Model Theory of Tame Classes of Finite Structures

Part 3: Stability and Independence

Anuj Dawar

Department of Computer Science and Technology, University of Cambridge

(with assistance from Ioannis Eleftheriadis)

Logic Colloquium, Gothenburg 25 June 2024

Locality

Gaifman's Theorem: Every first-order sentence is equivalent to a Boolean combination of *basic local sentences*.

To evaluate a formula φ in a structure $\mathbb{A}$ it suffices to look at *local neighbourhoods* of elements of $\mathbb{A}$.

If the local neighbourhoods are strucurally simpler, this can allow for a *recursive* procedure for evaluating a formula efficiently.

But, sometimes the local neighbourhoods are *not* simpler.

Complete Graphs

Consider the class of *complete graphs*.
K_n: the complete graph on n vertices.

$N_1(v)$ is the *whole graph* for each v.
So, Gaifman's theorem provides no simplification.

Yet, evaluating *first-order sentences* in complete graphs is *easy*.

For a first-order formula φ in the *language* of graphs, let $\overline{\varphi}$ be the formula obtained by from φ by replacing every subformula $E(x, y)$ by $x \neq y$.

Then $K_n \models \varphi$ *if, and only if,* $\overline{K_n} \models \overline{\varphi}$
where $\overline{K_n}$ is the *edgeless graph* on n vertices.

Neighbourhood Covers

For a graph $G = (V, E)$ and $r \in \mathbb{N}$, a family $\mathcal{X}$ of subsets of V is a *distance-r neighborhood cover* of G if for every $v \in V$, there is an $X \in \mathcal{X}$ with $N_r(v) \subseteq X$.

The *radius* of $\mathcal{X}$ is the largest *diameter* of any connected component in a graph induced by $X \in \mathcal{X}$.

The *overlap* of $\mathcal{X}$ is the largest m such that some $v \in V$ appears in m distinct sets in $\mathcal{X}$.

The complete graph K_n has a neighboorhood cover of radius 1 and overlap 1.

Interpretations

A *graph interpretation* $\mathcal{I}$ is a pair of formulas $\delta(x), \varepsilon(x, y)$.

For a structure $\mathbb{A}$, define $\mathcal{I}(\mathbb{A})$ to be the graph with:

vertices $\{a \in \mathbb{A} \mid \mathbb{A} \models \delta[a]\}$; and

edges $\{\{a, b\} \subseteq \mathbb{A} \mid a \neq b \text{ and } \mathbb{A} \models \varepsilon[a, b]\}$.

We say a class $\mathcal{C}$ *interprets* $\mathcal{D}$ (or $\mathcal{D}$ is *interpretable* in $\mathcal{C}$) if there is an *interpretation* $\mathcal{I}$ with $\mathcal{D} \subseteq \mathcal{I}(\mathcal{C})$.

For $G \in \mathcal{D}$, with $G = \mathcal{I}(\mathbb{A})$,
$G \models \varphi$ *if, and only if,* $\mathbb{A} \models \hat{\varphi}$
where $\hat{\varphi}$ is obtained from φ by replacing $E(x, y)$ with $\varepsilon(x, y)$ and relativizing quantifiers to $\delta(x)$.

Transductions

Say that a class $\mathcal{C}$ of σ-structures *transduces* a class of graphs $\mathcal{D}$ if there is an *expansion*

$$\sigma^+ = \sigma \cup \{C_1, \ldots, C_m\}$$

with finitely many unary relation symbols

and a σ^+ graph interpretation $\mathcal{I}$ such that

$$\mathcal{D} \subseteq \mathcal{C}^+$$

where $\mathcal{C}^+$ is the class of σ^+-expansions of structures in $\mathcal{C}$.

Composition and Transduction Order

Transductions *compose*.

If $\mathcal{C}$ transduces $\mathcal{D}$ and $\mathcal{D}$ transduces $\mathcal{E}$ then $\mathcal{C}$ transduces $\mathcal{E}$.

If $\mathcal{C}$ transduces $\mathcal{D}$ then, *in some sense*, $\mathcal{D}$ is structurally "*no more complicated*" than $\mathcal{C}$.

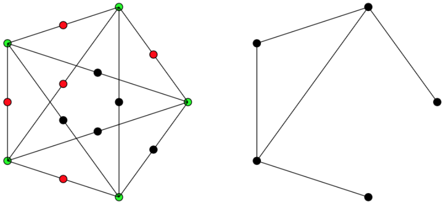
Thus, if a class $\mathcal{C}$ transduces the *class of all graphs*, it is as complicated as it gets.

Every *infinite* class of structures transduces the class of all *complete graphs*.

The class of complete graphs is as *simple* as it gets.

Subdivided Cliques

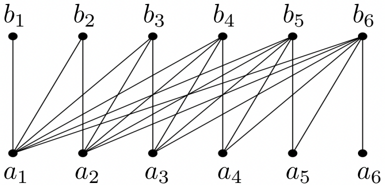
The class of 1-*subdivided cliques*: $\{K_n^1 \mid n \in \omega\}$ transduces the class of *all graphs*.



Half Graphs

A *half-graph of order* n is a *bipartite graph* on two sets of n vertices: $\{a_i \mid 1 \leq i \leq n\}$ and $\{b_i \mid 1 \leq i \leq n\}$ with edges:

$$\{a_i, b_j\}_{i \leq j}$$



Cliques and Half Graphs

The class of *complete graphs* does *not transduce* the class of *half graphs*.

In any expansion of K_n with m unary relations, the automorphism group has at most 2^{2m} *orbits* on pairs of elements.

In the half-graph or order n, we can define a linear order on $\{a_i \mid 1 \leq i \leq n\}$, giving an *unbounded* number of orbits.

Half-Graphs and Powerset graphs

The class of *half graphs* does not *transduce* the class of *all graphs*.

In particular, it does not transduce the class of *powerset graphs*.

The *powerset graph* of order n is the *bipartite graph* on two sets:

$\{a_i \mid i \in \{1, \ldots, n\}\}$ *and* $\{b_J \mid J \subseteq \{1, \ldots, n\}\}$

with edges: $\{a_i, b_J\}_{i \in J}$

The class of *powerset graphs* transduces the class of *all graphs*.

Structurally Tame Classes

A class of graphs $\mathcal{C}$ is *structurally nowhere dense* if it is *transduced* by a *nowhere dense* class.

A class of graphs $\mathcal{C}$ is *monadically stable* if it does *not* transduce the class of *half graphs*.

A class of graphs $\mathcal{C}$ is *monadically NIP* if it does not tranduce the class of *all graphs*.

Inclusions

Any *nowhere dense* class is *structurally nowhere dense*.

The complete graphs are an example of a *structurally nowhere dense* class that is not *nowhere dense*.

(Adler and Adler) showed that any *nowhere dense* class is *stable*.

It follows that such a class is *monadically stable*.

It also follows that any *structurally nowhere dense* class is *monadically stable*

Any *monadically stable* class is *monadically NIP*.

Separations

The *complete graphs* are *structurally nowhere dense* but not *nowhere dense*.

The *half graphs* are *monadically NIP* but not *monadically stable*.

We do not have an example of a class of graphs that is *monadically stable* but not *structurally nowhere dense*.

> **Sparsification Conjecture:**
> *Every monadically stable class is structurally nowhere dense*

Monotone Classes

Every *nowhere dense* class of graphs is *monadically stable*

(Adler and Adler)

Every *monotone*, *monadically NIP* class is *nowhere dense*

This follows from the fact that if $\mathcal{C}$ is not nowhere dense, there is a fixed r such that for infinitely many n, K_n^r appears as a subgraph of a graph in $\mathcal{C}$.

Thus, for a *monotone* class $\mathcal{C}$ TFAE:

1. $\mathcal{C}$ is *nowhere dense*
2. $\mathcal{C}$ is *structurally nowhere dense*
3. $\mathcal{C}$ is *monadically stable*
4. $\mathcal{C}$ is *monadically NIP*

Edge Stable

Say a class $\mathcal{C}$ of graphs is *edge stable* if it does not contain arbitrarily large *half graphs* as *semi-induced subgraphs*.

A half graph of order n is a *semi-induced* subgraph of G if G contains two disjoint sets of n vertices: $A = \{a_i \mid 1 \leq i \leq n\}$ and $B = \{b_i \mid 1 \leq i \leq n\}$ so that the *only* edges between A and B are: $\{a_i, b_j\}_{i \leq j}$

Theorem (Nešetřil et al.)
A graph class $\mathcal{C}$ is *monadically stable* if, and only if, it is *monadically NIP* and *edge stable*.

Flips

The *monadically stable* classes admit a characerization in terms of *two player games*.

For this, we first define the notion of a *flip*.

Given a graph $G = (V, E)$ and a set $A \subseteq V$, the *flip* $G \oplus A$ is the graph (V, E') where
$\{u, v\} \in E'$ *if, and only if,*

$u, v \in A$ and $\{u, v\} \notin E$; or

$u \notin A$ or $v \notin A$ and $\{u, v\} \in E$.

We *complement* the graph induced by A.

Flip Rank

For a graph G and $r \in \mathbb{N}$, we define the *flip rank* of G by the following recursion.

$$\mathrm{rank}_r^F(G) = \begin{cases} 1, & \text{if } |G| = 1; \\ 1 + \min \mathrm{rank}_r^F(G \oplus A), & \text{if } G \text{ has radius} \leq r \text{ and } |G| > 1; \\ \max \mathrm{rank}_r^F(B_r(G, v)) & \text{otherwise.} \end{cases}$$

where the $\min$ is taken over all sets A of vertices, and the $\max$ over all vertices v.

Flipper-Connector Game

- $G_0 := G$;
- If G_i contains *no edges*, *Flipper* has won;
- Otherwise: *Connector* chooses a *vertex* u of G_i and *Flipper* chooses a set $A \subseteq V(G_i)$.
- $G_{i+1} := N_r^{G_i}(u) \oplus A$.

$\mathrm{rank}_r^F(G) \leq k$ *if, and only if,* there is a *strategy* for *Flipper* to win in no more than k rounds.

Examples

If v is a vertex in G and N the set of its *neighbours*, then

$$(G \oplus (N \cup \{v\})) \oplus N$$

is the graph G with all edges incident on v removed.

Thus $\mathrm{rank}_r(G) \leq 2\mathrm{rank}_r^F(G)$.

$\mathrm{rank}_r^F(K_n) = 2$

Flipper Rank and Monadic Stability

Theorem (Gajarsky et al.)
A class of graphs $\mathcal{C}$ is *monadically stable* if, and only if, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{rank}_r^F(G) \leq f(r)$ for every $r \in \mathbb{N}$ and every $G \in \mathcal{C}$.

Note: This, with $\mathrm{rank}_r(G) \leq 2\mathrm{rank}_r^F(G)$ gives another proof that every nowhere dense class is monadically stable.

First-Order Evaluation

The *recursive* definition of monadic stability enables a strategy for evaluating *first-order formulas*:

Theorem (Dreier et al.)
For any *monadically stable* class $\mathcal{C}$, the problem of evaluating first-order sentences ($G \models \varphi$) for graphs $G \in \mathcal{C}$ is FPT.

This is based on a *combinatorial* construction showing that in a *monadically stable* class $\mathcal{C}$, distance-r neighbourhood covers of small diameter and overlap can be found.

Main Conjecture

Conjecture
For any *hereditary* class of structures $\mathcal{C}$, the evaluation of first-order sentences on $\mathcal{C}$ is FPT *if, and only if,* $\mathcal{C}$ is *monadically NIP*.

One direction has been very recently proved (subject to the usual *complexity-theoretic assumptions*):

Theorem (Dreier, Mählmann, Toruńczyk)
If $\mathcal{C}$ is hereditary and *not* monadically NIP, then the evaluation of first-order sentences on $\mathcal{C}$ is AW$[\star]$-hard.

What is missing is an *algorithmic tractability* result for all *monadically NIP* classes.

Tractability Results

If $\mathcal{C}$ is *monotone* and *monadically NIP*, the evaluation of first-order sentences on $\mathcal{C}$ is FPT.

If $\mathcal{C}$ is *edge-stable* and *monadically NIP*, the evaluation of first-order sentences on $\mathcal{C}$ is FPT.

Theorem (Bonnet et al.)
If $\mathcal{C}$ is a hereditary class of *ordered graphs monadically NIP*, the evaluation of first-order sentences on $\mathcal{C}$ is FPT. and

Conclusion

The complexity of the evaluation of first-order sentences is a *benchmark* of the *tameness* of a class of finite structures.

The tame *monotone, sparse* classes of structures have been completely mapped out.

An ongoing effort to completely map the *hereditary* classes.

Borrowing notions and methods from *stability theory* and combining with some hard combinatorics.