Fixed-Point Logic with Counting.

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Finite Model Theory

In *finite model theory* we are concerned with studying the *definability* of classes of finite relational structures by means of formulas of logic.

$\mathrm{Mod}(\varphi)$ denotes the *finite* models of φ

Specifically, the study of *descriptive complexity* relates definability to the *computational complexity* of the decision problem:

Given $\mathbb{A}$ *decide if* $\mathbb{A} \models \varphi$

Many of the examples in this talk concern finite structures over a vocabulary with *one binary relation*, which we think of as *finite graphs*:

$$G = (V, E)$$

First Order Formulas

$$\forall x \forall y \forall z (\neg E(x,y) \vee \neg E(x,z) \vee \neg E(y,z))$$

defines the graphs that do not contain a triangle.

For any first-order sentence φ, $\mathrm{Mod}(\varphi)$ is trivially decidable (in *polynomial time* and *logarithmic space*).

There are computationally easy classes that are not defined by any first-order sentence.

- The class of graphs with an even number of vertices.
- The class of graphs that are connected.

First Order Theories

For every finite structure $\mathbb{A}$, there is a first-order sentence $\varphi_{\mathbb{A}}$ defining the structures isomorphic to $\mathbb{A}$.

Every *isomorphism-invariant* class S of finite structures is definable by a *first-order theory* T:

$$T = \{\neg\varphi_{\mathbb{A}} \mid \mathbb{A} \not\in S\}$$

The interesting definability questions are obtained by considering:

- *extensions* of first-order logic; or equivalently
- *restricted* first-order theories.

Fixed-Point Logic with Counting

FPC is the extension of first-order logic with a mechanism for *iteration* and a mechanism for *counting*.

It was proposed by **Immerman** as a possible logic for *capturing* P:

It was proved **(Cai, Fürer, Immerman 1992)** that there are polynomial-time graph properties that are *not* expressible in FPC.

A number of other results about the limitations of FPC followed.

Still, FPC forms a *natural* and *powerful* fragment of P.
In this talk, we look at three recent, *positive* results on the expressive power of FPC.

Fixed-Point Logic

The logic FP is formed by closing first-order logic under the rule:

If φ is a formula, positive in the relational variable R, then so is

$$[\mathbf{lfp}_{R,\mathbf{x}}\varphi](\mathbf{t}).$$

The formula is read as:

the tuple $\mathbf{t}$ is in the least fixed point of the operator that maps R to $\varphi(R, \mathbf{x})$.

Connectivity

The formula

$$\forall u \forall v [\mathbf{lfp}_{T,xy}(x = y \lor \exists z(E(x,z) \land T(z,y)))](u,v)$$

is satisfied in a graph (V, E) if, and only if, it is connected.

The expressive power of FP properly extends that of first-order logic.

On structures which come equipped with a linear order FP expresses exactly the classes that are decidable in *polynomial time*.

(Immerman; Vardi)

In the *absence* of order, there is no formula of FP that defines the graphs with an even number of vertices.

Fixed-Point Logic with Counting

FPC is a logic formulated to add the ability to count to FP.

If $\varphi(x)$ is a formula with free variable x, then $\#x\varphi$ is a term denoting the number of elements satisfying φ.

Formulae of FPC:

- all atomic formulae as in FP;
- $\tau_1 < \tau_2$; $\tau_1 = \tau_2$ where τ_i is a term of numeric sort;
- $\exists x\, \varphi$; $\exists \nu\, \varphi$; where ν is a variable ranging over numbers up to the size of the domain;
- $[\mathbf{lfp}_{X,\mathbf{x},\nu}\varphi](\mathbf{t})$; and
- $\varphi \land \psi$; $\neg\varphi$.

Counting Quantifiers

C^k is the logic obtained from *first-order logic* by allowing:

- *counting quantifiers*: $\exists^i x\, \varphi$; and
- only the variables $x_1, \ldots . x_k$.

Every formula of C^k is equivalent to a formula of first-order logic, albeit one with more variables.

For every sentence φ of FPC, there is a k such that φ is equivalent to a *theory* of C^k.

Indeed, for any fixed n, there is a formula of C^k equivalent to φ on structures with at most n elements.

Cai-Fürer-Immerman

There are polynomial-time decidable properties of graphs that are not definable in FPC. **(Cai, Fürer, Immerman, 1992)**

Other inexpressiblity results for FPC follow, either as a consequence of **(Cai, Fürer, Immerman, 1992)** or by similar methods:

- *Hamiltonian Cycle* and *Satisfiability* are not definable in FPC.
- *3-Colourability* is not definable in FPC.

  (D. 1998)
- Solvability of systems of linear equations (over any fixed finite Abelian group) is not definable in FPC

  (Atserias, Bulatov, D. 2009)

All of these are shown, in fact, to be not axiomatizable in C^k, for any k.

Restricted Graph Classes

If we restrict the class of structures we consider, FPC may be powerful enough to express all polynomial-time decidable properties.

1. FPC captures P on *trees*. **(Immerman and Lander 1990)**.
2. FPC captures P on any class of graphs of *bounded treewidth*. **(Grohe and Mariño 1999)**.
3. FPC captures P on the class of *planar graphs*. **(Grohe 1998)**.
4. FPC captures P on any *proper minor-closed class of graphs*. **(Grohe 2010)**.

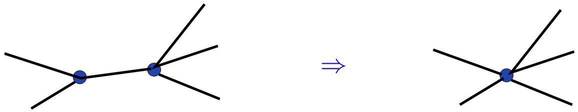
In each case, the proof proceeds by showing that for any G in the class, a *canonical, ordered* representaton of G can be interpreted in G using FPC.

Graph Minors

Graph Minors

We say that a graph H is a minor of graph G (written $H \preceq G$) if H can be obtained from G by repeated applications of the operations:

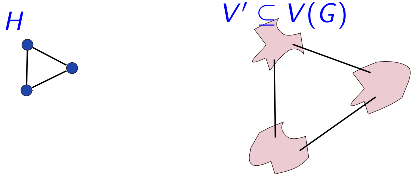
- *delete an edge*;
- *delete a vertex* (and all incident edges); and
- *contract an edge*



Graph Minors

Alternatively, $H = (U, F)$ is a minor of $G = (V, E)$, if there is a set $V' \subseteq V$ and a surjective map $M : V' \to U$ such that

- for each $u \in U$, $M^{-1}(u)$ is a connected subgraph of G; and
- for each edge $(u, v) \in F$, there is an edge in E between some $x \in M^{-1}(u)$ and some $y \in M^{-1}(v)$.



Robertson-Seymour

Recall: G is planar if, and only if, $K_5 \not\preceq G$ and $K_{3,3} \not\preceq G$.

Theorem (Robertson-Seymour)
In any infinite collection $\{G_i \mid i \in \omega\}$ of graphs, there are i, j with $G_i \preceq G_j$.

Corollary
For any class $\mathcal{C}$ *closed under minors*, there is a finite collection $\mathcal{F}$ of graphs such that $G \in \mathcal{C}$ *if, and only if,* $F \not\preceq G$ for all $F \in \mathcal{F}$.

A consequence is that any $\mathcal{C}$ *closed under minors* is decidable in polynomial-time.

The proof relies on Robertson and Seymour's *structure theorem*:

> *A graph G that excludes a minor K_k admits a tree-decomposition in which each bag is almost embeddable in a surface of genus k'*

Treelike Decompositions

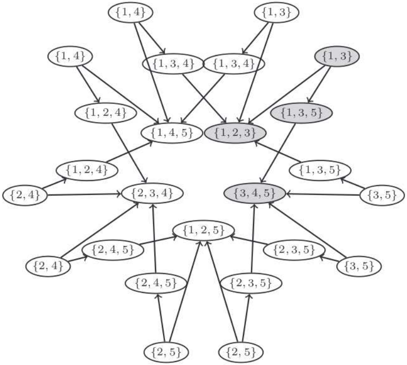
Grohe's proof is a version of the structure theorem with *definable decompositions*.

A *treelike decomposition* of a graph G is a *directed acyclic graph* D, with a *bag* $\beta(d) \subseteq V(G)$ of vertices associated with each node of D and satisfying certain *connectedness* and *consistency* conditions.

A treelike decomposition of G can be obtained (for instance) from a *tree decomposition* by closing it under the *automorphisms* of G—starting at leaves and working upwards.

Treelike Decomposition of a 5-cycle

The picture shows a treelike decomposition of a 5-cycle C_5.
The *grey nodes* form a tree decomposition.



picture credit: M. Grohe: JACM, 59(5), 27.

Definable Treelike Decompositions

Grohe shows that there is an FPC-definable decomposition of *planar graphs* into their *3-connected* components.

This is lifted into a decomposition of graphs *embeddable* in an arbitrary surface.

More heavy lifting is required to obtain a *definable treelike decomposition* of the class of graphs *excluding a K_k-minor* into components that can be almost embedded in a surface.

This is used to show that for each k, there is a k' such that on graphs excluding K_k as a minor, $C^{k'}$ defines isomorphism.

As a consequence, *every* class of graphs closed under taking minors is definable in FPC.

Linear Programming

Linear Programming

We can represent an instance of a linear programming feasibility problem as a *relational structure* over a suitable vocabulary.

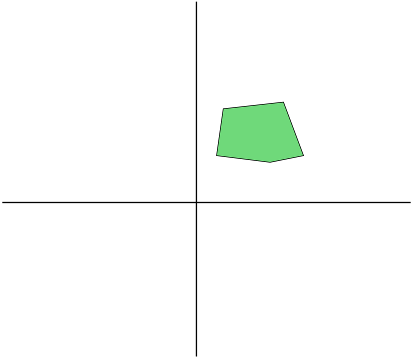
We have a set C of *constraints* over a set V of *variables*.
Each $c \in C$ consists of $a_c \in \mathbb{Q}^V$ and $b_c \in \mathbb{Q}$.
The numbers are encoded in *binary* over an ordered set of *bit positions*.

Feasibility Problem: Given a linear programming instance, determine if there is an $x \in \mathbb{Q}^V$ such that:

$$a_c^T x \leq b_c \quad \text{for all } c \in C$$

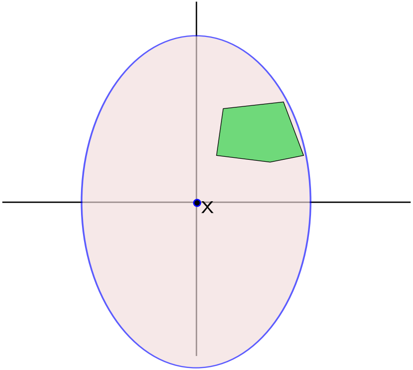
In **Anderson, D., Holm (2013)** we show that this, and the corresponding *optimization problem* are expressible in FPC.

Ellipsoid Method



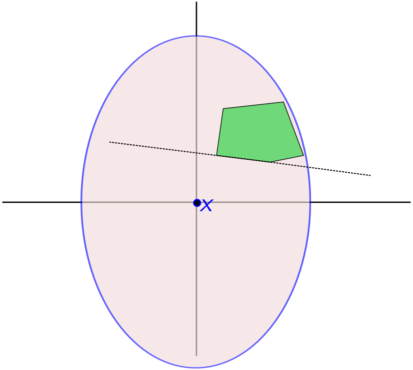
The set of constraints determines a *polytope*

Ellipsoid Method



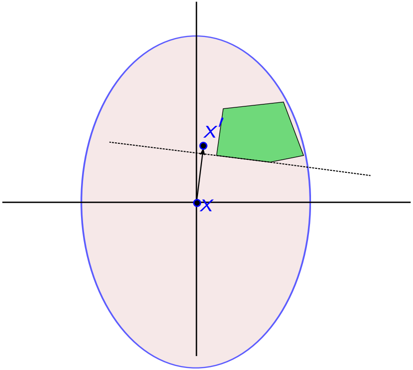
Start at the origin and calculate an *ellipsoid* enclosing it.

Ellipsoid Method



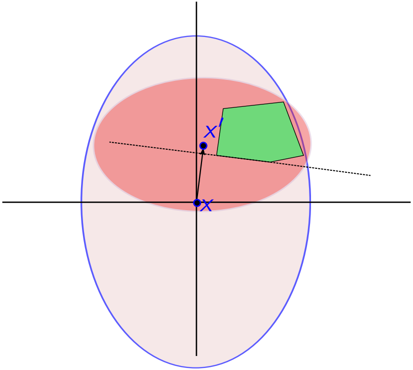
If the centre is not in the polytope, choose a constraint it *violates*.

Ellipsoid Method



Calculate a new *centre*.

Ellipsoid Method



And a new ellipsoid around the centre of at most *half* the volume.

Ellipsoid Method in FPC

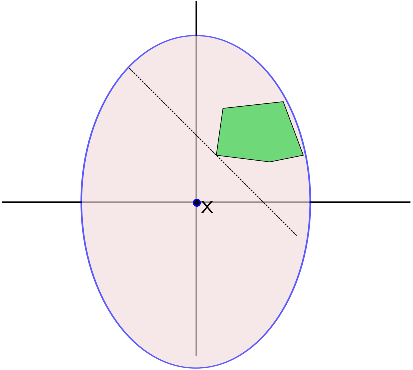
We can encode all the calculations involved in FPC.

This relies on expressing algebraic manilpulations of *unordered* matrices.

What is not obvious is how to *choose* the violated constraint on which to project.

However, the ellipsoid method works as long as we can find, at each step, some *separating hyperplane*.

Ellipsoid Method in FPC



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So, we can take:

$$(\sum_{c \in S} a_c)^T x \leq \sum_{c \in S} b_c$$

where S is the *set* of all violated constraints.

Separation Oracle

More generally, the ellipsoid method can be used, even when the *constraint matrix* is not given explicitly, as long as we can always determine a *separating hyperplane*.

In particular, the polytope represented may have *exponentially many* facets.

Anderson, D., Holm (2013) shows that as long as the *separation oracle* can be defined in FPC, the corresponding *optimization problem* can be solved in FPC.

Matching

In a *graph* $G = (V, E)$ a matching $M \subset E$ is a set of edges such that each vertex is incident on *at most* one edge in M.

The problem of finding a *maximum matching* in G can be represented by a linear program with *exponentially many* constraints.

We show that a *separation oracle* for this polytope is definable by an FPC formula interpreted in the graph G.

As a consequence, there is an FPC formula defining the *size* of the maximum matching in G.

Blass, Gurevich and Shelah (2001) had shown that matching on *bipartite* graphs is definable in FPC and conjectured that this was *not* true on general graphs.

Symmetric Circuits

Circuit Complexity

A *language* $L \subseteq \{0,1\}^*$ can be described by a family of *Boolean functions*:

$$(f_n)_{n \in \omega} : \{0,1\}^n \to \{0,1\}.$$

Each f_n may be computed by a *circuit* C_n made up of

- Gates labeled by Boolean operators: $\wedge, \vee, \neg$,
- Boolean inputs: $x_1, \ldots, x_n$, and
- A distinguished gate determining the output.

If there is a polynomial $p(n)$ bounding the *size* of C_n, i.e. the number of gates in C_n, the language L is in the class $\mathrm{P/poly}$.

If, in addition, the function $n \mapsto C_n$ is computable in *polynomial time*, L is in P.

Note: For these classes it makes no difference whether the circuits only use $\{\wedge, \vee, \neg\}$ or a richer basis with *threshold* or *majority* gates.

Circuits for Graph Properties

A property of *graphs* (or other relational structures) in P is recognised by a family of Boolean circuits $(C_n)_{n\in\omega}$ where:

- inputs to C_n are n^2 *potential edges*, each taking value 0 or 1;
- the size of C_n is bounded by a polynomial $p(n)$; and
- the family is uniform, so the function $n \mapsto C_n$ is in P (or $\mathrm{DLogTime}$).

C_n is *invariant* if, for every input graph, the output is unchanged under a permutation of the inputs induced by a permutation of $[n]$.

Symmetric Circuits

Say C_n is *symmetric* if any permutation of $[n]$ applied to its inputs can be extended to an automorphism of C_n.

- Any symmetric circuit is invariant, but *not* conversely.
- Any formula of *first-order logic* translates into a uniform family of *constant-depth*, *polynomial-size symmetric* Boolean circuits.

  For each subformula $\psi(\overline{x})$ and each assignment $\overline{a}$ of values to the free variables, we have a gate.

- Any formula φ of FP translates into a uniform family of polynomial-size *symmetric* Boolean circuits.
- Any formula of FPC translates into a uniform family of polynomial-size *symmetric* threshold (or majority) circuits.

Circuits and Fixed-Point Logic

We established the following in **Anderson, D. (2014)**:

Theorem

A class of graphs is accepted by a P-uniform symmetric family of Boolean circuits *if, and only if,* it is definable by an FP formula interpreted in $G \uplus ([n], <)$.

Theorem

A class of graphs is accepted by a P-uniform symmetric family of threshold circuits *if, and only if,* it is definable in FPC.

Main Technical Tools

For a symmetric circuit C_n we can assume *w.l.o.g.* that the automorphism group is the symmetric group S_n acting in the natural way.

For a gate g in C_n, $\mathrm{Stab}(g)$ denotes the *stabilizer group of* g, i.e.,

$$\mathrm{Stab}(g) = \{\pi \in S_n \mid \pi(g) = g\}.$$

Say a set $X \subseteq [n]$ *supports* g if

$$\mathrm{Stab}^{\bullet}(X) \subseteq \mathrm{Stab}(g),$$

where $\mathrm{Stab}^{\bullet}(X) := \{\pi \in S_n \mid \pi(x) = x \text{ for all } x \in X\}$ is the *pointwise stabilizer* of X.

Note: For the family of circuits $(C_n)_{n\in\omega}$ obtained from an FPC formula there is a constant k such that all gates in each C_n have a support of size at most k.

Support Theorem

Our main technical theorem shows that in *sub-exponential size* symmetric circuits, all gates have *small* support.

Theorem
For any $1 > \epsilon \geq \frac{2}{3}$, let C be a symmetric s-gate circuit over $[n]$ with $n \geq 2^{\frac{56}{\epsilon^2}}$, and $s \leq 2^{n^{1-\epsilon}}$. Then every gate g of C has a support of size at most $\frac{33}{\epsilon}\frac{\log s}{\log n}$.

Corollary
Polynomial-size symmetric circuits have constant support.

Some Consequences

There is no polynomial-size family of symmetric Boolean circuits deciding if an n vertex graph has an even number of edges.

Polynomial-size families of uniform symmetric *threshold circuits* are more powerful than Boolean circuits.

There is no translation of invariant circuit into equivalent symmetric threshold circuits, with only *polynomial blow-up*.

We get a natural and purely circuit-based characterisation of FPC definability.

Inexpressibility results for FPC are also lower bound results against a natural circuit class.

Conclusions

The intuition behind the conjecture that FPC captures P was that algorithmic techniques that are *obviously* polynomial-time are all expressible in the logic.

The **Cai-Fürer-Immerman** construction and related results show that one important polynomial-time algorithmic technique—*Gaussian elimination*—is not captured by the logic.

Recent results show that some very *non-trivial* and *non-obvious* polynomial-time problems can be expressed in FPC:

- *Linear Programming*
- *Arbitrary minor-closed classes*
- *Maximum Matching*

And, there is a *natural* circuit complexity class corresponding to FPC.