cases.

Strings

Structures $\mathbb{A}$ where the binary relation E forms a connected graph, with each node having *in-degree* and *out-degree* at most 1, can be viewed as words over the alphabet $\mathscr{P}(\mathcal{U})$, where $\mathcal{U}$ is the collection of unary relation symbols.

Theorem (Büchi, Elgot, Trakhtenbrot)

For any sentence φ of MSO, the language $L_\varphi = \{s \mid s \text{ a string and } s \models \varphi\}$ is regular.

A particularly perspicuous proof of this is obtained by using the *Myhill-Nerode theorem*.

Myhill-Nerode Theorem

Theorem (Myhill-Nerode)

A language L is regular *if, and only if,* there is an equivalence relation $\sim$ on strings such that:

1. $\sim$ has finite index on the set of all strings;
2. $\sim$ is a congruence for string concatenation, i.e.

$$s_1 \sim t_1 \text{ and } s_2 \sim t_2 \quad \Rightarrow \quad s_1 \cdot s_2 \sim t_1 \cdot t_2;$$

   and

3. L is the union of some number of $\sim$-equivalence classes.

MSO Languages

φ—an MSO sentence of quantifier rank m.

$\mathbb{A} \equiv_m^{(\text{MSO})} \mathbb{B}$ if they cannot be distinguished by any first-order (MSO) sentence of quantifier rank m.

- $\equiv_m^{\text{MSO}}$ has finite index since there are, up to logical equivalence, only finitely many MSO sentences of quantifier rank at most m.
- $\equiv_m^{\text{MSO}}$ is a congruence for concatenation by an easy argument using *Ehrenfeucht-Fraïssé games* (a special case of the *Feferman-Vaught theorem*).
- It is immediate that L_φ is closed under $\equiv_m^{\text{MSO}}$.

Tree-Width

Tree-width is a measure of how *tree-like* a structure is.

For a graph $G = (V, E)$, a *tree decomposition* of G is a relation $D \subset V \times T$ with a tree T such that:

- for each $v \in V$, the set $\{t \mid (v, t) \in D\}$ forms a connected subtree of T; and
- for each edge $(u, v) \in E$, there is a $t \in T$ such that $(u, t), (v, t) \in D$.

The *tree-width* of G is the least k such that there is a tree T and a tree decomposition $D \subset V \times T$ such that for each $t \in T$,

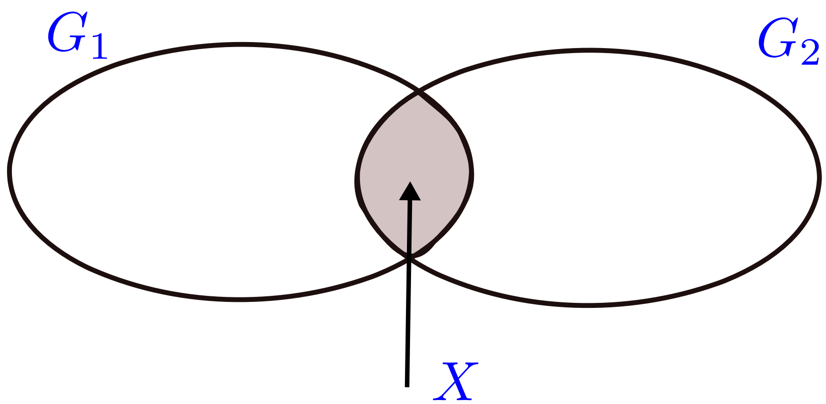
$$|\{v \in V \mid (v, t) \in D\}| \leq k + 1.$$

Tree-Width

Looking at the decomposition *bottom-up*, a graph of tree-width k is obtained from graphs with at most $k+1$ nodes through a finite sequence of applications of the operation of taking *sums over sets* of at most k elements.

$$G_1 \oplus_X G_2$$

$$|X| \leq k$$



We let $\mathcal{T}_k$ denote the class of structures $\mathbb{A}$ such that $\mathrm{tw}(G\mathbb{A}) \leq k$.

Courcelle's Theorem

Theorem (Courcelle)

For any MSO sentence φ and any k there is a linear time algorithm that decides, given $\mathbb{A} \in \mathcal{T}_k$ whether $\mathbb{A} \models \varphi$.

A proof relies on the fact (proved by **Bodlaender**) that there is an algorithm (linear in n, exponential in k) that given a graph $G \in \mathcal{T}_k$ computes a tree decomposition of G of width k.

Given $\mathbb{A} \in \mathcal{T}_k$ and φ, compute:

- from $\mathbb{A}$ a labelled tree T; and
- from φ a bottom-up tree automaton $\mathcal{A}$

such that $\mathcal{A}$ accepts T if, and only if, $\mathbb{A} \models \varphi$.

The Labelled Tree

$C = \{c_0, \ldots, c_k\}$ a set of $k+1$ new constants.

$(\mathbb{A}, \rho)$—expansion of $\mathbb{A}$ with $\rho : C \rightharpoonup V$, a partial map interpreting some of the constants in C.

Let

- $\mathcal{B}_k$—the collection of $(\mathbb{A}, \rho)$ such that $\mathbb{A}$ has at most $k+1$ elements.
- erase$_i$—an operation which takes $(\mathbb{A}, \rho)$ to $(\mathbb{A}, \rho')$, where ρ' is as ρ but without c_i.
- a binary operation of union disjoint over C:

$$(\mathbb{A}_1, \rho_1) \oplus_C (\mathbb{A}_1, \rho_2)$$

Congruence

- Any $\mathbb{A} \in \mathcal{T}_k$ is obtained from $\mathcal{B}_k$ by finitely many applications of the operations erase_i and $\oplus_C$.
- If $\mathbb{A}_1, \rho_1 \equiv_m^{\text{MSO}} \mathbb{A}_2, \rho_2$, then

$$\text{erase}_i(\mathbb{A}_1, \rho_1) \equiv_m^{\text{MSO}} \text{erase}_i(\mathbb{A}_2, \rho_2)$$

- If $\mathbb{A}_1, \rho_1 \equiv_m^{\text{MSO}} \mathbb{A}_2, \rho_2$, and $\mathbb{B}_1, \sigma_1 \equiv_m^{\text{MSO}} \mathbb{B}_2, \sigma_2$ then

$$(\mathbb{A}_1, \rho_1) \oplus_C (\mathbb{B}_1, \sigma_1) \equiv_m^{\text{MSO}} (\mathbb{A}_2, \rho_2) \oplus_C (\mathbb{B}_2, \sigma_2)$$

Model-Checking on $\mathcal{T}_k$

Any $\mathbb{A} \in \mathcal{T}_k$ can be represented as a finite tree, with leaves labelled by elements of $\mathcal{B}_k$, internal nodes labelled by operations erase_i and $\oplus_C$.

We can then compute the $\equiv_m^{\mathsf{MSO}}$ type of $\mathbb{A}$ bottom-up.

This establishes the following:

> The model-checking problem for MSO is decidable in time $f(l, k)n$, where
>
> - f is some computable function
> - l is the length of the input formula
> - k is the tree-width of the input structure
> - n is the size of the input structure.

The Method of Automata

Suppose $\mathcal{C}$ is a class of structures such that there is a finite class $\mathcal{B}$ and a finite collection Op of operations such that:

- $\mathcal{C}$ is contained in the closure of $\mathcal{B}$ under the operations in Op;
- there is a polynomial-time algorithm which computes, for any $\mathbb{A} \in \mathcal{C}$, an Op-decomposition of $\mathbb{A}$ over $\mathcal{B}$; and
- for each m, the equivalence class $\equiv_m^{(\mathsf{MSO})}$ is an *effective* congruence with respect to to all operations $o \in \mathsf{Op}$ (i.e., the $\equiv_m^{(\mathsf{MSO})}$-type of $o(\mathbb{A}_1, \ldots, \mathbb{A}_s)$ can be computed from the $\equiv_m^{(\mathsf{MSO})}$-types of $\mathbb{A}_1, \ldots, \mathbb{A}_s$).

Then, FO (MSO) model-checking is fixed-parameter tractable on $\mathcal{C}$.

Relaxations of the Method

1. Instead of requiring $\mathcal{B}$ be finite, require that *model-checking is in FPT over* $\mathcal{B}$.
2. In place of $\equiv_m^{(\text{MSO})}$, we can take any sequence of equivalence relations $\sim_m$ $(m \in \mathbb{N})$ satisfying
   - for every φ there is an m such that models of φ are closed under $\sim_m$; and
   - for all m, $\sim_m$ has finite index.

Note: letting $\mathbb{A} \sim_m \mathbb{B}$ if $\mathbb{A}, \mathbb{B}$ cannot be distinguished by a formula of *length* m, does not yield a congruence with respect to disjoint union.

There is no elementary function e such that $\mathbb{A}_1 \sim_{e(m)} \mathbb{B}_1$ and $\mathbb{A}_2 \sim_{e(m)} \mathbb{B}_2$ implies $\mathbb{A}_1 \oplus \mathbb{A}_2 \sim_m \mathbb{B}_1 \oplus \mathbb{B}_2$.

(D.,Grohe, Kreutzer, Schweikardt 2007)

Bounded Degree

$\mathcal{D}_k$—the class of structures $\mathbb{A}$ in which every element has degree (in-degree $+$ out-degree) at most k.

Theorem (Seese)

For every sentence φ of FO and every k there is a linear time algorithm which, given a structure $\mathbb{A} \in \mathcal{D}_k$ determines whether $\mathbb{A} \models \varphi$.

Note: this is not true for MSO unless P $=$ NP.

The proof is based on *locality* of first-order logic. Specifically, *Hanf's theorem*.

Hanf Types

For an element a in a structure $\mathbb{A}$, define

> $N_r^{\mathbb{A}}(a)$—the substructure of $\mathbb{A}$ generated by the elements whose distance from a (in $G\mathbb{A}$) is at most r.

We say $\mathbb{A}$ and $\mathbb{B}$ are *Hanf equivalent* with radius r and threshold q ($\mathbb{A} \simeq_{r,q} \mathbb{B}$) if, for every $a \in A$ the two sets

$$\{a' \in a \mid N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{A}}(a')\} \quad \text{and} \quad \{b \in B \mid N_r^{\mathbb{A}}(a) \cong N_r^{\mathbb{B}}(b)\}$$

either have the same size or both have size greater than q;

and, similarly for every $b \in B$.

Hanf Locality Theorem

Theorem (Hanf)

For every vocabulary σ and every m there are $r \leq 3^m$ and $q \leq m$ such that for any σ-structures $\mathbb{A}$ and $\mathbb{B}$: if $\mathbb{A} \simeq_{r,q} \mathbb{B}$ then $\mathbb{A} \equiv_m \mathbb{B}$.

In other words, if $r \geq 3^m$, the equivalence relation $\simeq_{r,m}$ is a refinement of $\equiv_m$.

For $\mathbb{A} \in \mathcal{D}_k$:

$N_r^{\mathbb{A}}(a)$ has at most $k^r + 1$ elements

each $\simeq_{r,m}$ has finite index.

Each $\simeq_{r,m}$-class t can be characterised by a finite table, I_t, giving isomorphism types of neighbourhoods and numbers of their occurrences up to threshold m.

Model-Checking on $\mathcal{D}_k$

For a sentence φ of FO, we can compute a set of tables $\{I_1, \ldots, I_s\}$ describing $\simeq_{r,m}$-classes consistent with it.

This computation is independent of any structure $\mathbb{A}$.

Given a structure $\mathbb{A} \in \mathcal{D}_k$,

for each a, determine the isomorphism type of $N_r^{\mathbb{A}}(a)$

construct the table describing the $\simeq_{r,m}$-class of $\mathbb{A}$.

compare against $\{I_1, \ldots, I_s\}$ to determine whether $\mathbb{A} \models \varphi$.

For fixed k, r, m, this requires time *linear* in the size of $\mathbb{A}$.

Note: model-checking for FO is in $O(f(l, k)n)$.

Local Tree-Width

Let $t : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function.

LTW$_t$—the class of structures $\mathbb{A}$ such that for every $a \in A$:

$GN_r^{\mathbb{A}}(a)$ has tree-width at most $t(r)$. **(Eppstein; Frick-Grohe)**.

We say that $\mathcal{C}$ has *bounded local tree-width* if there is some function t such that $\mathcal{C} \subseteq$ LTW$_t$.

Examples:

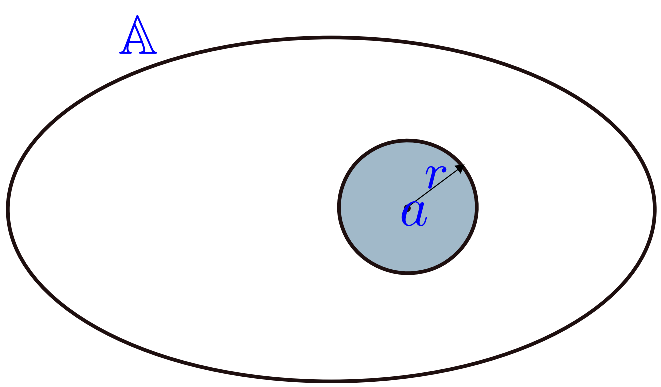
1. $\mathcal{T}_k$ has local tree-width bounded by the constant function $t(r) = k$.
2. $\mathcal{D}_k$ has local tree-width bounded by $t(r) = k^r + 1$.
3. Planar graphs have local tree-width bounded by $t(r) = 3r$.

Bounded Local Tree-Width

Theorem (Frick-Grohe)

For any class $\mathcal{C}$ of bounded local tree-width and any $\varphi \in$ FO, there is a *quadratic* time algorithm that decides, given $\mathbb{A} \in \mathcal{C}$, whether $\mathbb{A} \models \varphi$.

The idea:



For each a, the structure $N_r^{\mathbb{A}}(a)$ has tree-width bounded by $t(r)$.
Use the linear time algorithm on $T_{t(r)}$ to determine $\equiv_m$-type of $N_r^{\mathbb{A}}(a)$.

Hanf's theorem uses *isomorphism types* of $N_r^{\mathbb{A}}(a)$. We use *Gaifman's locality theorem* instead.

Gaifman's Theorem

We write $\delta(x, y) > d$ for the formula of FO that says that the distance between x and y is greater than d.

We write $\psi^N(x)$ to denote the formula obtained from $\psi(x)$ by relativising all quantifiers to the set N.

A *basic local sentence* is a sentence of the form

$$\exists x_1 \cdots \exists x_s \left(\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \wedge \bigwedge_i \psi^{N_r(x_i)}(x_i) \right)$$

Theorem (Gaifman)

Every first-order sentence is equivalent to a Boolean combination of basic local sentences.

Using Gaifman's Theorem

How do we evaluate a basic local sentence
$\exists x_1 \cdots \exists x_s \left(\bigwedge_{i \neq j} \delta(x_i, x_j) > 2r \wedge \bigwedge_i \psi^{N_r(x_i)}(x_i) \right)$ in a structure $\mathbb{A}$?

For each $a \in A$, determine whether

$$N_r^{\mathbb{A}}(a) \models \psi[a]$$

using the linear time model-checking algorithm on $\mathcal{T}_{t(r)}$.

Label a red if so.

We now want to know whether there exists a $2r$-*scattered* set of red vertices of size s.

Finding a Scattered Set

Choose red vertices from $\mathbb{A}$ in some order, removing the $2r$-neighbourhood of each chosen vertex.

$a_1 \in \mathbb{A}$,
$a_2 \in \mathbb{A} \setminus N_{2r}^{\mathbb{A}}(a_1)$,
$a_3 \in \mathbb{A} \setminus (N_{2r}^{\mathbb{A}}(a_1) \cup N_{2r}^{\mathbb{A}}(a_2)), \ldots$

If the process continues for s steps, we have found a $2r$-scattered set of size s.

Otherwise, for some $u < s$ we have found $a_1, \ldots, a_u$ such that all red vertices are contained in

$$N_{2r}^{\mathbb{A}}(a_1, \ldots, a_u)$$

This is a structure of tree-width at most $t(2rs)$ and the property of containing a $2r$-scattered set of *red* vertices of size s can be stated in FO.

Method of Locality

- Suppose we have a function, associating a parameter $k_{\mathbb{A}} \in \mathbb{N}$ with each structure $\mathbb{A}$.
- Suppose we have an algorithm which, given $\mathbb{A}$ and φ decides $\mathbb{A} \models \varphi$ in time

$$g(l, k_{\mathbb{A}})n^c$$

  for some computable function g and some constant c.

- Let $\mathcal{C}$ be a class of structures of *bounded local* k, i.e.

  there is a computable function $t : \mathbb{N} \to \mathbb{N}$ such that for every $\mathbb{A} \in \mathcal{C}$ and $a \in \mathbb{A}$, $k_{N_r^{\mathbb{A}}(a)} < t(r)$.

Then, there is an algorithm which, given $\mathbb{A} \in \mathcal{C}$ and φ decides whether $\mathbb{A} \models \varphi$ in time

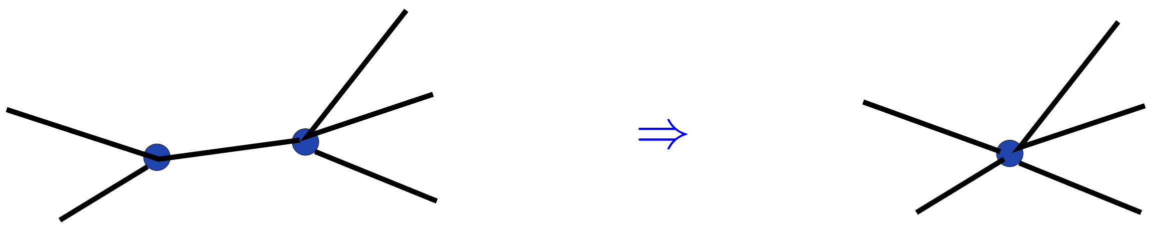
$$f(l)n^{c+1}$$

for some computable function f.

Graph Minors

We say that a graph G is a minor of graph H (written $G \prec H$) if G can be obtained from H by repeated applications of the operations:

- *delete an edge*;
- *delete a vertex* (and all incident edges); and
- *contract an edge*

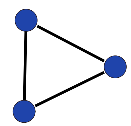


Graph Minors

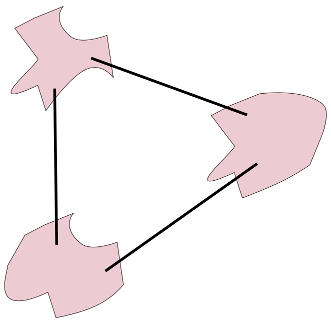
Alternatively, $G = (V, E)$ is a minor of $H = (U, F)$, if there is a graph $H' = (U', F')$ with $U' \subseteq U$ and $F' \subseteq F$ and a surjective map $M : U' \to V$ such that

- for each $v \in V$, $M^{-1}(v)$ is a connected subgraph of H'; and
- for each edge $(u, v) \in E$, there is an edge in F' between some $x \in M^{-1}(u)$ and some $y \in M^{-1}(v)$.

G



H'



Facts about Graph Minors

- G is planar if, and only if, $K_5 \not\prec G$ and $K_{3,3} \not\prec G$.
- If $G \subset H$ then $G \prec H$.
- The relation $\prec$ is transitive.
- If $G \prec H$, then $\mathrm{tw}(G) \leq \mathrm{tw}(H)$.
- If $\mathrm{tw}(G) < k - 1$, then $K_k \not\prec G$.

Say that a class of structures $\mathcal{C}$ *excludes* H *as a minor* if $H \not\prec G\mathbb{A}$ for all $\mathbb{A} \in \mathcal{C}$.

$\mathcal{C}$ has *excluded minors* if it excludes some H as a minor (equivalently, it excludes some K_k as a minor).

- $\mathcal{T}_k$ excludes K_{k+2} as a minor.

More Facts about Graph Minors

Theorem (Robertson-Seymour)

In any infinite collection $\{G_i \mid i \in \omega\}$ of graphs, there are i, j with $G_i \prec G_j$.

Corollary

For any class $\mathcal{C}$ *closed under minors*, there is a finite collection $\mathcal{F}$ of graphs such that $G \in \mathcal{C}$ *if, and only if,* $F \not\prec G$ for all $F \in \mathcal{F}$.

Theorem (Robertson-Seymour)

For any G there is an $O(n^3)$ algorithm for deciding , given H, whether $G \prec H$.

Corollary

Any class $\mathcal{C}$ closed under minors is decidable in *cubic time*.

Decomposing Graphs with Excluded Minors

Write $\mathcal{M}_k$ for the class of graphs G such that $K_k \not\prec G$.

from now on, we elide the distinction between restrictions on $\mathbb{A}$ and $G\mathbb{A}$.

Robertson and Seymour show how to obtain a decomposition of graphs in $\mathcal{M}_k$.

Grohe shows that this can be done over graphs of *almost bounded local tree-width*.

Let

$$\mathcal{L}_\lambda = \{G \mid \forall H \prec G: \ \mathrm{ltw}_r(H) \leq \lambda r\}$$

$$\mathcal{L}_{\lambda,\mu} = \{G \mid \exists v_1, \ldots, v_\mu: \ G \setminus \{v_1, \ldots, v_\mu\} \in \mathcal{L}_\lambda\}$$

Almost Bounded Local Tree-width

Classes $\mathcal{L}_\lambda$ and $\mathcal{L}_{\lambda,\mu}$ are *minor-closed* and so decidable in cubic time.

Given $G \in \mathcal{L}_{\lambda,\mu}$, we can find $v_1, \ldots, v_\mu$ witnessing this in time $O(n^4)$.

For each v, check if $G - v$ is in $\mathcal{L}_{\lambda,\mu-1}$.

If so, add v to the list and proceed with $G - v$ and $\mathcal{L}_{\lambda,\mu-1}$.

Question: Is this algorithm in time $O(f(\lambda,\mu)n^4)$ for a *computable* function f?

There is a polynomial-time computable map taking a structure $\mathbb{A} \in \mathcal{L}_{\lambda,\mu}$ to $\mathbb{A}' \in \mathcal{L}_\lambda$ so that the FO-type of $\mathbb{A}$ is determined by that of $\mathbb{A}'$.

$\mathbb{A}'$ is obtained from $\mathbb{A} \setminus \{v_1, \ldots, v_\mu\}$ by adding new relations $S_1, \ldots, S_\mu$ interpreted by the neighbours of $v_1, \ldots, v_\mu$.

Decomposition Theorem

$\forall k \exists \lambda \exists \mu$

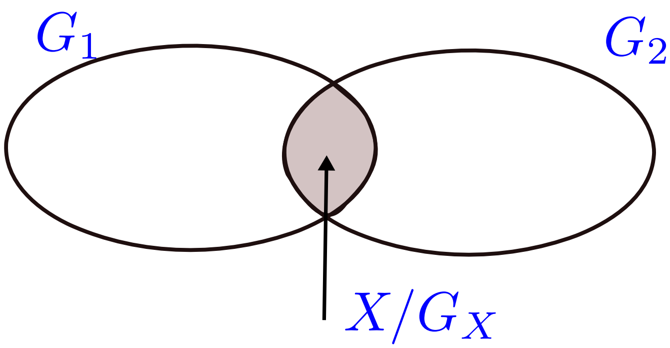
Any $G \in \mathcal{M}_k$ can be obtained from graphs in $\mathcal{L}_{\lambda,\mu}$ by a finite sequence of *clique sum* operations.

And the decomposition can be computed in time $O(n^4)$

Clique Sum: G_1, G_2 graphs with $X \subseteq G_1 \cap G_2$ a set of vertices that induces a clique in each of G_1 and G_2.

$$G_1 \oplus_{X,G_X} G_2$$

Take the disjoint sum of G_1 and G_2, identifying the two copies of X and replacing the clique by the graph G_X.



Congruences

For graphs $G \in \mathcal{L}_{\lambda,\mu}$, if X is a clique in G,

$$|X| < \lambda + \mu + 1$$

Thus, there are only finitely many operations of the form $\oplus_{X,G_X}$.

We have nearly satisfied the requirements for an application of the *automata-theoretic method*, but . . . .

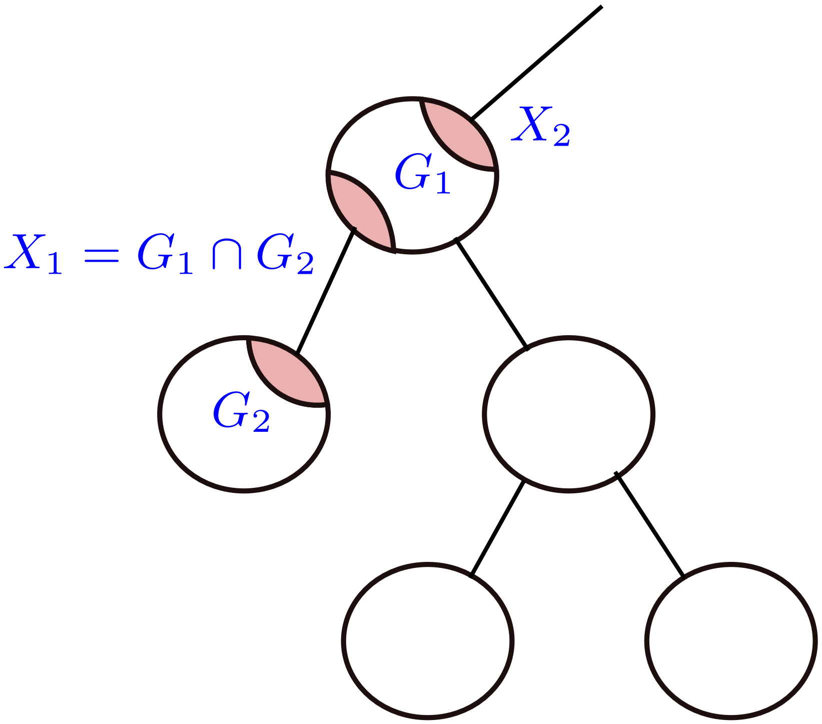
If $X = x_1, \ldots, x_s$, the $\equiv_m$-type of $(G, x_1, \ldots, x_s)$, where

$$G = G_1 \oplus_{X,G_X} G_2,$$

is given by the $\equiv_m$-types of $(G_1, x_1, \ldots, x_s)$ and $(G_2, x_1, \ldots, x_s)$.

However, different clique-sum operations may apply to different cliques X.

Bounding decompositions



While in a *bounded-width* tree-decomposition of G, the size of the individual bags is bounded, here we only have a bound on the size of the *intersections* between bags.

What we do have is a bound on the *local tree-width* of the bags G_1 (by replacing structures in $\mathcal{L}_{\lambda,\mu}$ by their encodings in $\mathcal{L}_\lambda$).

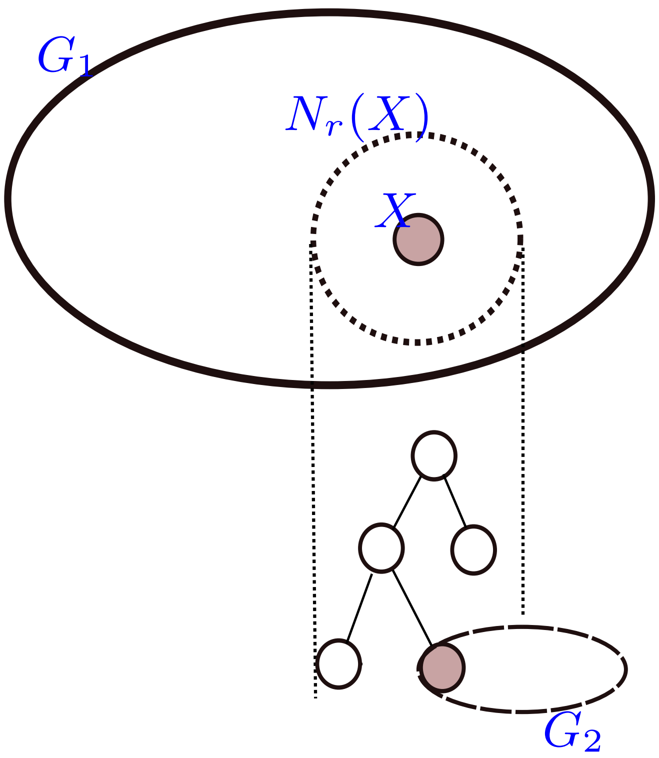
Idea: the type of X_2 in $G_1 \oplus_X G_2$ is determined by the type of $(G_1, \bar{x_2})$, the type of $(G_2, \bar{x_1})$ and the *local neighbourhood* of the clique X_1 in G_1.

Typing the Sum

The tree-decomposition of $N_r^{G_1}(X)$ determines a function θ that takes the $\equiv_m$-type of $(G_2, \bar{x_2})$ to the $\equiv_m$-type of $N_r^{G_1}(X) \oplus_X (G_2, \bar{x_2})$

There are only finitely many such functions θ.

Define the asymmetric clique-sum *of type* θ:



$$(G_1, \bar{y}) \oplus^{\theta}_{X, G_X} (G_2, \bar{x})$$

of taking the clique-sum of the two graphs, joining $\bar{x}$ to a clique in G_1 whose neighbourhood has type θ.

Automata on $\mathcal{M}_k$

Given a first-order sentence φ, it determines a radius of locality r and quantifier rank m.

- We have a finite collection of operations $\oplus^{\theta}_{X,G_X}$ (depending on r and m).
- We have structures $(\mathbb{A}, \bar{x})$, where the length of x is bounded by s (depending only on k).

  Thus, there are only finitely many $\equiv_m$ classes.
- $\equiv_m$ is a congruence for each operation $\oplus^{\theta}_{X,G_X}$.

Thus, *first-order logic* is *fixed-parameter tractable* on $\mathcal{M}_k$.

(Flum-Grohe)

Locally Excluded Minors

Let $t : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function.

LEM$_t$—the class of structures $\mathbb{A}$ such that for every $a \in A$:

$$K_{t(r)} \not\preceq GN_r^{\mathbb{A}}(a)$$

We say that $\mathcal{C}$ *locally excludes minors* if there is some function t such that $\mathcal{C} \subseteq$ LEM$_t$.

Theorem (D., Grohe, Kreutzer)

First-order logic is fixed-parameter tractable on every class $\mathcal{C}$ that locally excludes minors.

Application of Locality Method?

The result would be an easy application of the *locality method* if we had established:

> There is an algorithm deciding $\mathbb{A} \models \varphi$ in time $f(l,k)n^c$
> where k is the least value such that $K_k \not\prec G\mathbb{A}$

While the **Flum-Grohe** theorem does give a $O(n^5)$ algorithm for the class of structures that exclude K_k as a minor, *it is not clear if the constants are bounded by a computable function of* k.

Potential Sources of Uncomputability

1. The algorithm decomposing a graph in $\mathcal{M}_k$ over the class $\mathcal{L}_{\lambda,\mu}$ relies on a membership test in a *minor-closed superclass* of $\mathcal{M}_k$. It is not known whether the excluded minors for this class are given by a computable function of k.

2. The algorithm for reducing structures in $\mathcal{L}_{\lambda,\mu}$ to $\mathcal{L}_{\lambda}$ relies on membership tests for $\mathcal{L}_{\lambda,\mu'}$ (for $\mu' \leq \mu$) and it is not known if the excluded minors for these classes are given by a computable function of k.

(D., Grohe, Kreutzer) gives constructive solutions to both these problems.

Constructive Decomposition

There is a *uniform in k* algorithm which computes a decomposition of a graph $G \in \mathcal{M}_k$ over $\mathcal{L}_{\lambda,\mu}$ in time $O(n^4)$.

Instead of *clique-sums*, the decomposition uses *neighbourhood-sums*.

$$(G_1, x) \odot_x (G_2, x)$$

is the graph obtained by taking the *disjoint sum* of G_1 and G_2 while identifying $N_1(x)$ and *deleting* x.

It is also shown that given $G \in \mathcal{L}_{\lambda,\mu}$, we can effectively find $v_1, \ldots, v_\mu$ such that $G \setminus \{v_1, \ldots, v_\mu\} \in \mathcal{L}_{\lambda'}$ for some λ' computable from λ.

Review

We have gone from graphs of *bounded size* to *locally excluded minors* in four steps, alternating decomposition steps with localisation steps.

In general, these steps may be carried out for logics more expressive than first-order logic.

- Each decomposition gives rise to a notion of automata. What is the full power of these automata?
- Locality steps would work, not just for first-order logic, but the “Gaifman closure” of the logic. What is its power?